Strengthening and weakening

Strengthening

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always when P has truth value 1,
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always when P is true, Q is also true

Q is weaker than P

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi\{\xi/P\} \overset{val}{\models} \psi\{\xi/P\}}$$

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

Substitution

just holds

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Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

EVERY occurrence of P is substituted!

Simultaneous

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P,\eta/Q] \overset{val}{\models} \psi[\xi/P,\eta/Q]}$$

The rule of Leibnitz

does not hold for weakening!

Leibnitz

$$\cancel{\dfrac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}}$$

formula that has ϕ as a sub formula

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Leibniz

$$\cancel{\dfrac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}}$$

Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$

Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

Every player except the winner looses a match

Unary predicate (example)

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Whether this statement is true or false depends on the value of m (and on the domain of values).

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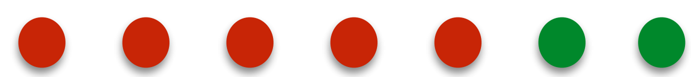

on $\mathbb{Z}$

… -3 -2 -1 0 1 2 3 …

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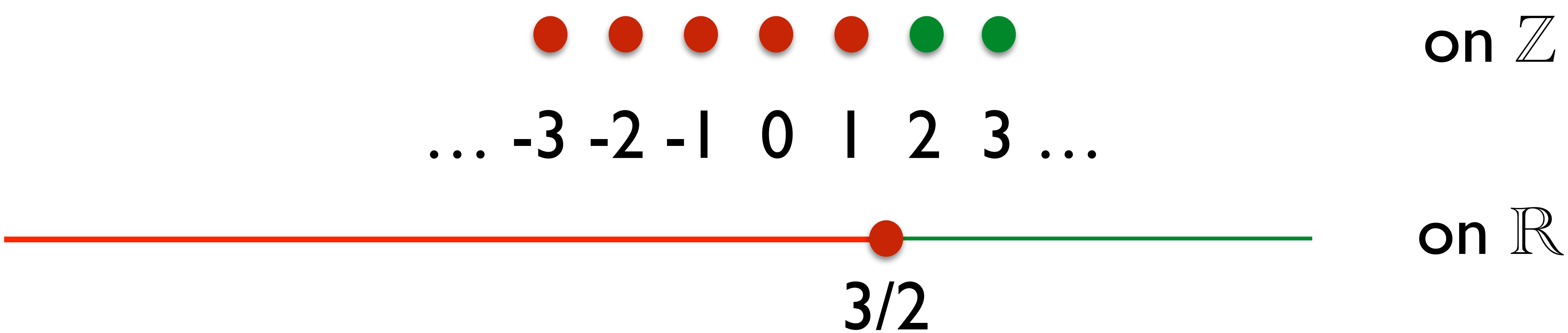

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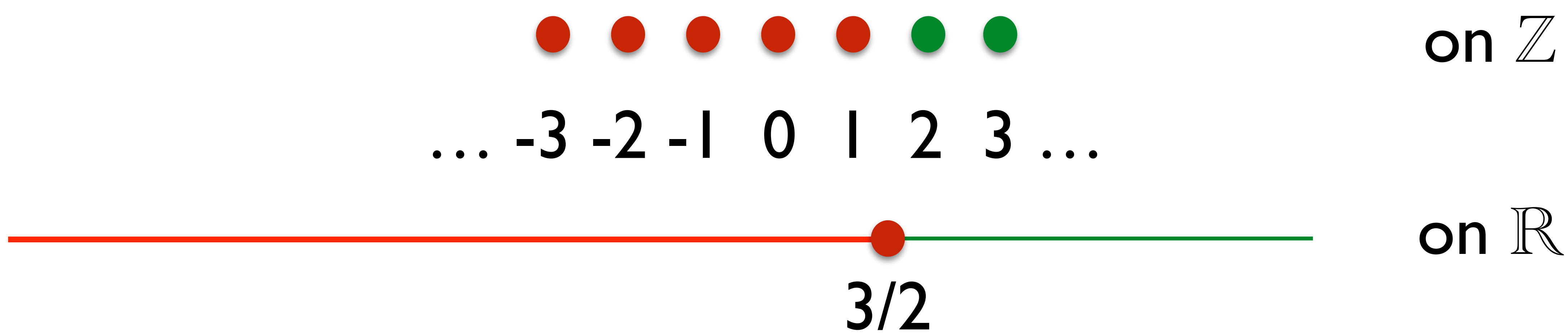

