Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

We can already prove these!

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

But not yet these two…

Theorem C4

The class of regular languages is closed under Kleene star

Equivalence of regular expressions and regular languages

Theorem (Kleene)

A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

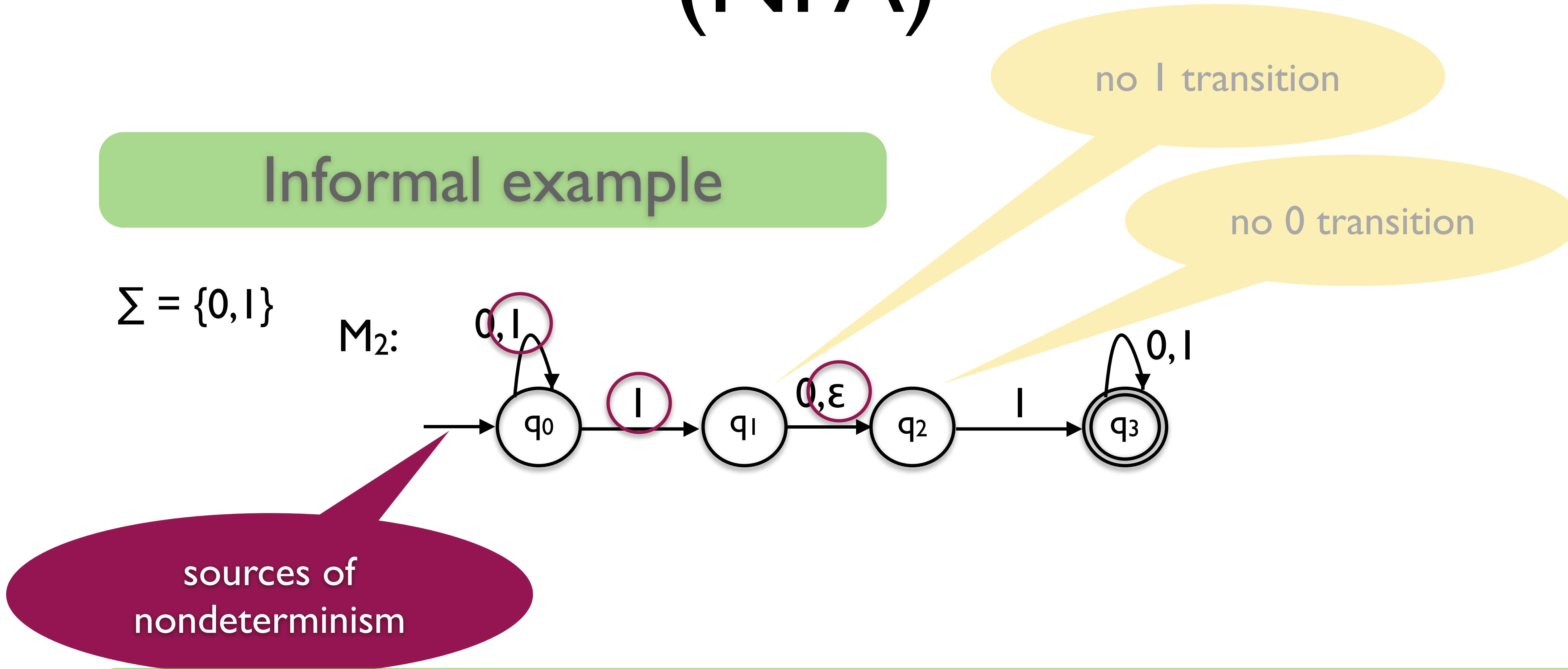
needs nondeterminism

Proof $\Leftarrow$ easy, as the constructions for the closure properties,
$\Rightarrow$ not so easy, we'll skip it for now…

Nondeterministic Automata (NFA)

Informal example

$\Sigma = \{0,1\}$



Accepts a word iff there **exists** an accepting run

NFA

Definition

A nondeterministic automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

Q is a finite set of states
Σ is a finite alphabet
$\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function
q_0 is the initial state, $q_0 \in Q$
F is a set of final states, $F \subseteq Q$

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

In the example M_2

$M_2 = (Q, \Sigma, \delta, q_0, F)$ for

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$ $F = \{q_3\}$

$\delta(q_0, 0) = \{q_0\}$
$\delta(q_0, 1) = \{q_0, q_1\}$
$\delta(q_0, \varepsilon) = \emptyset$
…..

NFA

ε-closure of q, all states reachable by ε-transitions from q

$E(q) = \{q' \mid q' = q \vee \exists n \in \mathbb{N}^+ . \exists q_0, .., q_n \in Q . q_0 = q, q_n = q', q_{i+1} \in \delta(q_i, \varepsilon), \text{ for } i = 0, .., n-1\}$

The extended transition function

Given an NFA $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta : Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ to

$$\delta^* : Q \times \Sigma^* \longrightarrow \mathcal{P}(Q)$$

$E(X) = \bigcup_{x \in X} E(x)$

In M_2, $\delta^*(q_0, 0110) = \{q_0, q_2, q_3\}$

inductively, by:

$$\delta^*(q, \varepsilon) = E(q) \quad \text{and} \quad \delta^*(q, wa) = E\left(\bigcup_{q' \in \delta^*(q,w)} \delta(q', a)\right)$$

Definition

The language recognised / accepted by a automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \varnothing\}$$

$L(M_2) = \{u101w \mid u,w \in \{0,1\}^*\}$

$\cup$

$\{u11w \mid u,w \in \{0,1\}^*\}$

Equivalence of automata

Definition

Two automata M_1 and M_2 are equivalent if $L(M_1) = L(M_2)$

Theorem NFA ~ DFA

Every NFA has an equivalent DFA

Proof via the "powerset construction" / determinization

Corollary

A language is regular iff it is recognised by a NFA

Closure under regular operations

Theorem C1

The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

Now we can prove these too

Theorem C4

The class of regular languages is closed under Kleene star

Nonregular languages

every long enough word of a regular language can be pumped

Theorem (Pumping Lemma)

If L is a regular language, then there is a number $p \in \mathbb{N}$ (the pumping length) such that for any $w \in L$ with $|w| \geq p$, there exist $x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $xy^iz \in L$, for all $i \in \mathbb{N}$
2. $|y| > 0$
3. $|xy| \leq p$

Proof easy, using the pigeonhole principle

Example "corollary"

$L = \{ 0^n1^n \mid n \in \mathbb{N}\}$ is nonregular.

Note the logical structure!