



LOGIC and SETS

# Formale Systeme

3VO + 2PS

**Lecturer:** Dr. Ana Sokolova

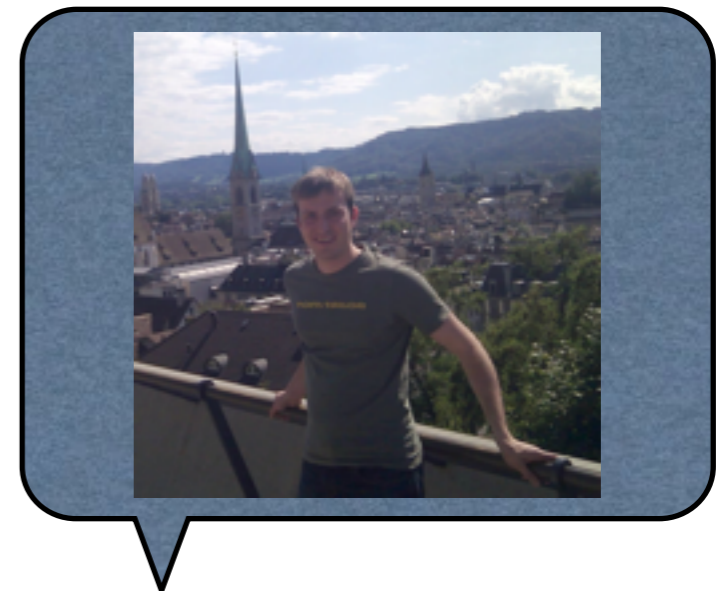
**Instructions:** Dr. Ana Sokolova + Andreas Haas

[http://cs.uni-salzburg.at/~anas/Ana\\_Sokolova/FormaleSysteme2014/](http://cs.uni-salzburg.at/~anas/Ana_Sokolova/FormaleSysteme2014/)

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# The Rules of the Game

- **Lectures** Wednesday 2:15 pm - 3 pm in T01  
Thursday 10:15 am - 12 am in T01
- **Instructions**  
Group 1, Thursday 1:15 pm - 3 pm (AS) in T01  
Group 2, Thursday 1:15 pm - 3 pm (AH) in T02  
Group 3, Thursday 3:15 pm - 5 pm (AH/AS) in T01
- **Tutors** Markus Reiter and Sarah Sophie Salinger  
Tuesday 12am-1pm in T06
- **Books**  
Logical Reasoning: A First Course by R. Nederpelt and F. Kamaraddine  
  
Modellierung: Grundlagen und formale Methoden by U. Kastens and H. Kleine Büning  
  
Introduction to Automata Theory, Languages, and Computation by J. E. Hopcroft, R. Motwani and J.D. Ullman

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starting  
next week

# The Rules... Instructions (PS)

- **Instruction exercises** on the web  
[http://cs.uni-salzburg.at/~anas/Ana\\_Sokolova/  
FormaleSystemeProseminar2014/](http://cs.uni-salzburg.at/~anas/Ana_Sokolova/FormaleSystemeProseminar2014/)  
on Thursday afternoons
- To be solved by the students (in groups of at most 3 students) and handed in as homework to the instruction lecturer before Wednesday 11am
- In class we will present a sample solution and the students will be asked to present solutions/discuss the exercises

# The Rules... Instructions

## (PS)

- One randomly chosen exercise will be graded each week
- The graded exercise will be returned to you in class
- **Grade** based on
  - (1) the grades of the corrected exercise and
  - (2) activity in class (ability to present solutions)
- All information about the course / rules / exams / grading is / will be on the course webpage

# The Rules... Exam (VO)

- Written exams
- **Written exam** in February, April, and July or **two partial tests during the semester**
- Grade based on the # of points on the written exam (sum of the points on the partial tests)
- For better grade **oral exam** after the written one **upon appointment**
- You can pass the course if you have 55% of the maximal points on the exam.

# The Rules... Tests (VO)

- One test end of November, one beginning of February
- The tests are **partial** (half material)
- You can pass via tests if the sum of your points on both tests is at least 55% of the sum of maximal points on the tests **and** if on each test you have at least 20% of the maximal points
- The tests and the exams consist of exercises / questions related to the material taught in class



# Some advice

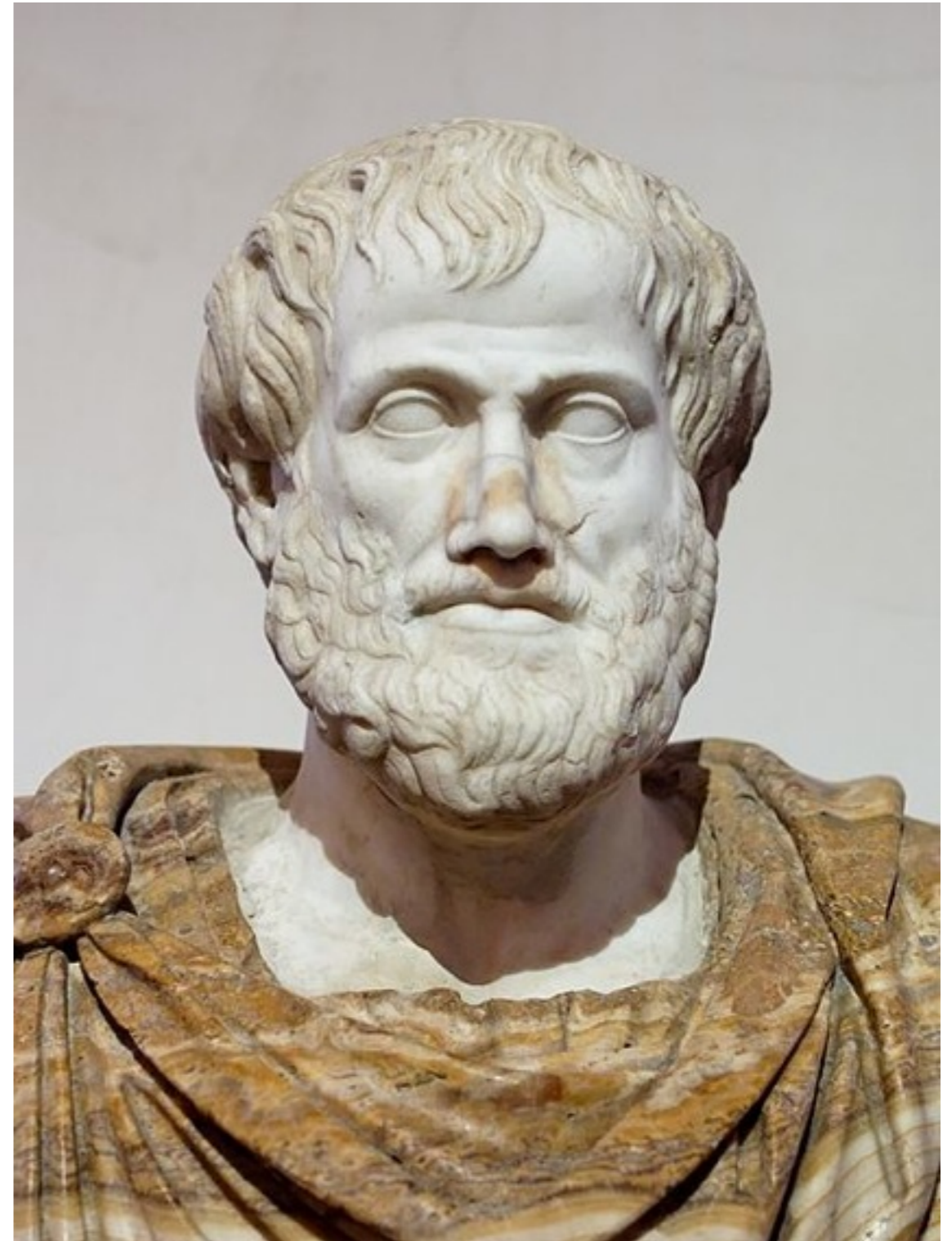
- It starts easy, but soon it gets more difficult
- There accumulates lots of material for the exam
- **Best is** to **regularly** study, practice, solve the exercises **yourself!**

# In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



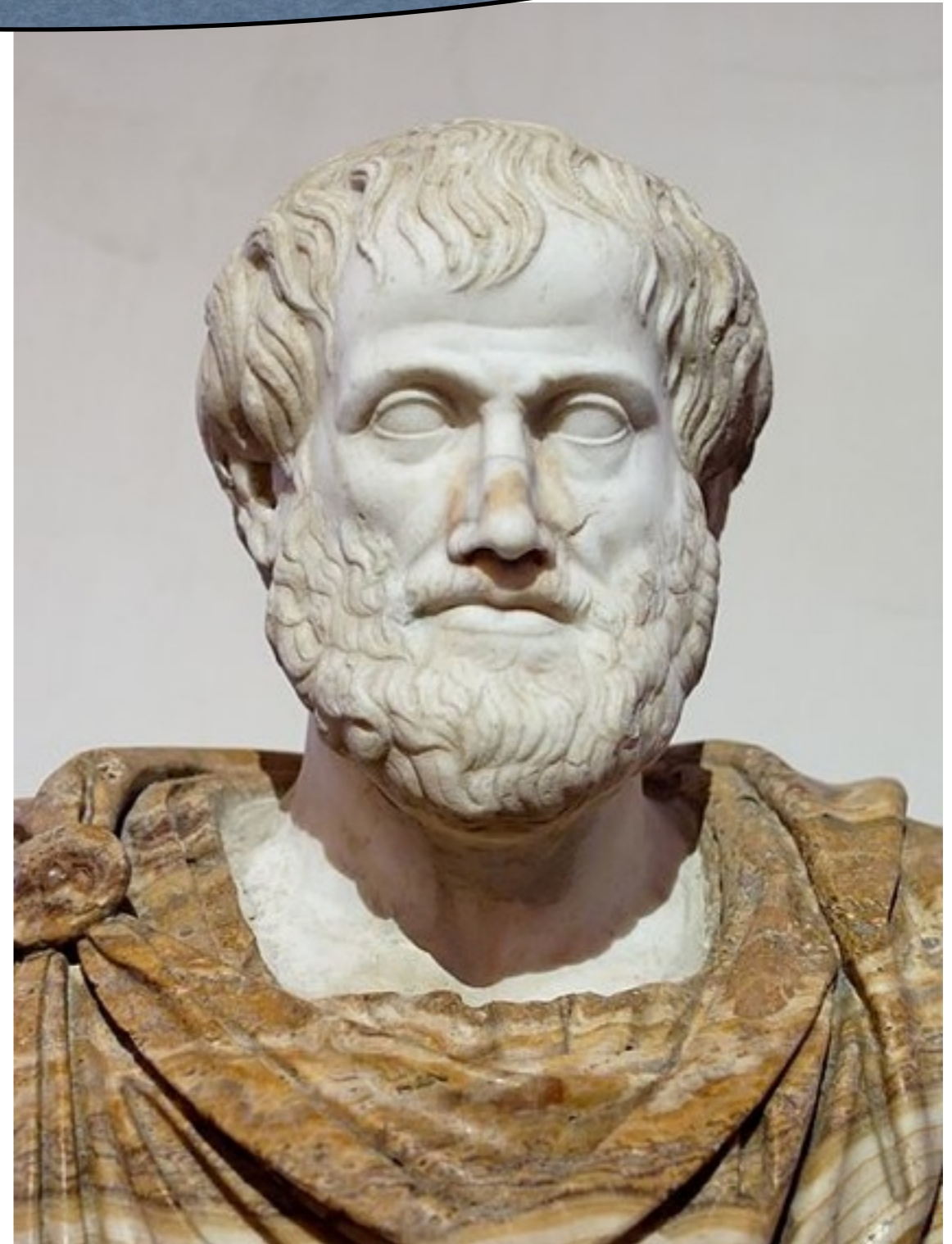
Logic = study of correct reasoning

In the beginning

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# Formal Logic

Gottfried Wilhelm Leibnitz  
(1646 - 1716)

