Truth-functions

Def. A truth-function or Boolean function is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ induces a truth-function.

by its inductive structure, using the truth tables

Notation in the book…

$$\begin{array}{l} \quad a, b \\ \left\{ \begin{array}{l} (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array} \right. \end{array}$$

P(a,b): $(a \wedge b) \vee b$

Truth-functions

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c)$: $(a \wedge b) \vee b$

induces

$$
\begin{array}{l}
\quad a,\ b,\ c \\
\left\{
\begin{array}{l}
(0,0,0) \longmapsto 0 \\
(0,0,1) \longmapsto 0 \\
(0,1,0) \longmapsto 1 \\
(0,1,1) \longmapsto 1 \\
(1,0,0) \longmapsto 0 \\
(1,0,1) \longmapsto 0 \\
(1,1,0) \longmapsto 1 \\
(1,1,1) \longmapsto 1
\end{array}
\right.
\end{array}
$$

Equivalence of propositions

Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\overset{val}{=}$ is an equivalence on the set of all abstract propositions

i.e., for all abstract propositions P, Q, R,
(1) $P \overset{val}{=} P$; (2) if $P \overset{val}{=} Q$, then $Q \overset{val}{=} P$; and
(3) if $P \overset{val}{=} Q$ and $Q \overset{val}{=} R$, then $P \overset{val}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1			
0	1	1			
1	0	0			
1	1	0			

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	
0	1	1	0	0	
1	0	0	1	0	
1	1	0	0	0	

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent

$$b \wedge \neg b \overset{val}{=} c \wedge \neg c$$

Tautologies and contradictions

Def. An abstract proposition P is a tautology iff
its truth-function is constant 1.

all tautologies are equivalent

Def. An abstract proposition P is a contradiction iff
its truth-function is constant 0.

all contradictions are equivalent

but not all contingencies!

Def. An abstract proposition P is a contingency iff
it is neither a tautology nor a contradiction.

Abstract propositions

Definition

Basis (Case 1) T and F are abstract propositions.
Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive definition

Propositional Logic
Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \overset{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$
$$P \vee Q \stackrel{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \overset{val}{=} P$$

$$P \vee P \overset{val}{=} P$$

$$P \Rightarrow P \overset{val}{\neq} P$$

$$P \Leftrightarrow P \overset{val}{\neq} P$$

Double negation

$$\neg\neg P \overset{val}{=} P$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \overset{val}{=} F$$

Excluded Middle

$$P \vee \neg P \overset{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \wedge T \stackrel{val}{=}$$

$$P \wedge F \stackrel{val}{=}$$

$$P \vee T \stackrel{val}{=}$$

$$P \vee F \stackrel{val}{=}$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$

$$P \wedge F \stackrel{val}{=} F$$

$$P \vee T \stackrel{val}{=} T$$

$$P \vee F \stackrel{val}{=} P$$

Distributivity, De Morgan

Distributivity

$$P \land (Q \lor R) \overset{val}{=} (P \land Q) \lor (P \land R)$$

$$P \lor (Q \land R) \overset{val}{=} (P \lor Q) \land (P \lor R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$$

common
mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=}$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \overset{val}{=} T$$

Calculating with equivalent propositions
(the use of standard equivalences)

Recall…

Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

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i.e., for all abstract propositions P, Q, R,
(1) $P \overset{val}{=} P$; (2) if $P \overset{val}{=} Q$, then $Q \overset{val}{=} P$; and
(3) if $P \overset{val}{=} Q$ and $Q \overset{val}{=} R$, then $P \overset{val}{=} R$

Substitution

meta rule

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]}$$

The rule of Leibnitz

meta rule

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!