Renaming bound variables

Bound variables

$$\forall_x [P:Q] \stackrel{val}{=} \forall_y [P[y/x]:Q[y/x]]$$

$$\exists_x [P:Q] \stackrel{val}{=} \exists_y [P[y/x]:Q[y/x]]$$

if y does not occur in P or Q (not even in $\forall y, \exists y$)

Domain splitting

Examples:

$$\forall_{x} [x \le 1 \lor x \ge 5 \colon x^{2} - 6x + 5 \ge 0]$$

$$\stackrel{val}{=} \forall_{x} [x \le 1 \colon x^{2} - 6x + 5 \ge 0] \land \forall_{x} [x \ge 5 \colon x^{2} - 6x + 5 \ge 0]$$

Domain splitting

Examples:

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$$\stackrel{val}{=} \forall_{x} [x \le 1 \colon x^{2} - 6x + 5 \ge 0] \land \forall_{x} [x \ge 5 \colon x^{2} - 6x + 5 \ge 0]$$

$$\exists_{k} [0 \le k \le n : k^{2} \le 10]$$

$$\stackrel{val}{=} \exists_{k} [0 \le k \le n - 1 \lor k = n : k^{2} \le 10]$$

$$\stackrel{val}{=} \exists_{k} [0 \le k \le n - 1 : k^{2} \le 10] \lor \exists_{k} [k = n : k^{2} \le 10]$$

Domain splitting

Examples:

$$\forall_{x} [x \le 1 \lor x \ge 5 \colon x^{2} - 6x + 5 \ge 0]$$

$$\stackrel{val}{=} \forall_{x} [x \le 1 \colon x^{2} - 6x + 5 \ge 0] \land \forall_{x} [x \ge 5 \colon x^{2} - 6x + 5 \ge 0]$$

$$\exists_{k} [0 \le k \le n : k^{2} \le 10]$$

$$\stackrel{val}{=} \exists_{k} [0 \le k \le n - 1 \lor k = n : k^{2} \le 10]$$

$$\stackrel{val}{=} \exists_{k} [0 \le k \le n - 1 : k^{2} \le 10] \lor \exists_{k} [k = n : k^{2} \le 10]$$

Domain splitting

$$\forall_x [P \lor Q : R] \stackrel{val}{=} \forall_x [P : R] \land \forall_x [Q : R]$$
$$\exists_x [P \lor Q : R] \stackrel{val}{=} \exists_x [P : R] \lor \exists_x [Q : R]$$

One-element domain

$$\forall_x [x = n \colon Q] \stackrel{val}{=} Q[n/x]$$

$$\exists_x [x = n \colon Q] \stackrel{val}{=} Q[n/x]$$

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Example:

$$\forall_x [x = 3: 2 \cdot x \geqslant 1] \stackrel{val}{=} 2 \cdot 3 \geqslant 1$$

One-element domain

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Example:

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Empty domain

$$\forall_x [F:Q] \stackrel{val}{=} T$$

$$\exists_x [F:Q] \stackrel{val}{=} F$$

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One-element domain

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"All Marsians are green"

Empty domain

$$\forall_x [F:Q] \stackrel{val}{=} T$$

$$\exists_x [F:Q] \stackrel{val}{=} F$$

Domain weakening

Intuition: The following are equivalent

$$\forall_x [x \in D : A(x)]$$
 and $\forall_x [x \in D \Rightarrow A(x)]$
 $\exists_x [x \in D : A(x)]$ and $\exists_x [x \in D \land A(x)]$

The same can be done to parts of the domain

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Domain weakening

$$\begin{cases} \forall_x [P \land Q : R] \stackrel{val}{=} \forall_x [P : Q \Rightarrow R] \\ \exists_x [P \land Q : R] \stackrel{val}{=} \exists_x [P : Q \land R] \end{cases}$$

Domain weakening

Intuition: The following are equivalent

$$\forall_x [x \in D : A(x)]$$
 and $\forall_x [x \in D \Rightarrow A(x)]$
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The same can be done to parts of the domain