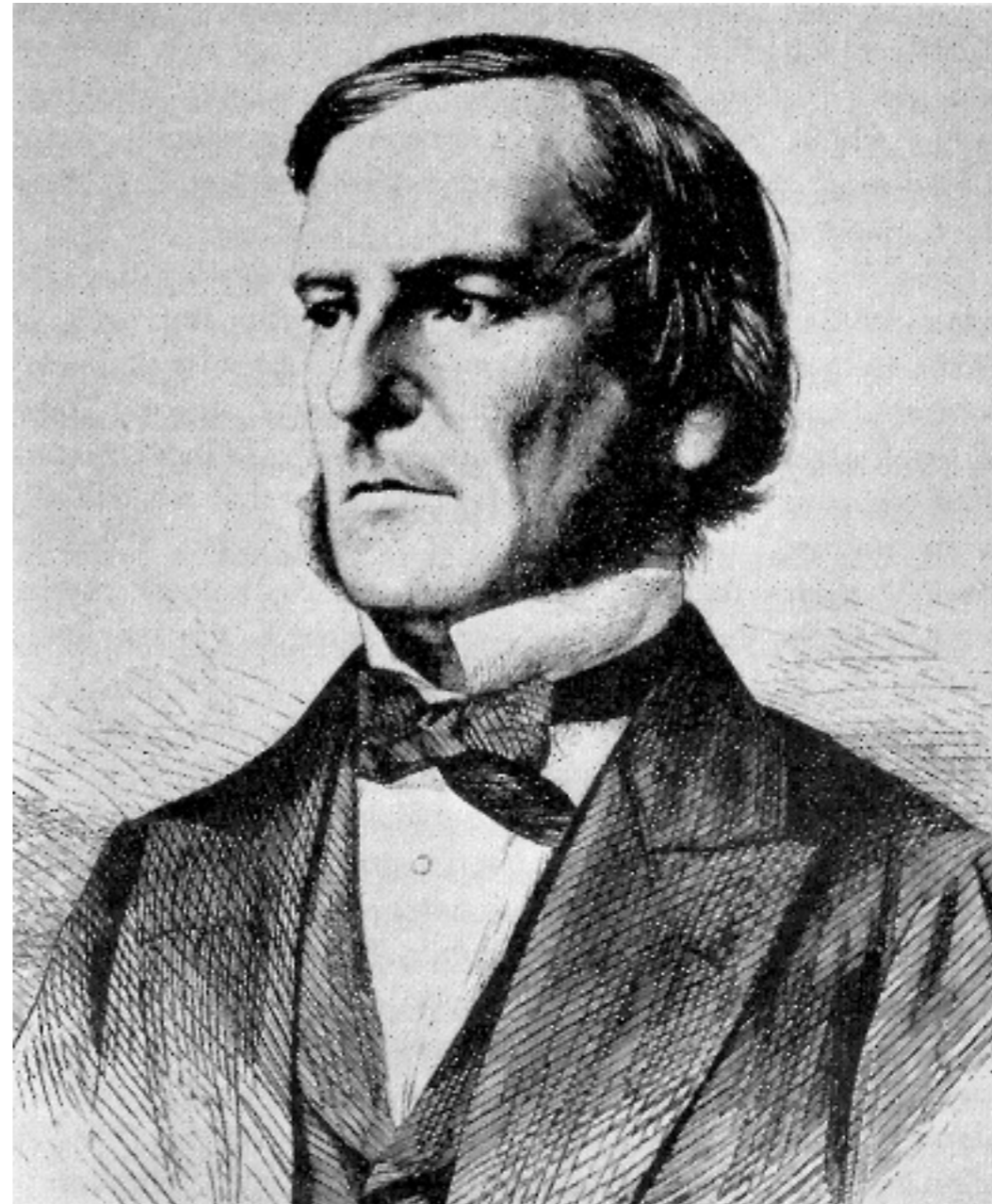
Beginnings of *symbolic logic*



# Boolean Logic

George Boole  
(1815 - 1864)

Boolean logic



# We will learn

- **Naive Set Theory** - sets, relations, mappings, numbers and structures, ordered sets
- **Logical Calculations** - propositional logic, predicate logic
- **Logical Derivations** - reasoning
- **Basics of formal models** - finite automata, transition systems, graphs, grammars...

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Starting this week

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# Why formal models/ methods?

- For better understanding of a complex system, problem, task,... models, **abstractions** are needed
- For rigorous precise **reasoning** about a complex system, problem, task



# The river-crossing puzzle

- A man stands with a wolf, a goat, and a cabbage at the left bank of a river, that he wants to cross.
- The man has a boat that is large enough to carry him and another object to the other side.
- If the man leaves the wolf and the goat, or the goat and the cabbage on one side without supervision, one of them will get eaten :-)
- Is it possible to cross the river so that neither the goat nor the cabbage is eaten?

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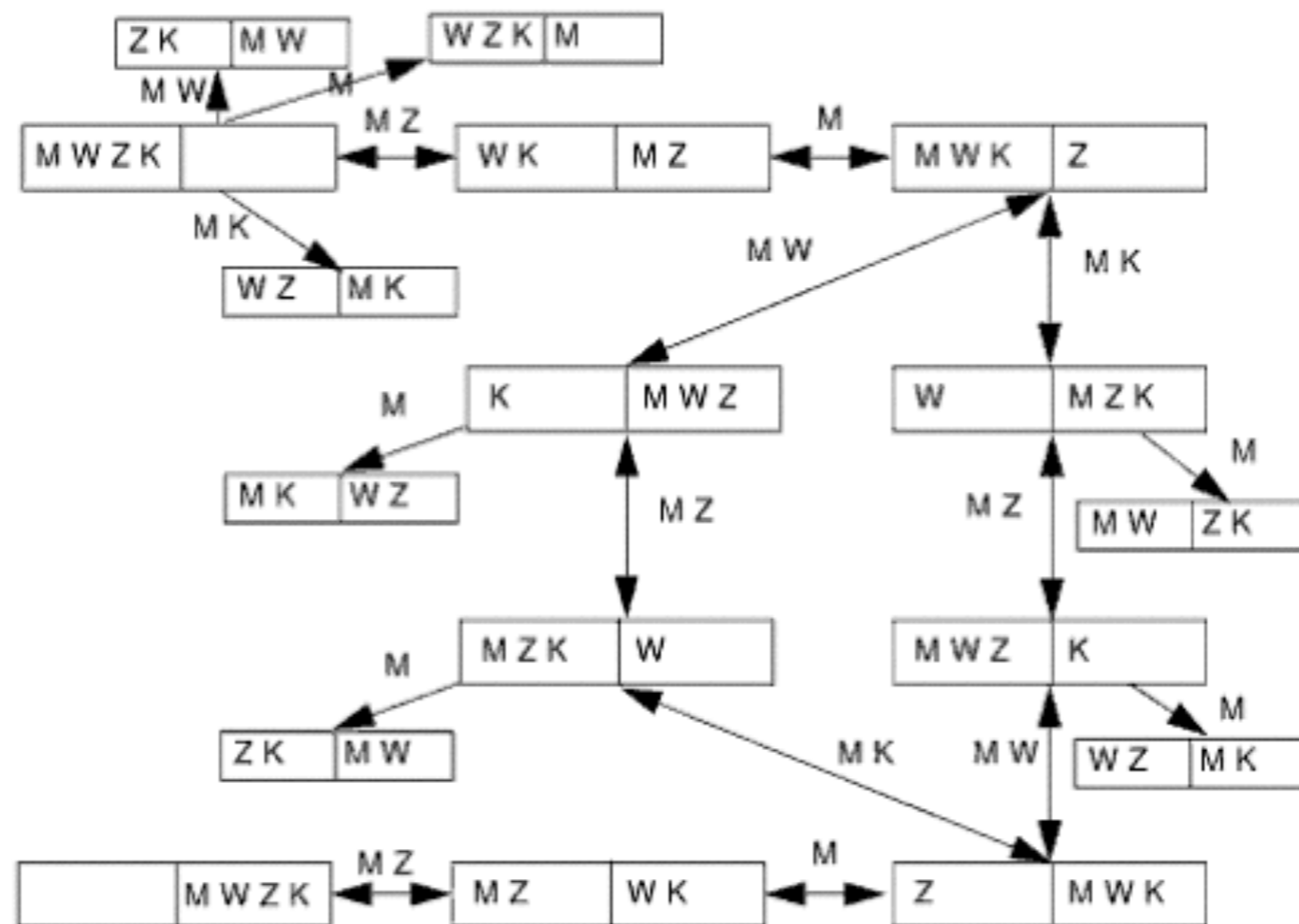
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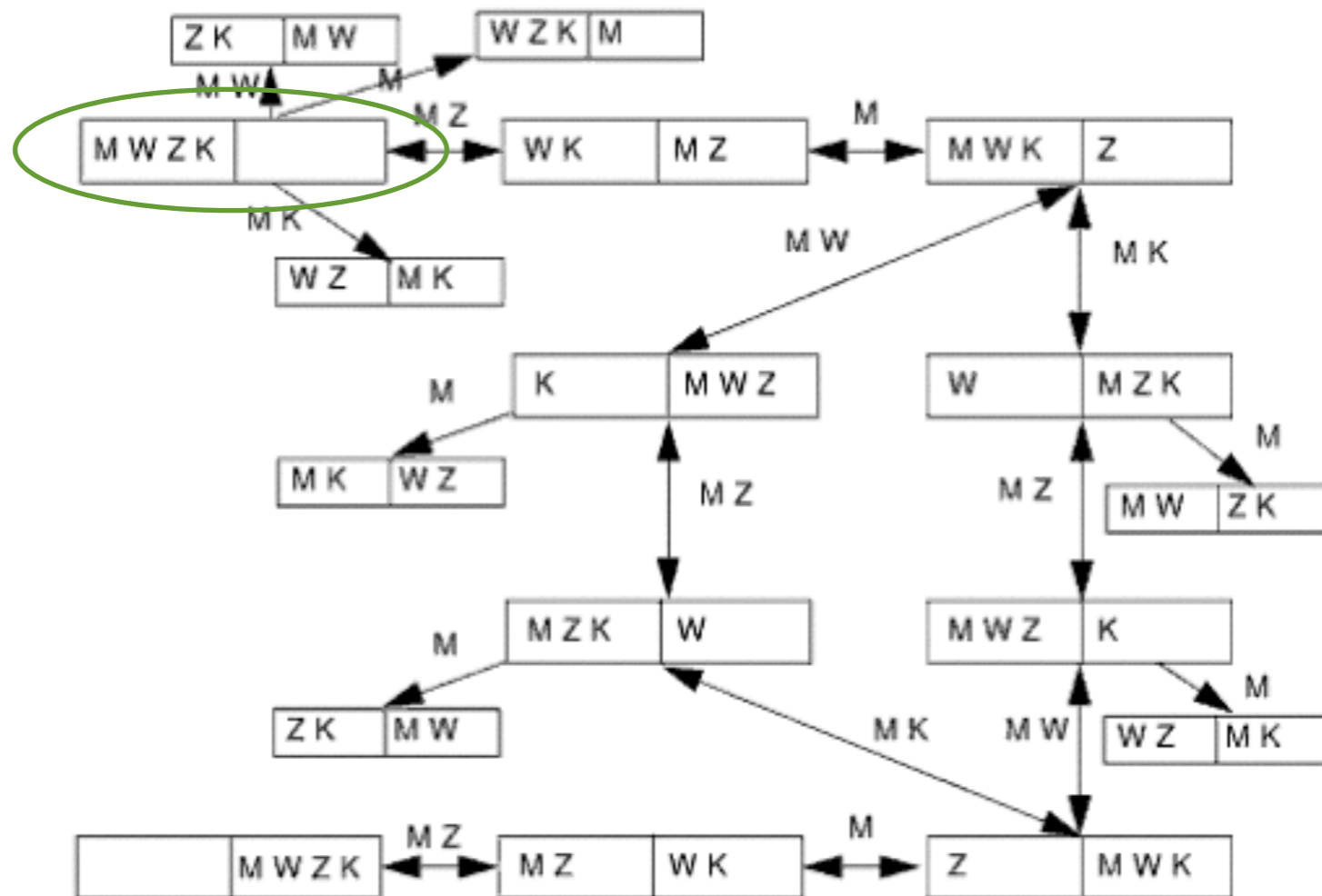
Formalization with a finite automaton [Kastens et al.] :