Note: $2m > 3 \stackrel{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \stackrel{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

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a unary relation

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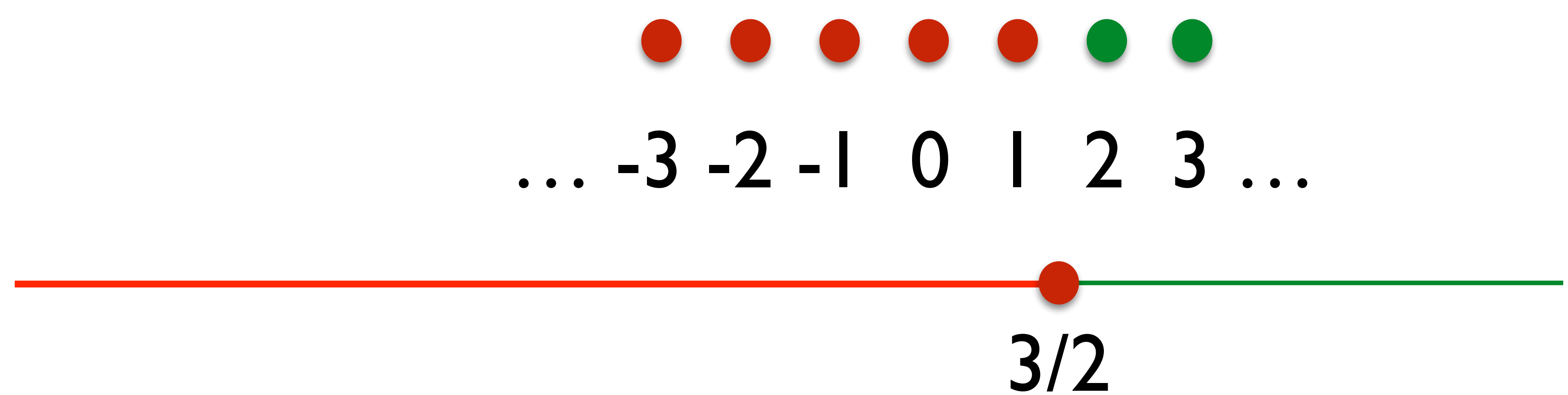

