Closure under regular operations

Theorem C1

The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

Theorem C4

The class of regular languages is closed under Kleene star

Now we can prove these too

Regular expressions

finite representation of infinite languages

inductive

example: $(ab \cup a)^*$

Definition

Let Σ be an alphabet. The following are regular expressions

1. a for $a \in \Sigma$
2. ε
3. $\varnothing$
4. $(R_1 \cup R_2)$ for R_1, R_2 regular expressions
5. $(R_1 \cdot R_2)$ for R_1, R_2 regular expressions
6. $(R_1)^*$ for R_1 regular expression

corresponding languages

$$L(a) = \{a\}$$
$$L(\varepsilon) = \{\varepsilon\}$$
$$L(\varnothing) = \varnothing$$
$$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$$
$$L(R_1 \cdot R_2) = L(R_1) \cdot L(R_2)$$
$$L(R_1^*) = L(R_1)^*$$

Equivalence of regular expressions and regular languages

Theorem (Kleene)

A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

Proof $\Leftarrow$ easy, as the constructions for the closure properties,
$\Rightarrow$ not so easy, we'll skip it for now…

Nonregular languages

every long enough word of a regular language can be pumped

Theorem (Pumping Lemma)

If L is a regular language, then there is a number $p \in \mathbb{N}$ (the pumping length) such that for any $w \in L$ with $|w| \geq p$, there exist $x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $xy^iz \in L$, for all $i \in \mathbb{N}$
2. $|y| > 0$
3. $|xy| \leq p$

Proof easy, using the pigeonhole principle

Example "corollary"

$L = \{ 0^n1^n \mid n \in \mathbb{N}\}$ is nonregular.

Note the logical structure!