states and transitions

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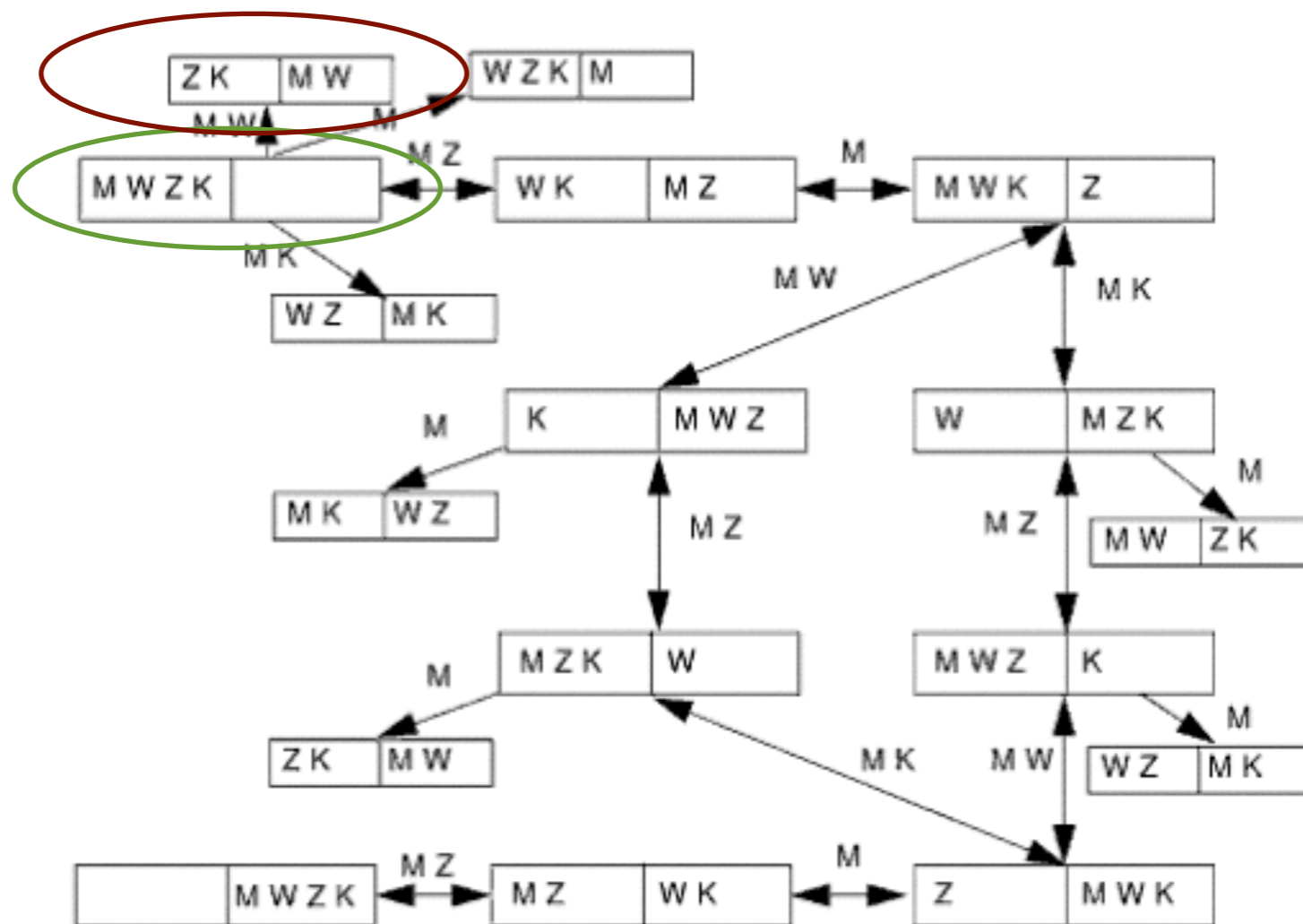
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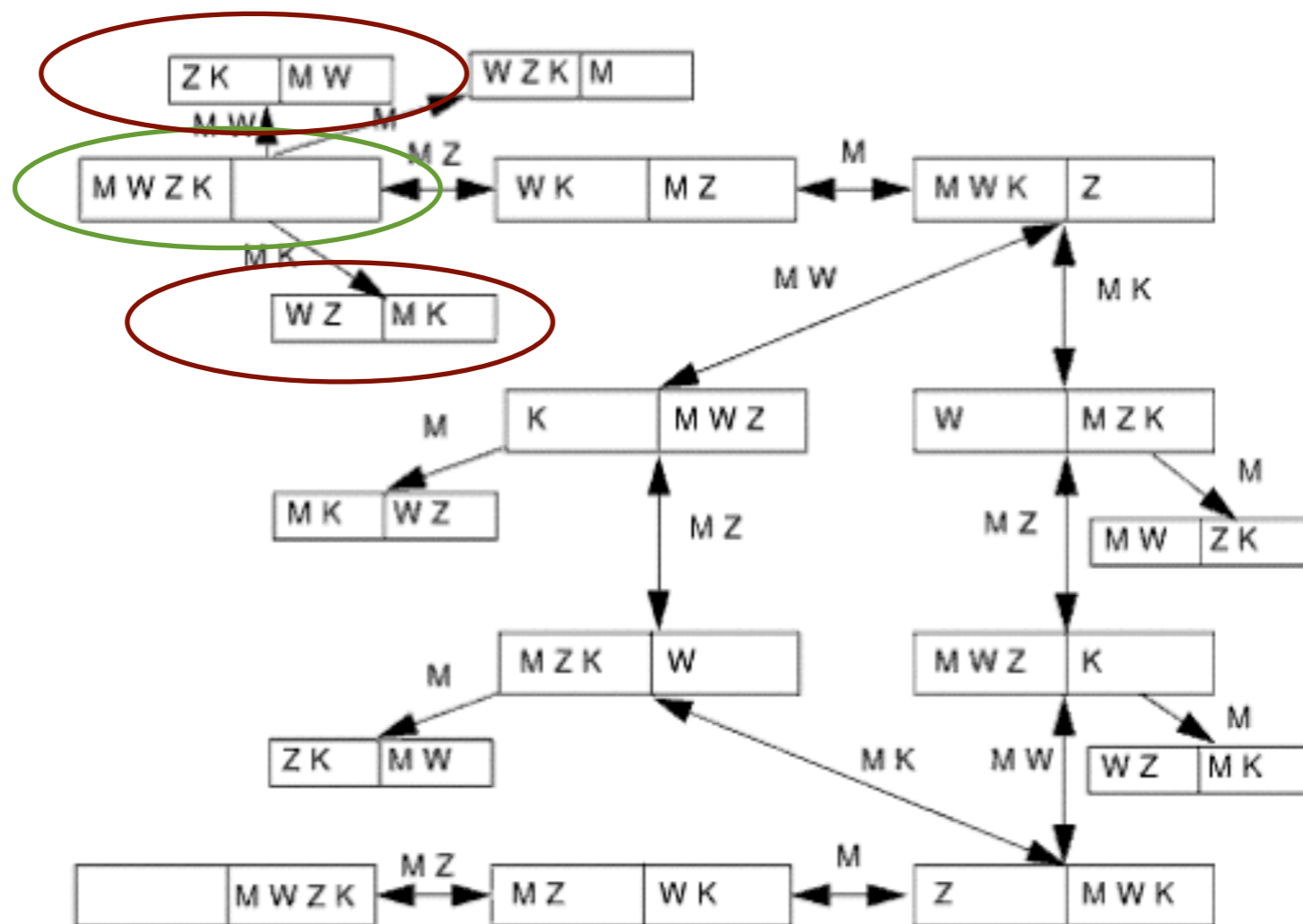
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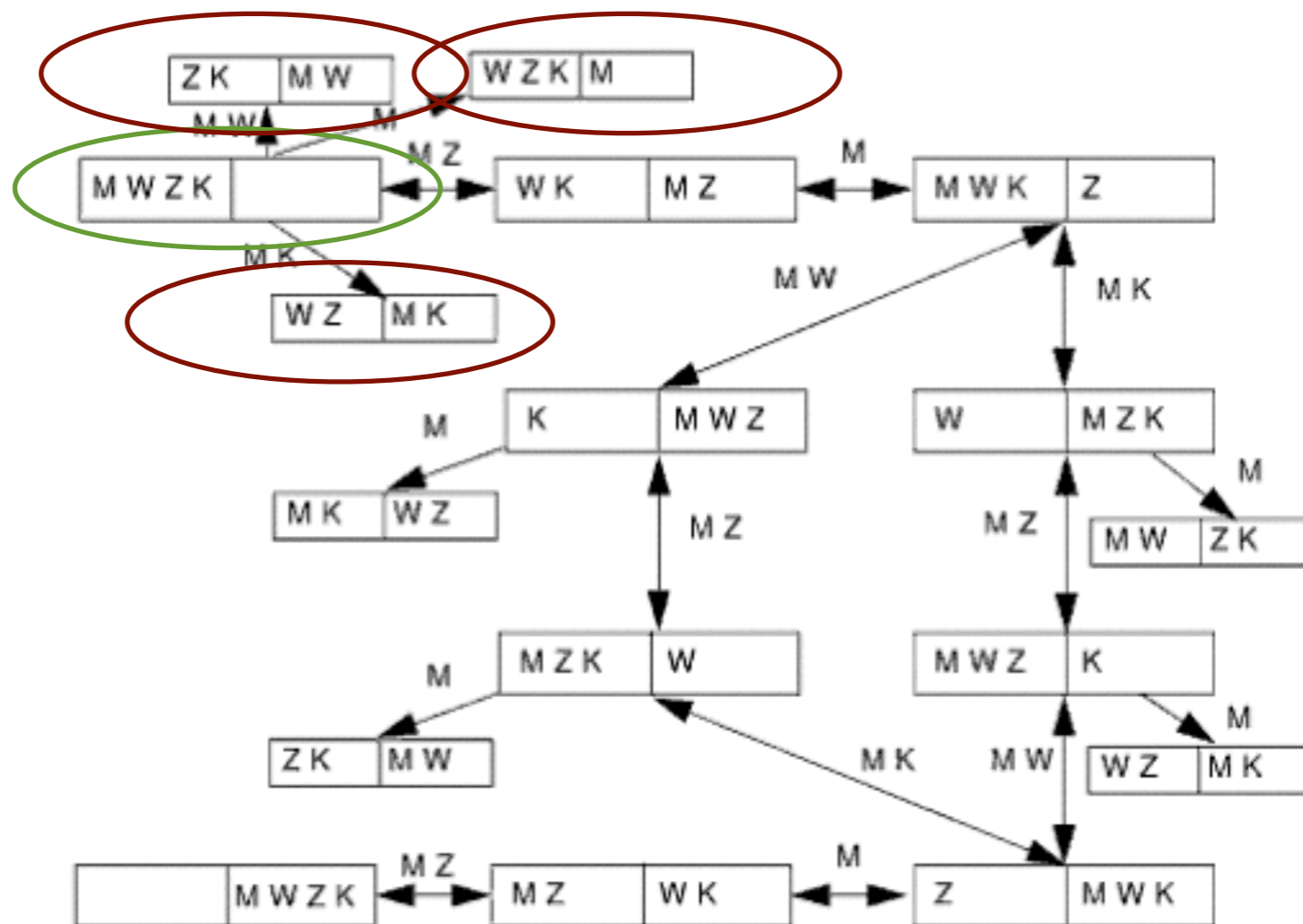