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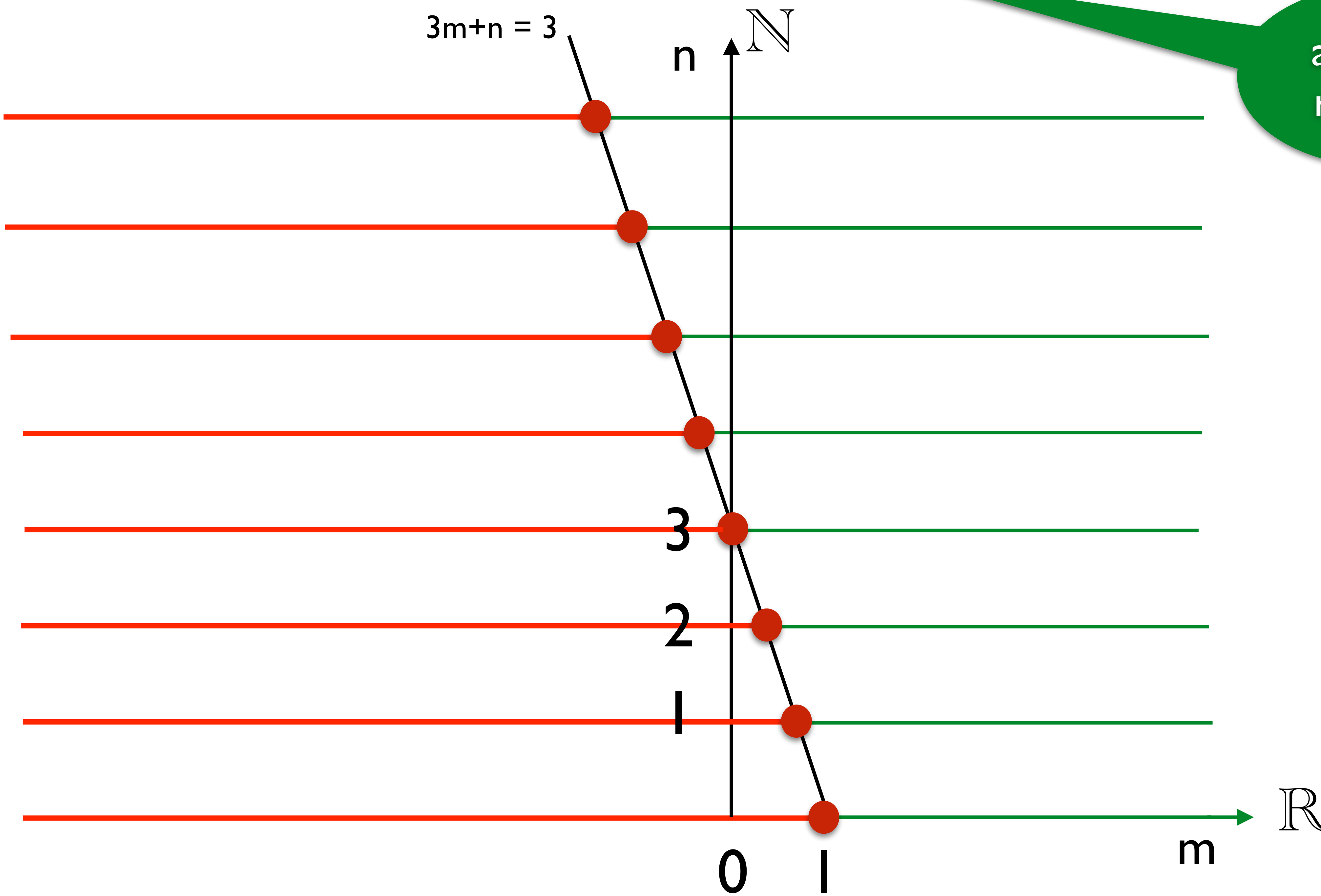
Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R} \times \mathbb{N}$.

a binary relation



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \to \{0,1\}$.

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for m=2

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

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predicate

domain (predicate)

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unary predicate

binary predicate

proposition, nullary predicate

standard (!) notation:

$\exists m\, (m \in \mathbb{R} \wedge \forall n. (n \in \mathbb{N} \Rightarrow 3m+n>3))$

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We also write $\exists_{m,} \forall_n [(m,n)\in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

for $\exists_m [m\in \mathbb{R} : \forall_n [n\in \mathbb{N} : 3m + n > 3]]$

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And even $\exists_{m,\,n}\ [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

for $\exists_m\ [m \in \mathbb{R} : \exists_n\ [n \in \mathbb{N} : 3m + n > 3]]$

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for $\exists_m [m \in \mathbb{R} : \exists_n [n \in \mathbb{N} : 3m + n > 3]]$

but only for the same quantifier!

Quantification - task

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

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Thanks to Bas Luttik

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Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \overset{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0]$$
$$\stackrel{val}{=} \forall_x[x \leqslant 1\colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0]$$

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$$\exists_k [0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
$$\stackrel{val}{=} \exists_k [0 \leqslant k \leqslant n - 1 \vee k = n : k^2 \leqslant 10]$$
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$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
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Domain splitting

$$\forall_x[P \vee Q : R] \overset{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
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