Domain weakening

$$\begin{vmatrix} \forall_x [P \land Q : R] \stackrel{val}{=} \forall_x [P : Q \Rightarrow R] \\ \exists_x [P \land Q : R] \stackrel{val}{=} \exists_x [P : Q \land R] \end{vmatrix}$$

$$P \wedge Q \models P$$

De Morgan

$$\neg \forall_x [P:Q] \stackrel{val}{=} \exists_x [P:\neg Q]$$
$$\neg \exists_x [P:Q] \stackrel{val}{=} \forall_x [P:\neg Q]$$

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```
not for all = at least for one not
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not exists = for all not

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Hence: $\neg \forall = \exists \neg \text{ and } \neg \exists = \forall \neg$

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not for all = at least for one not

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Hence: $\neg \forall = \exists \neg \text{ and } \neg \exists = \forall \neg$

It holds further that:

$$\neg \forall_x \neg = \exists_x \neg \neg = \exists_x$$
$$\neg \exists_x \neg = \forall_x \neg \neg = \forall_x$$

Substitution

meta rule

Simple

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

Sequential

$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]$$

Simultaneous

$$\phi \stackrel{val}{=} \psi$$

EVERY occurrence of P is substituted!

$$\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]$$

holds also for quantified formulas!

- Substitution

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The rule of Leibnitz

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$$\phi \stackrel{val}{=} \psi$$

$$C[\phi] \stackrel{val}{=} C[\psi]$$

formula that has ϕ as a sub formula

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single occurrence is replaced!

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$$C[\phi] \stackrel{val}{=} C[\psi]$$

formula that has ϕ as a sub formula meta rule

single occurrence is replaced!

Exchange trick

$$\forall_x [P:Q] \stackrel{val}{=} \forall_x [\neg Q:\neg P]$$

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$$\forall_x [P:Q] \stackrel{val}{=} \forall_x [\neg Q:\neg P]$$

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No wonder as

$$\forall_x [P:Q] \stackrel{val}{=} \forall_x [P \Rightarrow Q]$$

$$\exists_x [P:Q] \stackrel{val}{=} \exists_x [P \land Q]$$

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Term splitting

$$\forall_x [P:Q \land R] \stackrel{val}{=} \forall_x [P:Q] \land \forall_x [P:R]$$

$$\exists_x [P:Q \lor R] \stackrel{val}{=} \exists_x [P:Q] \lor \exists_x [P:R]$$

Monotonicity of quantifiers

$$\forall_x [P:Q \Rightarrow R] \Rightarrow (\forall_x [P:Q] \Rightarrow \forall_x [P:R]) \stackrel{val}{=} T$$

$$\forall_x [P:Q \Rightarrow R] \Rightarrow (\exists_x [P:Q] \Rightarrow \exists_x [P:R]) \stackrel{val}{=} T$$

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tautologies

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Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

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still hold (in predicate logic)

Lemma W5: If $Q \models R$ then $\forall_x [P:Q] \models \forall_x [P:R]$.

Derivations / Reasoning

Limitations of proofs by calculation

Proofs by calculation are formal and well-structured, but often undirected and not particularly intuitive.

Example

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} (P \vee F) \wedge (P \vee Q)$$

$$\stackrel{\text{val}}{=} P \vee (F \wedge Q)$$

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Conclusions

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} P \quad P \wedge (P \vee Q) \Leftrightarrow P \stackrel{\text{val}}{=} T$$

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$$\stackrel{\text{val}}{=} P \vee F$$

$$\stackrel{\text{val}}{=} P$$

we can prove this more intuitively by reasoning

Conclusions

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} P \wedge (P \vee Q) \Leftrightarrow P \stackrel{\text{val}}{=} T$$

An example of a mathematical proof

Theorem

If x^2 is even, then x is even $(x \in \mathbb{Z})$.

Proof

Let $x \in \mathbb{Z}$ be such that x^2 is even.

We need to prove that x is even too.

Assume that x is odd, towards a contradiction.