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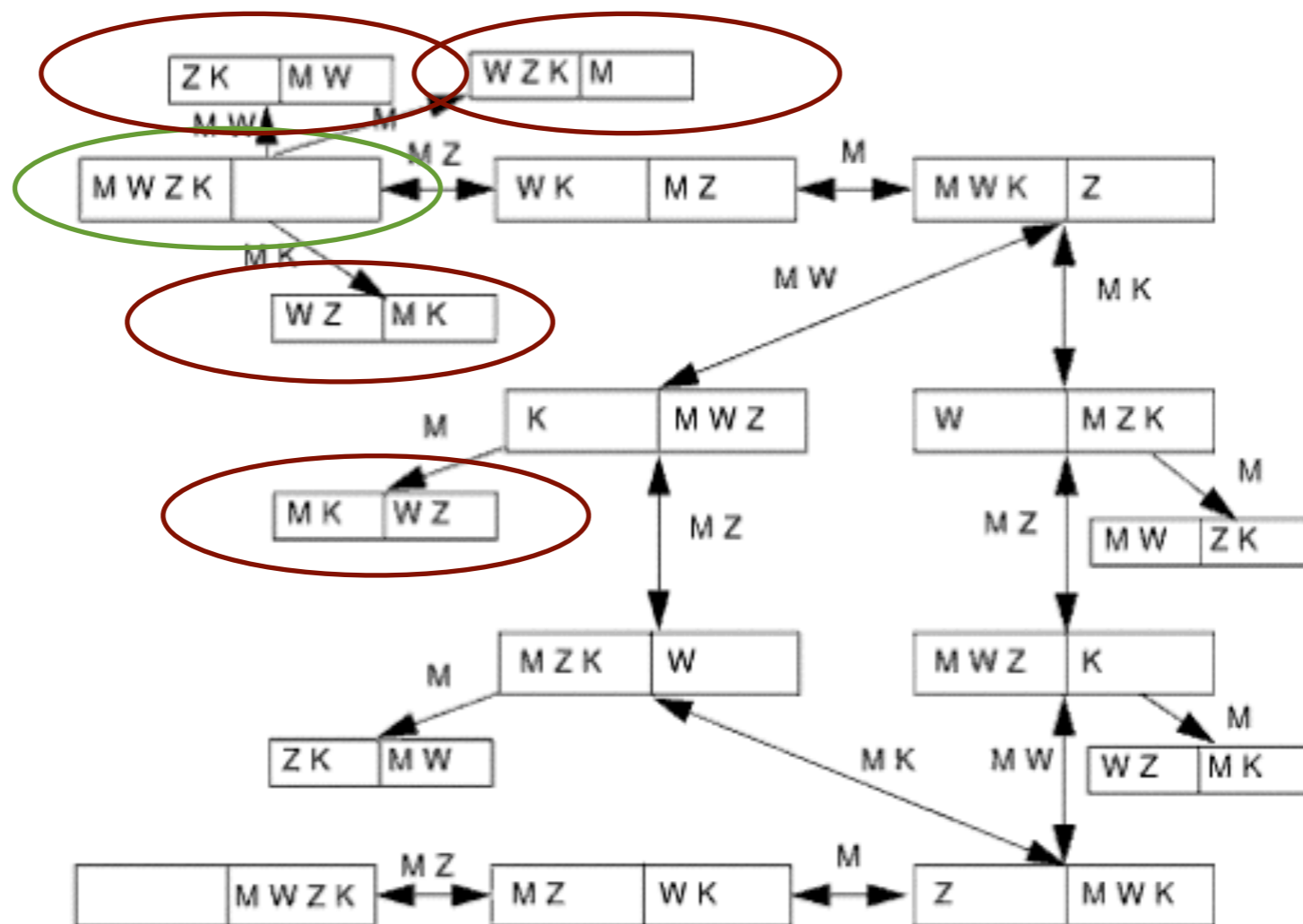
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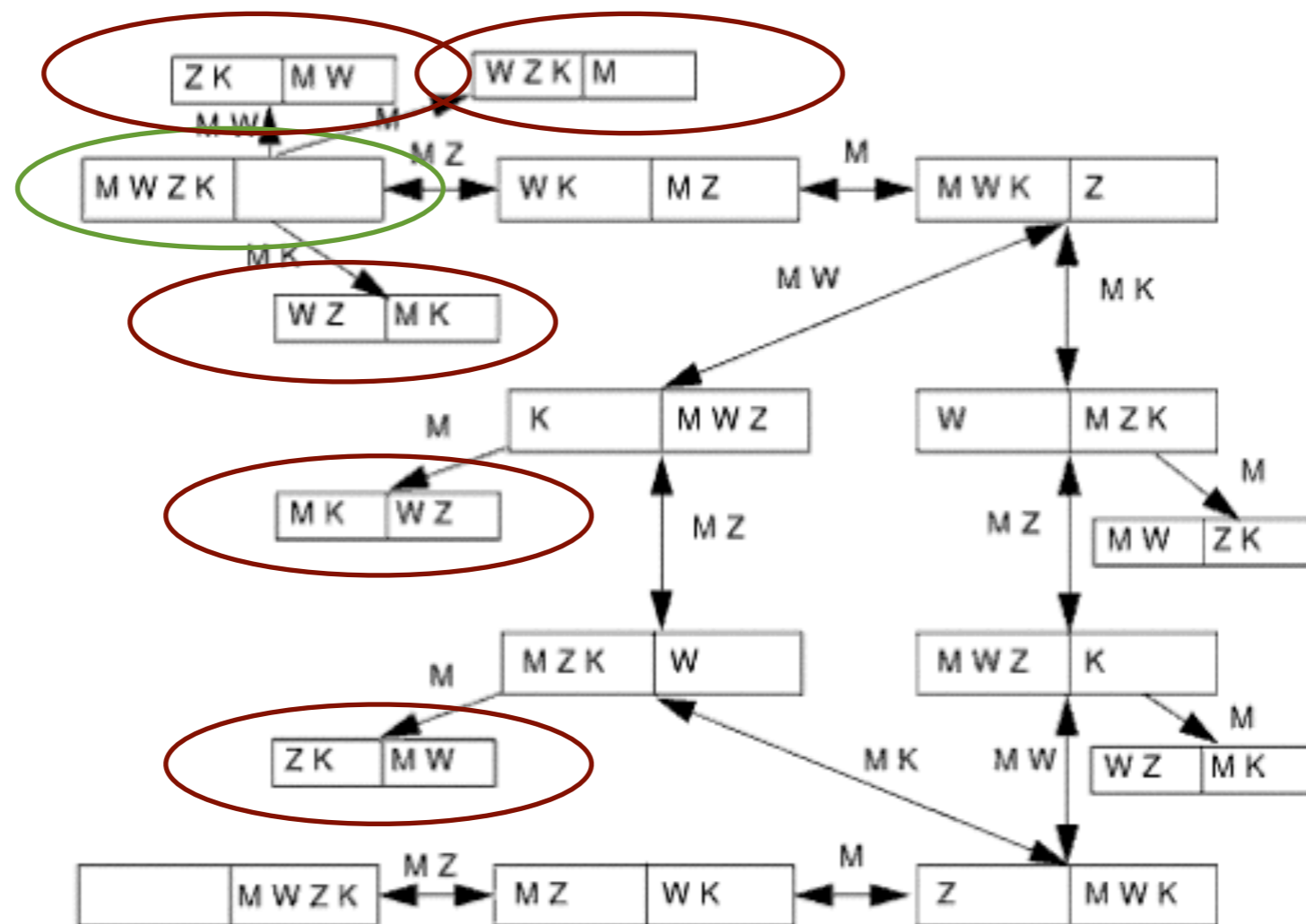
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