Propositional Logic
Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \overset{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation

$$\neg\neg P \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \overset{val}{=} F$$

Excluded Middle

$$P \vee \neg P \overset{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T/F - elimination

$$P \wedge T \stackrel{val}{=}$$

$$P \wedge F \stackrel{val}{=}$$

$$P \vee T \stackrel{val}{=}$$

$$P \vee F \stackrel{val}{=}$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$
$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \overset{val}{=} F$$

Excluded Middle

$$P \vee \neg P \overset{val}{=} T$$

T/F - elimination

$$P \wedge T \overset{val}{=} P$$
$$P \wedge F \overset{val}{=} F$$
$$P \vee T \overset{val}{=} T$$
$$P \vee F \overset{val}{=} P$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \overset{val}{\neq} \neg P \Rightarrow \neg Q$$

common mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=}$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=} T$$

Calculating with equivalent propositions

(the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation P $\overset{\text{val}}{=}$ Q, iff they induce the same truth-function.

on any sequence containing their common variables

Property: The relation $\overset{\text{val}}{=}$ is an equivalence on the set of all abstract propositions.

Substitution

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

Substitution

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Substitution

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

EVERY occurrence of P is substituted!

Substitution

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultanious

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

EVERY occurrence of P is substituted!

Substitution

meta rule

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultanious

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]}$$

The rule of Leibnitz

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

The rule of Leibnitz

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

single occurrence is replaced!

The rule of Leibnitz

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

The rule of Leibnitz

meta rule

Leibnitz

$$\frac{\phi \stackrel{val}{=} \psi}{C[\phi] \stackrel{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1, also Q has truth value 1, and

(2) Always when Q has truth value 1, also P has truth value 1.

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1, also Q has truth value 1, and

(2) Always when Q has truth value 1, also P has truth value 1.

if we relax this, we get strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff

~~(1)~~ Always when P has truth value 1, also Q has truth value 1~~, and~~

~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff

~~(1)~~ Always when P has truth value 1, also Q has truth value 1~~, and~~

~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Q is weaker than P

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff

always when P has truth value 1,
also Q has truth value 1.

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \models_{val} Q$, iff
always when P has truth value 1,
also Q has truth value 1.

always when P is true,
Q is also true

Strengthening

Definition: The abstract proposition P is stronger than Q,
notation $P \models_{val} Q$, iff
always when P has truth value 1,
also Q has truth value 1.

always when P is true,
Q is also true

Q is weaker
than P

Properties

Properties

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Properties

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \overset{val}{=} Q$ iff $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} P$.

Properties

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

Lemma W2: Weakening is a reflexive relation on abstract propositions.

Properties

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

Lemma W2: Weakening is a reflexive relation on abstract propositions.

Lemma W3: Weakening is a transitive relation on abstract propositions.

Properties

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \overset{val}{=} Q$ iff $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} P$.

Lemma W2: Weakening is a reflexive relation on abstract propositions.

Lemma W3: Weakening is a transitive relation on abstract propositions.

Lemma W4: $P \overset{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi\{\xi/P\} \overset{val}{\models} \psi\{\xi/P\}}$$

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultanious

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

Substitution

just holds

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi\{\xi/P\} \overset{val}{\models} \psi\{\xi/P\}}$$

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultanious

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

Substitution

just holds

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi\{\xi/P\} \overset{val}{\models} \psi\{\xi/P\}}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultanious

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

The rule of Leibniz

does not hold for weakening!

Leibniz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

formula that has ϕ as a sub formula

The rule of Leibniz

does not hold
for weakening!

Leibniz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$