If x is odd than x = 2y+1 for some $y \in \mathbb{Z}$. Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$ and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd too, and we have a contradiction.

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So, x^2 is odd too, and we have a contradiction.

Thanks to Bas Luttik

Exposing logical structure

Theorem

If x^2 is even, then x is even $(x \in \mathbb{Z})$.

Proof



Assume x^2 is even.

Assume that x is odd.

Then x = 2y+1 for some $y \in \mathbb{Z}$.

Then
$$x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$$
 and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

a contradiction.

So, x is even

(sub)goal

generating hypothesis

pure hypothesis

conclusion

Thanks to Bas Luttik

Q is a correct conclusion from n premises $P_1, ..., P_n$ iff $(P_1 \land P_2 \land \land P_n) \stackrel{\text{val}}{\vDash} Q$

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If n=0, then
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Q holds unconditionally

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a formal system
based on the single
inference rule
for proofs that closely
follow our
intuitive reasoning

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Two types of inference rules:

elimination rules

introduction rules

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(particularly useful) instances of the single inference rule

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and one new special rule!

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Two types of inference rules:

elimination rules

introduction rules

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for simplifying goals

(particularly useful) instances of the single inference rule

and one new special rule!

How do we use a conjunction in a proof?

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 $P \wedge Q \stackrel{\text{val}}{\models} P$

 $P \land Q \stackrel{\text{val}}{\models} Q$

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 $P \wedge Q \stackrel{\text{val}}{=} P$

 $P \land Q \stackrel{\text{val}}{\models} Q$

```
|| ||
(k) P∧Q
```

11.1

 $\{\land$ -elim on $(k)\}$

(m) P

 $(k \le m)$

How do we use a conjunction in a proof?

 $P \wedge Q \stackrel{\text{val}}{\models} P$

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```
\parallel \parallel
```

(k) $P \wedge Q$

 $\Pi\Pi$

 $\{\land$ -elim on $(k)\}$

(m) F

$$\parallel \parallel$$

(k) $P \wedge Q$

|| ||

 $\{\land$ -elim on $(k)\}$

(m) Q

(k < m)

How do we use a conjunction in a proof?

 $P \land Q \stackrel{\text{val}}{\models} Q$

 $P \wedge Q \stackrel{\text{val}}{\models} P$

∧-elimination

|| ||

(k) $P \wedge Q$

|| ||

 $\{\land$ -elim on $(k)\}$

(k < m)

(m) F

|| ||

(k) $P \wedge Q$

 $\{\land$ -elim on $(k)\}$

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How do we use an implication in a proof?

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$$P \Rightarrow Q \stackrel{\text{val}}{\models} ???$$

$$(P \Rightarrow Q) \land P \stackrel{\text{val}}{\models} Q$$

How do we use an implication in a proof?

$$P \Rightarrow Q \stackrel{\text{val}}{\models} ???$$

$$(P{\Rightarrow}Q) \wedge P \overset{\text{val}}{\vDash} Q$$

```
\{\Rightarrow-elim on (k) and (l)\}
(m)
```

$$(k \le m, l \le m)$$

How do we use an implication in a proof?

 $P \Rightarrow Q \stackrel{\text{val}}{\models} ???$

 $(P \Rightarrow Q) \land P \stackrel{\text{val}}{\models} Q$

⇒-elimination

$$(m)$$
 Q

How do we prove a conjunction?

How do we prove a conjunction?



How do we prove a conjunction?

 $P \land Q \stackrel{\text{val}}{\models} P \land Q$

```
(k) P
```

(I)

(k < m, l < m)

How do we prove a conjunction?

∧-introduction

• • •

(k) F

(I)

• • •

 $\{\land$ -intro on (k) and (l) $\}$

(m) $P \wedge Q$

(k < m, l < m)

 $P \land Q \stackrel{\text{val}}{\models} P \land Q$

How do we prove an implication?

How do we prove an implication?