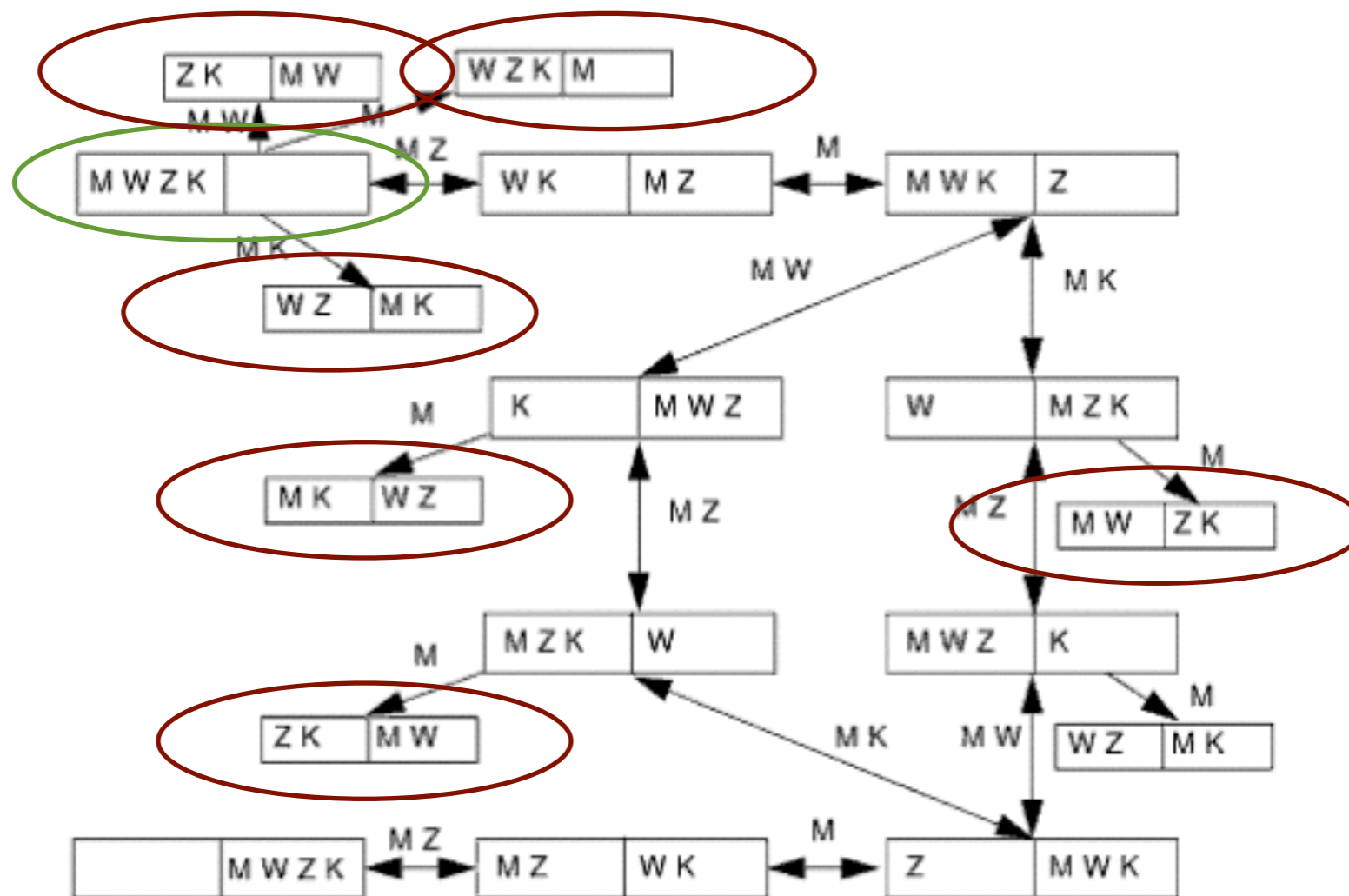
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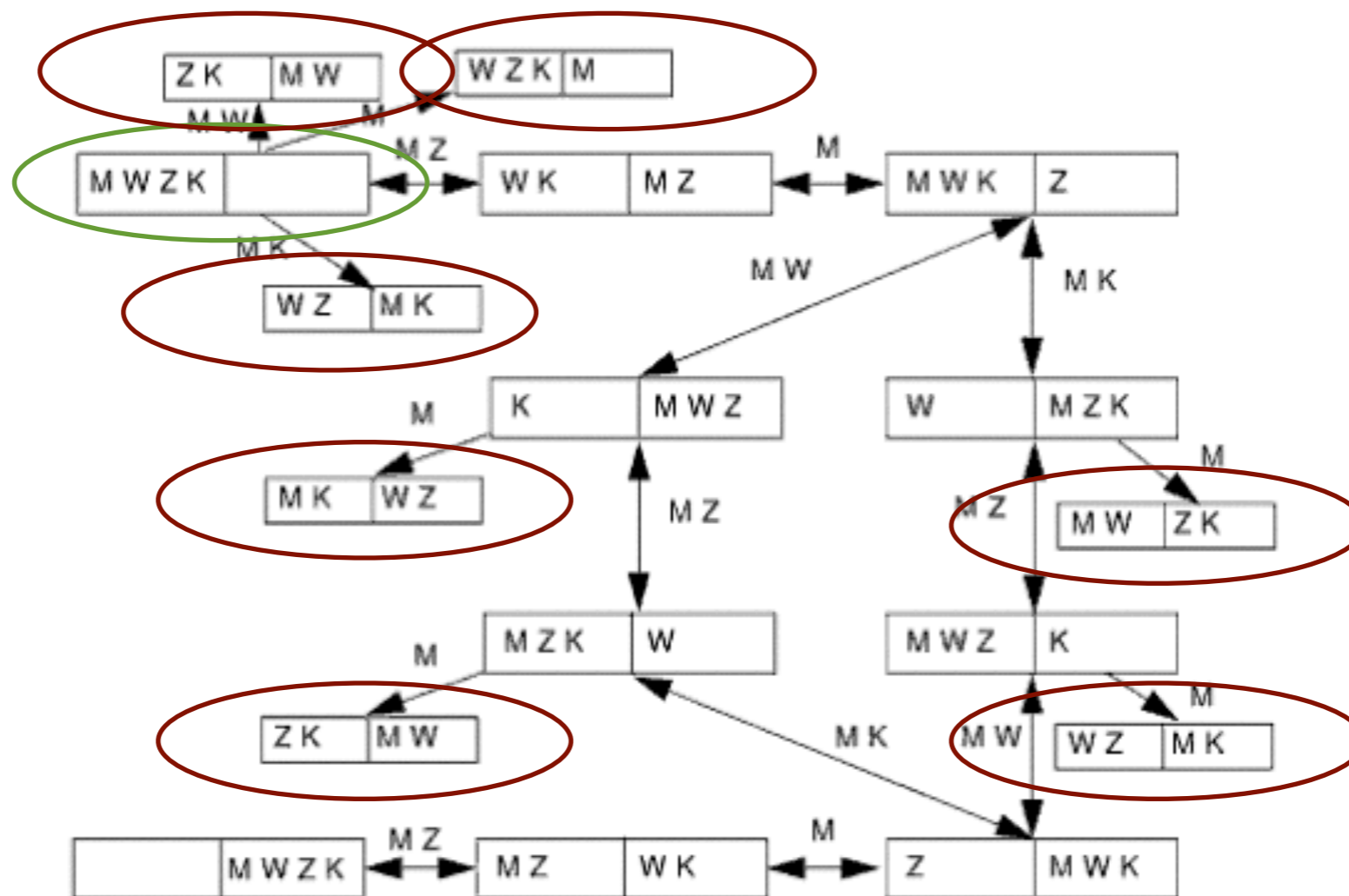
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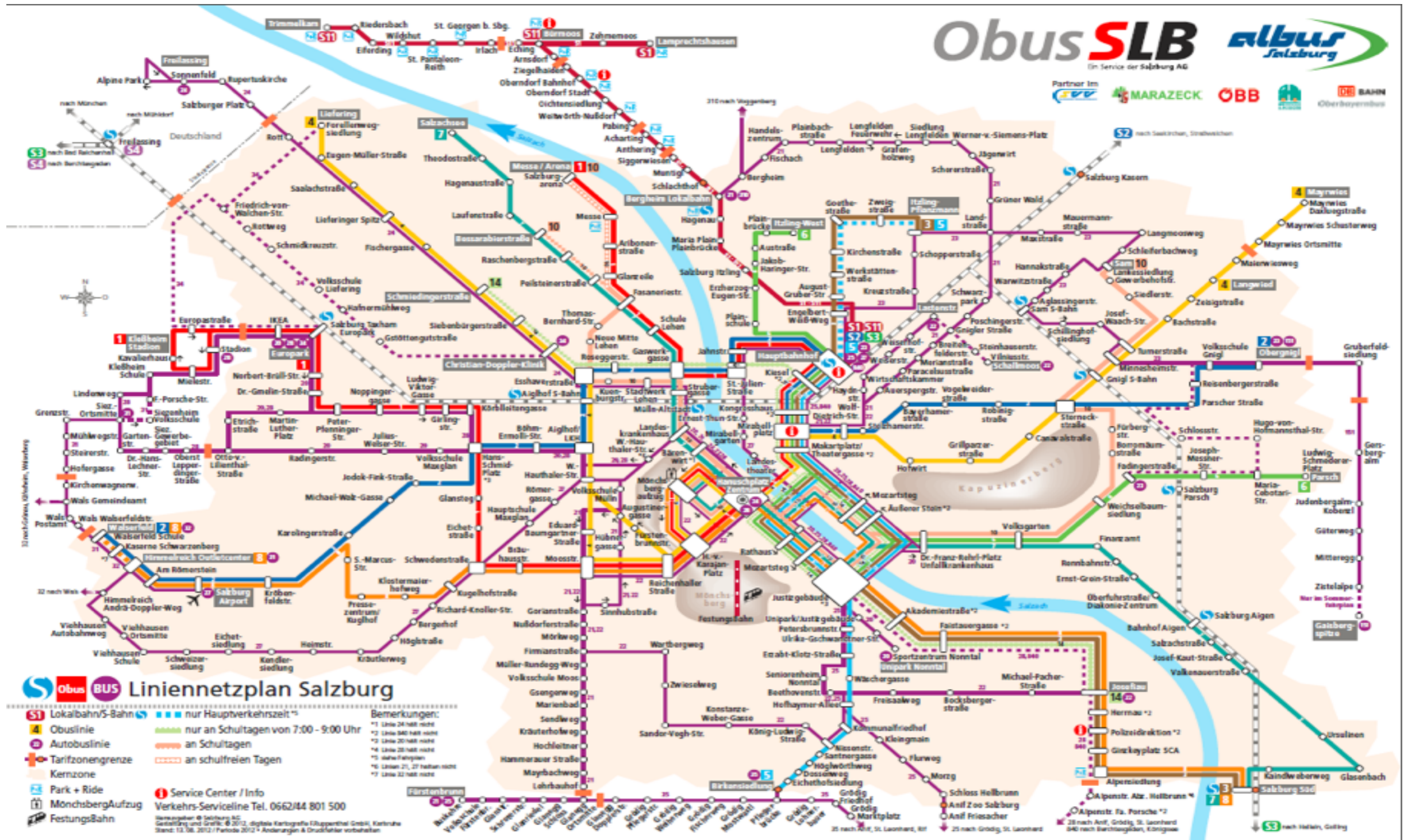


states and transitions





# Another model example



# Sets

- A **set**  $S$  is a collection of different objects, the elements of  $S$
- We write  $x \in S$  for 'x is an element of S'
- A set 'can' be specified by
  - (1) listing its elements, e.g.  $S = \{1, 3, 7, 18\}$
  - (2) **specifying a property**, e.g.  $S = \{x \mid P(x)\}$
- Sets can be **finite** e.g.  $\{\clubsuit, \heartsuit\}$  or **infinite** e.g.  $\mathbb{N}$
- The set with no elements is the **empty set**, notation  $\emptyset$
- The 'number' of elements in a set  $S$  is the **cardinality** of  $S$ , notation  $|S|$



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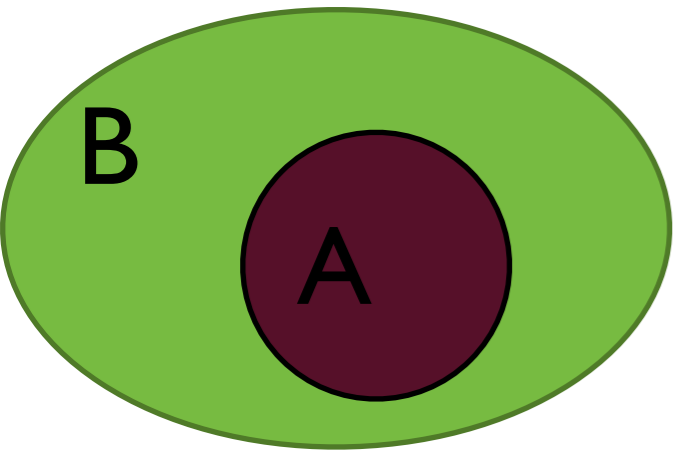
$P$  is a proposition over  $x$ , which is true or false

# Sets - properties

- All elements of a set are different
- The elements of a set are not ordered
- The same set can be specified in different ways, e.g.  
 $\{1,2,3,4\}$ ,  $\{2,3,1,4\}$ ,  $\{i \mid i \in \mathbb{N} \text{ and } 0 < i < 5\}$

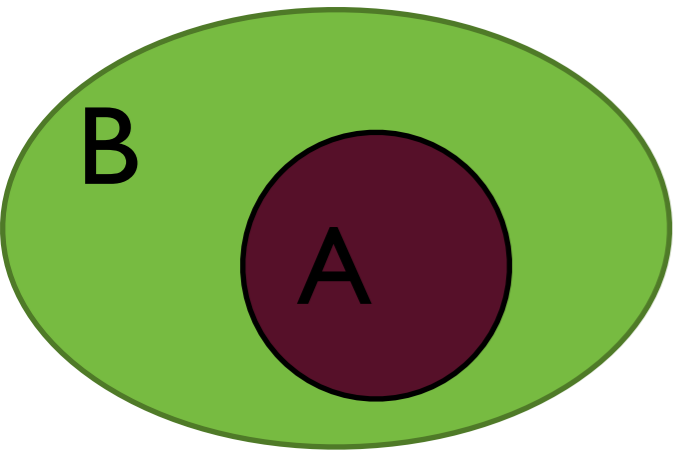
# Subsets, equality

**Def.**  $A \subseteq B$  iff all elements of  $A$  are elements of  $B$



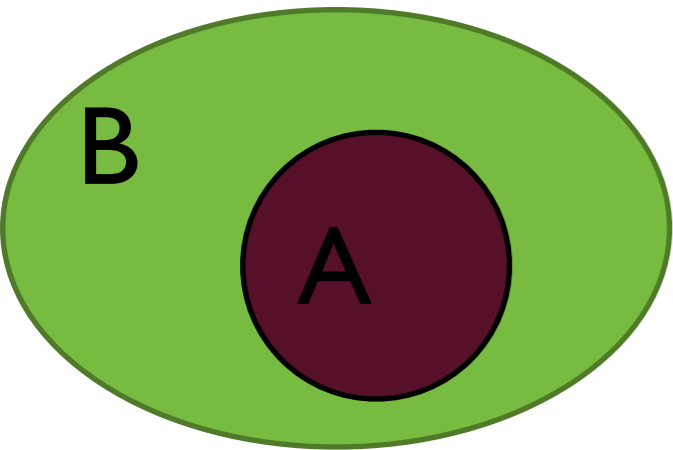
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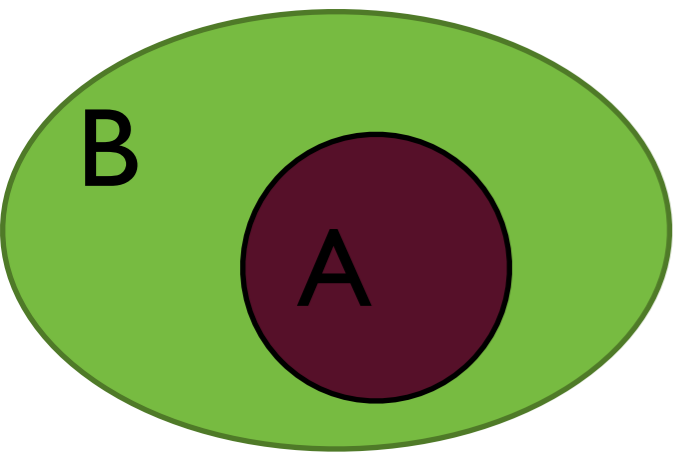
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[iff  $\forall a (a \in A \Rightarrow a \in B)$  ]

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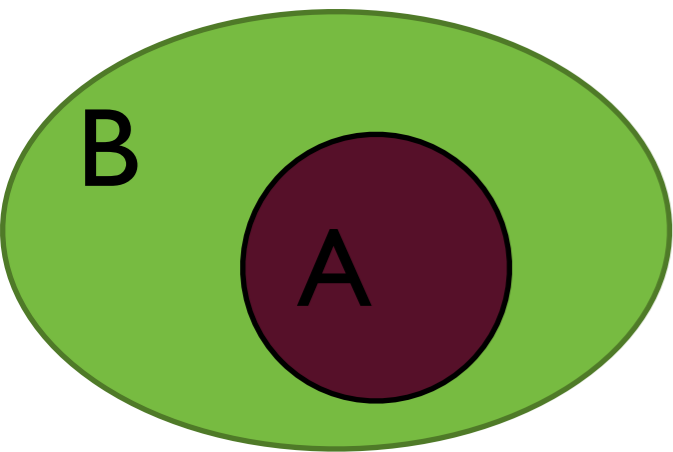


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logical formula

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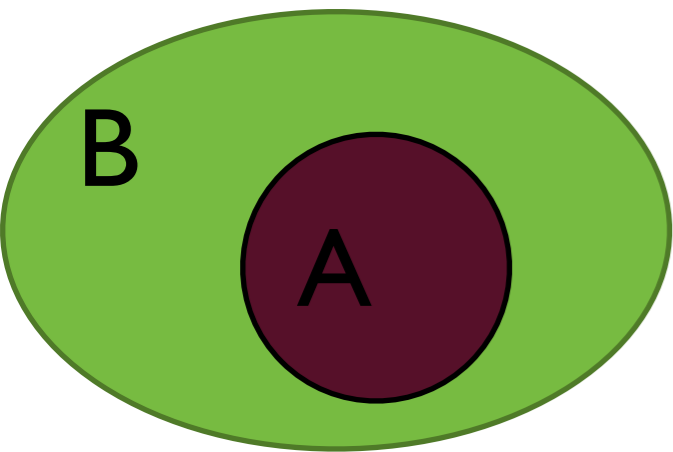
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quantifier