```
{Assume}
\{\Rightarrow-intro on (k) and (I-I)\}
```

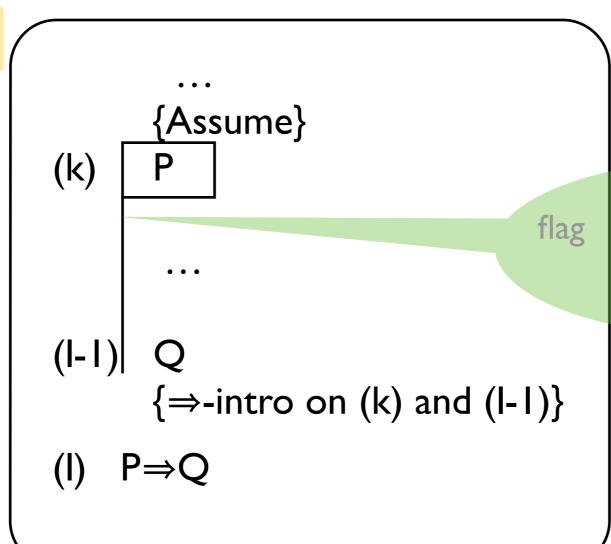
How do we prove an implication?

⇒-introduction

```
{Assume}
(k)
        \{\Rightarrow-intro on (k) and (I-I)\}
```

How do we prove an implication?

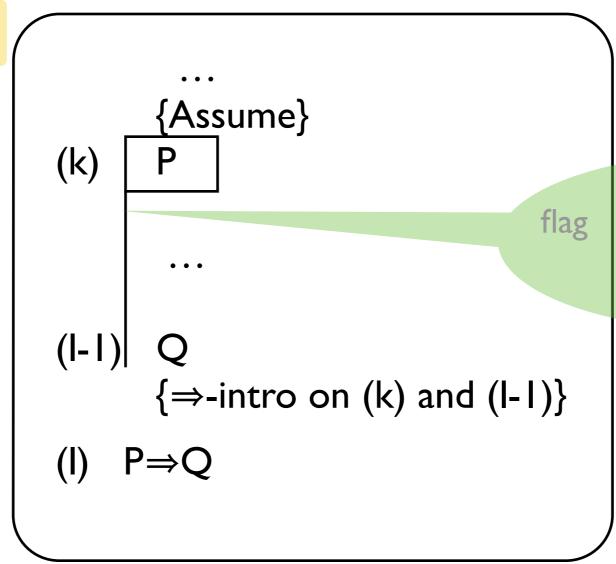
⇒-introduction



shows the validity of a hypothesis

How do we prove an implication?

⇒-introduction

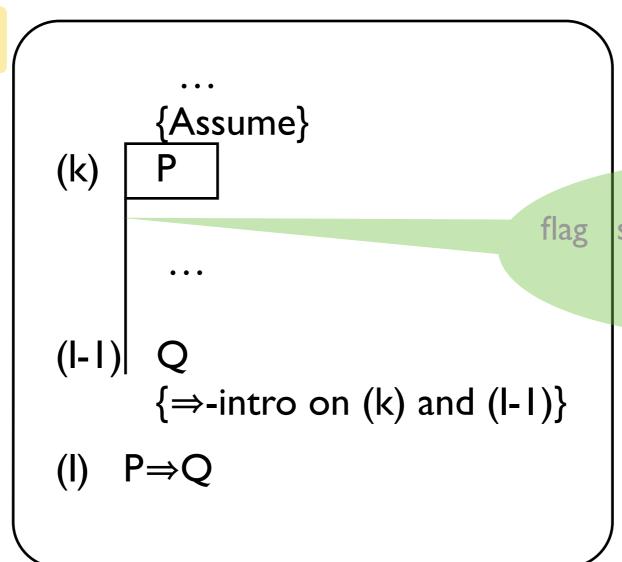


truly new and necessary for reasoning with hypothesis

shows the validity of a hypothesis

How do we prove an implication?

⇒-introduction



truly new and necessary for reasoning with hypothesis

shows the validity of a hypothesis

time for an example!