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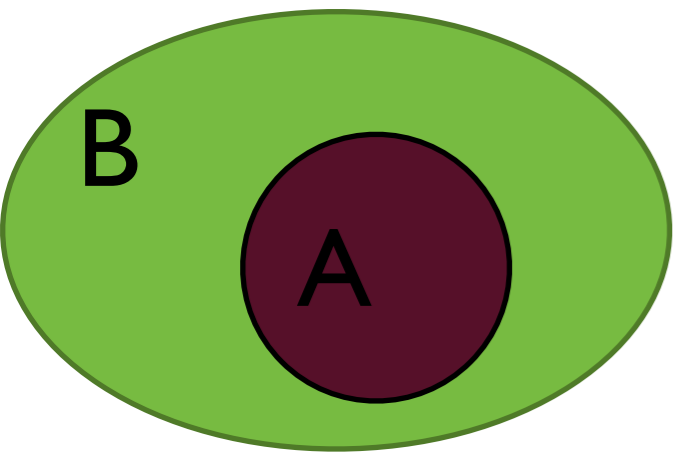
quantifier

logical  
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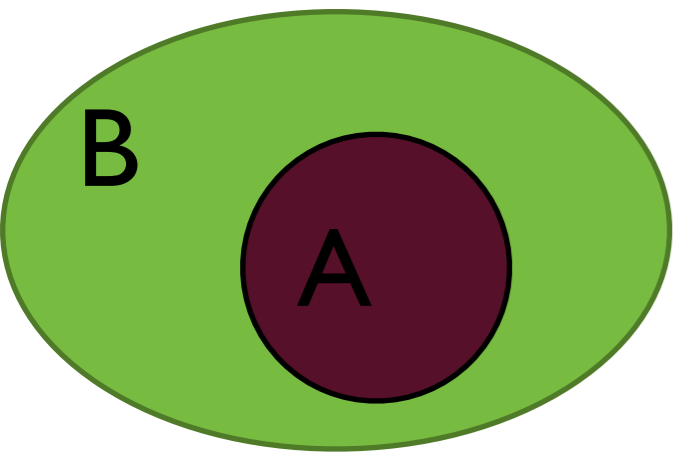
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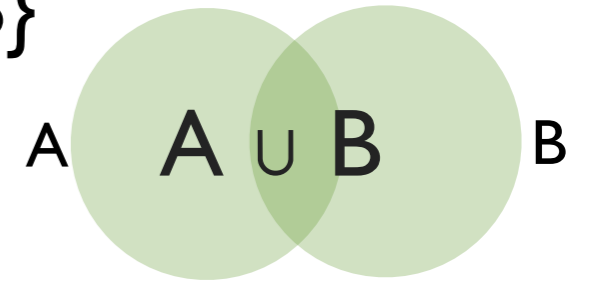
**Def.**  $A = B$  iff  $A \subseteq B$  and  $B \subseteq A$

**Def.**  $A \subset B$  iff  $A \subseteq B$  and  $A \neq B$

# Operations on sets

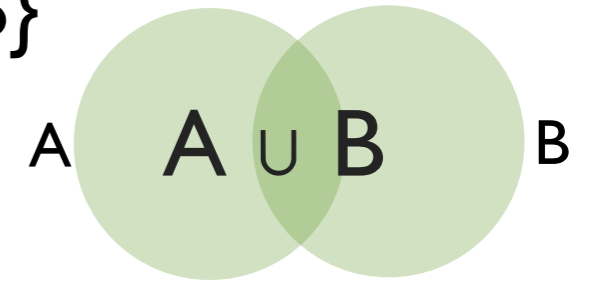
# Operations on sets

Union (Vereinigung)  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

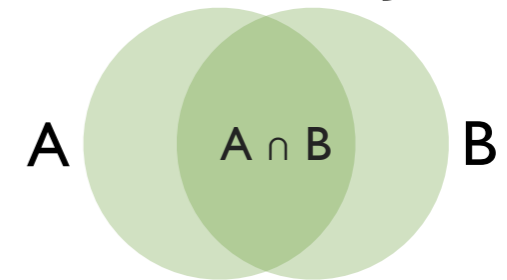


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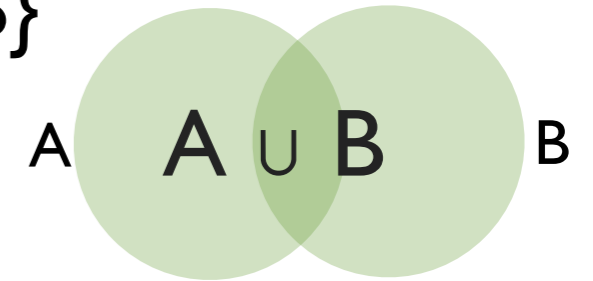


Intersection (**Durchschnitt**)  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



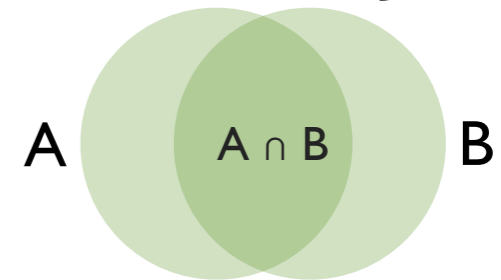
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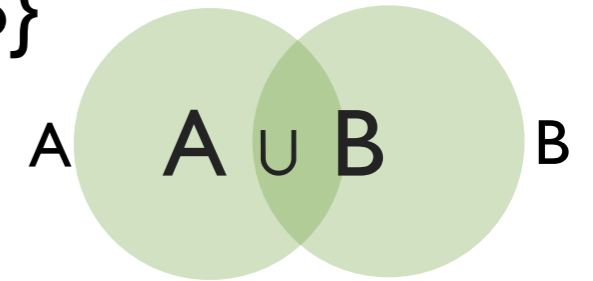
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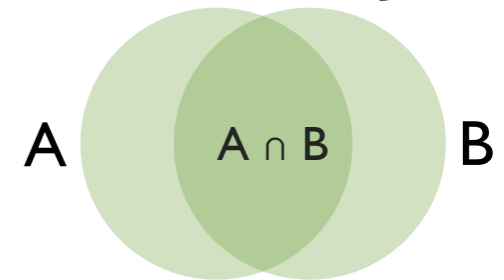
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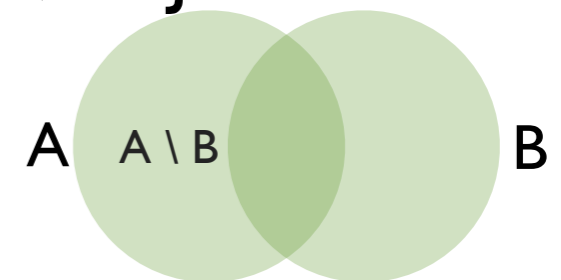


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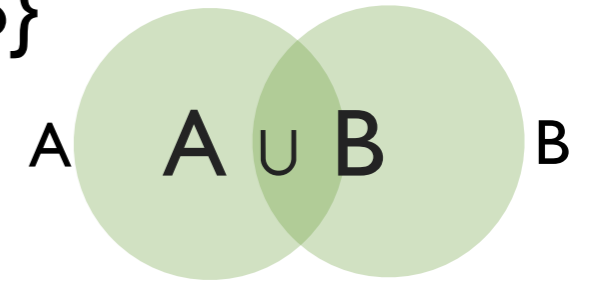
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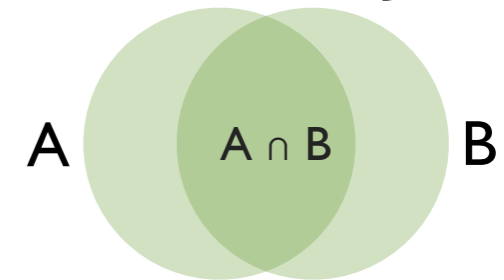
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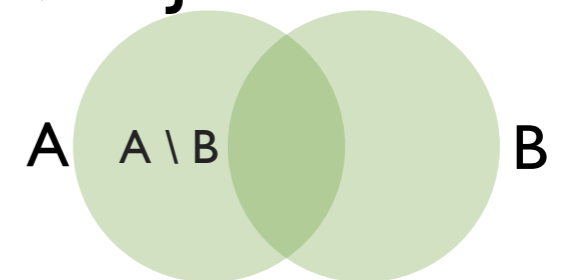


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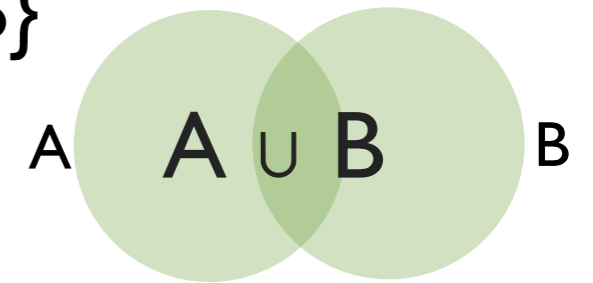


Direct product (**Kartesisches Produkt**)

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

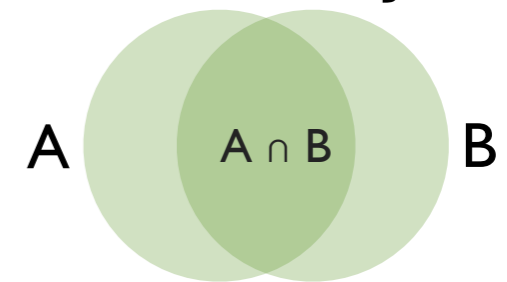
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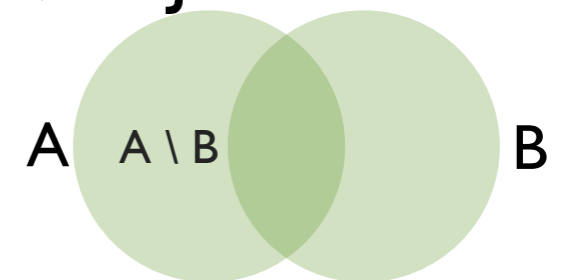


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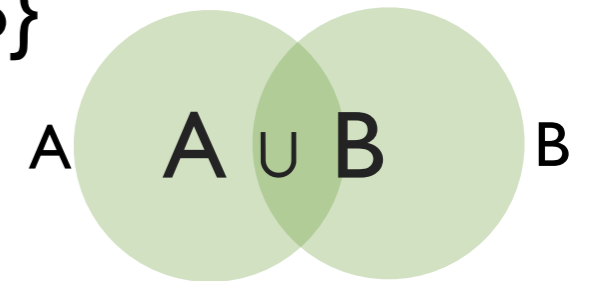
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ordered pairs

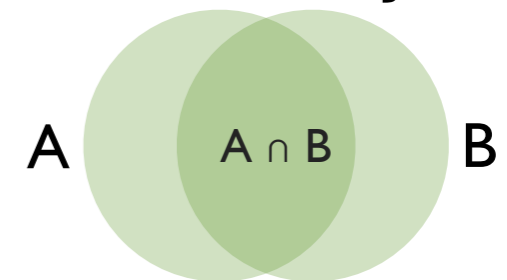
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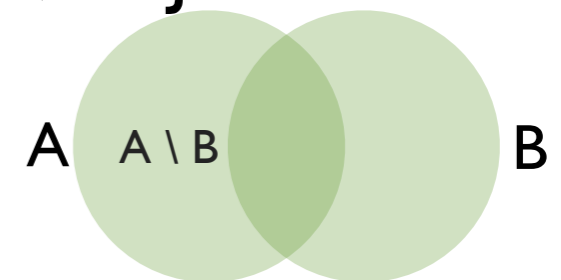


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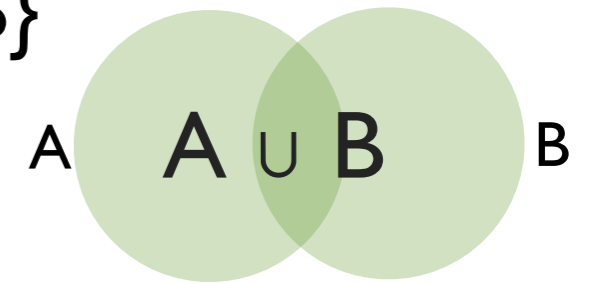
$(A \times B) \times C \neq A \times (B \times C)$

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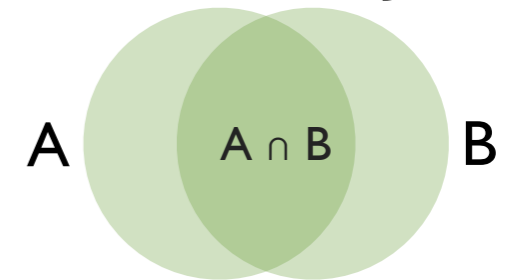
# Operations on sets

Union (**Vereinigung**)  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

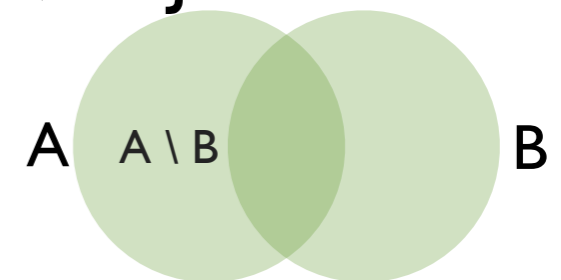


Intersection (**Durchschnitt**)  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are **disjoint** if  $A \cap B = \emptyset$



Difference (**Differenz**)  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Direct product (**Kartesisches Produkt**)

$(A \times B) \times C \neq A \times (B \times C)$

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

ordered pairs

Powerset (**Potenzmenge**)  $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

# Russell's paradox

- Let  $P$  be the set of all sets that are not an element of itself
- Hence,  $P = \{ x \mid x \notin x \}$
- Is  $P \in P$  ?
- **Contradiction!**

# Russell's paradox

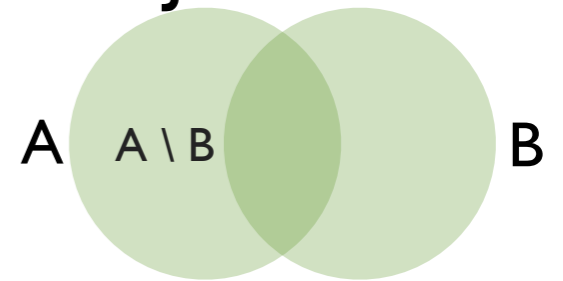
- Let  $P$  be the set of all sets that are not an element of itself
- Hence,  $P = \{x \mid x \notin x\}$
- Is  $P \in P$  ?
- **Contradiction!**

The need for a universal set  $U$

$$S = \{x \mid x \in U \text{ and } P(x)\}$$

# Operations on sets

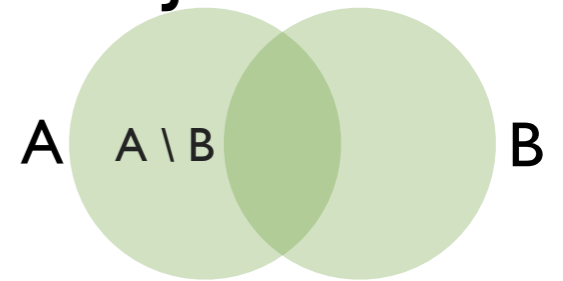
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# Operations on sets

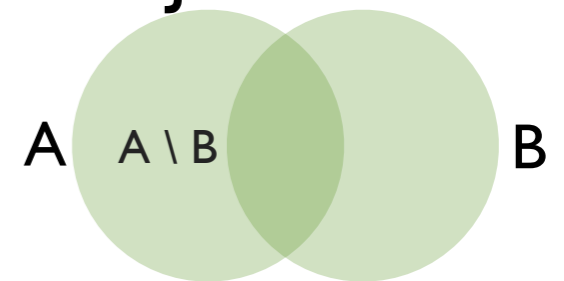
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Given a universal set U

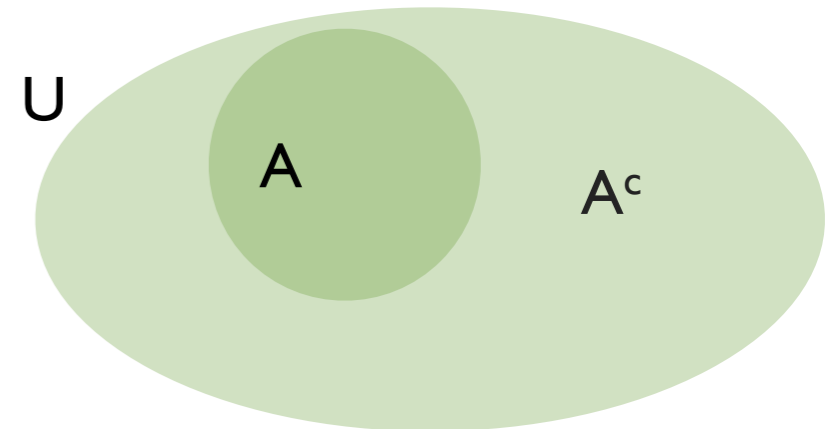
# Operations on sets

Difference (Differenz)  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



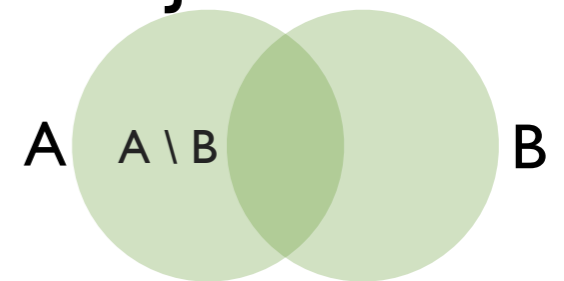
Given a universal set U

Complement (Komplement)  $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



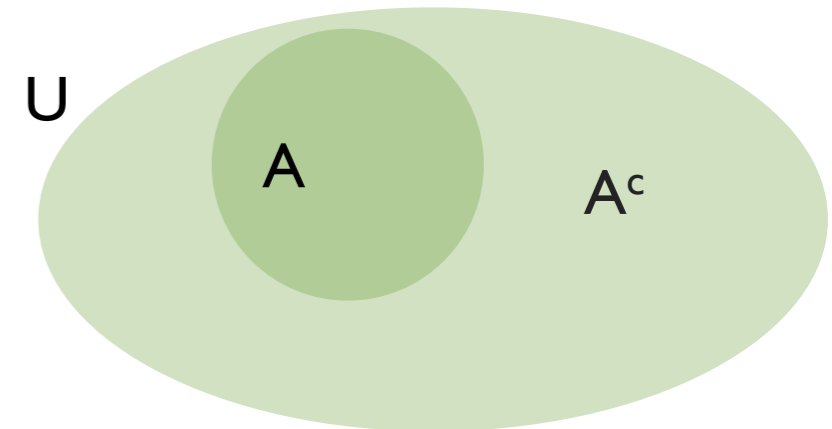
# Operations on sets

Difference (Differenz)  $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Given a universal set U

Complement (Komplement)  $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



Hence  $A^c = U \setminus A$

# Properties of sets

1.  $\emptyset \subseteq X$

2. If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$

3.  $X \cap Y \subseteq X$ ,  $X \cap Y \subseteq Y$

4.  $X \subseteq X \cup Y$ ,  $Y \subseteq X \cup Y$

5. If  $X$

6. If  $X$

7.  $X \cap Y = X$  iff  $X \subseteq Y$

8.  $X \cap X = X$  (idempotence)

9.  $X \cup X = X$  (idempotence)

10.  $X \cap \emptyset = \emptyset$

# Properties of sets

$$11. X \cup \emptyset = X$$

$$12. X \cap Y = Y \cap X \text{ (commutativity)}$$

$$13. X \cup Y = Y \cup X \text{ (commutativity)}$$

$$14. X \cap (Y \cap Z) = (X \cap Y) \cap Z \text{ (associativity)}$$

$$15. X \cup (Y \cup Z) = (X \cup Y) \cup Z \text{ (associativity)}$$

$$16. X \cap (X \cup Y) = X \text{ (absorption)}$$

$$17. X \cup (X \cap Y) = X \text{ (absorption)}$$

$$18. X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ (distributivity)}$$

$$19. X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z) \text{ (distributivity)}$$

$$20. X \setminus Y \subseteq X$$

# Properties of sets

$$21. (X \setminus Y) \cap Y = \emptyset$$

$$22. X \cup Y = X \cup (Y \setminus X)$$

$$23. X \setminus X = \emptyset$$

$$24. X \setminus \emptyset = X$$

$$25. \emptyset \setminus X = \emptyset$$

$$26. \text{If } X \subseteq Y, \text{ then } X \setminus Y = \emptyset$$

$$27. (X$$

$$28. (X$$

$$29. (X$$

$$30. X \times \emptyset = \emptyset \quad \emptyset \times X = \emptyset$$

$$31. \emptyset \times X = \emptyset$$

$$32. \text{If } X \subseteq Y, \text{ then } \mathcal{P}(X) \subseteq \mathcal{P}(Y)$$