Propositional Logic Standard Equivalences

$$\begin{array}{l} Commutativity \\ P \land Q \stackrel{val}{=} Q \land P \\ P \lor Q \stackrel{val}{=} Q \lor P \\ P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P \end{array}$$

Commutativity

$$P \land Q \stackrel{val}{=} Q \land P$$

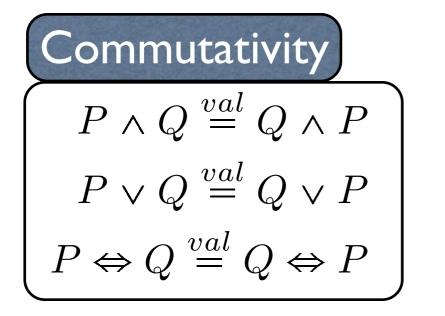
$$P \lor Q \stackrel{val}{=} Q \lor P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$$

$$P \quad Q \quad P \Rightarrow Q \quad Q \Rightarrow P$$

$$0 \quad 1 \quad 1 \quad 0$$



Associativity $(P \land Q) \land R \stackrel{val}{=} P \land (Q \land R)$ $(P \lor Q) \lor R \stackrel{val}{=} P \lor (Q \lor R)$ $(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$

Commutativity
$$P \land Q \stackrel{val}{=} Q \land P$$
 $P \lor Q \stackrel{val}{=} Q \lor P$ $P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$

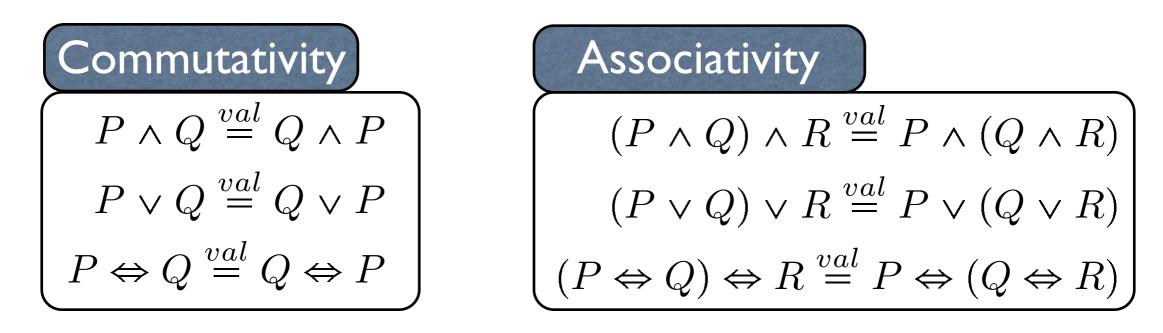
Associativity

$$(P \land Q) \land R \stackrel{val}{=} P \land (Q \land R)$$

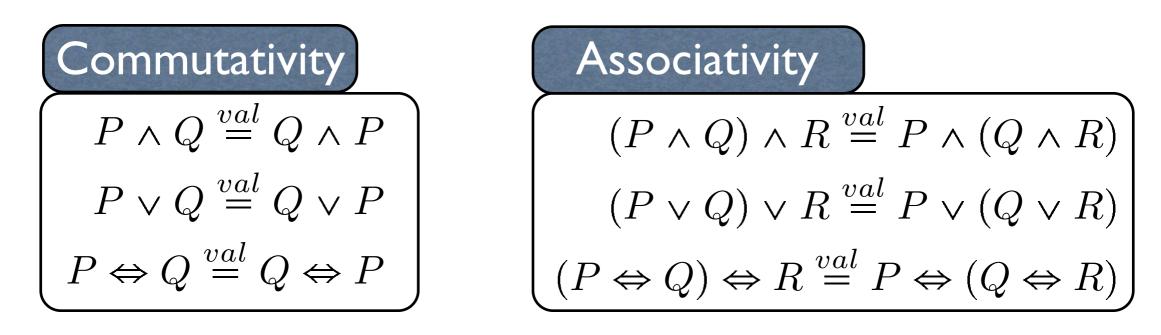
$$(P \lor Q) \lor R \stackrel{val}{=} P \lor (Q \lor R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

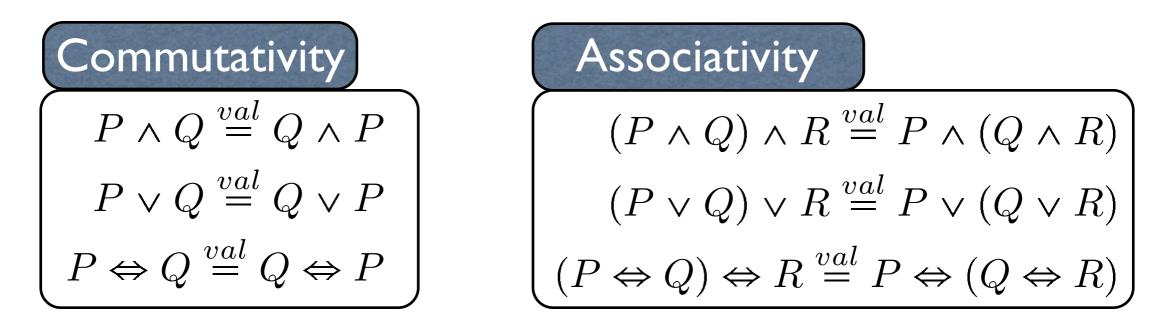
$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$



$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$



$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$



$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Idempotence and Double Negation

Idempotence	
$P \land P \stackrel{val}{=} P$	5
$P \lor P \stackrel{val}{=} P$	

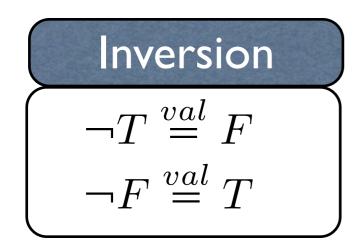
$$P \Rightarrow P \stackrel{val}{\neq} P$$
$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence
$$P \land P \stackrel{val}{=} P$$
 $P \lor P \stackrel{val}{=} P$

$$P \Rightarrow P \stackrel{val}{\neq} P$$
$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation
$$\neg \neg P \stackrel{val}{=} P$$



Inversion
$$\neg T \stackrel{val}{=} F$$
 $\neg F \stackrel{val}{=} T$

Negation
$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Inversion
$$\neg T \stackrel{val}{=} F$$
 $\neg F \stackrel{val}{=} T$

Negation
$$\neg P \stackrel{val}{=} P \Rightarrow F$$

$$\begin{array}{c} \textbf{Contradiction} \\ P \land \neg P \stackrel{val}{=} F \end{array}$$

Inversion
$$\neg T \stackrel{val}{=} F$$
 $\neg F \stackrel{val}{=} T$

Negation
$$\neg P \stackrel{val}{=} P \Rightarrow F$$

$$\begin{array}{c} \textbf{Contradiction} \\ P \land \neg P \stackrel{val}{=} F \end{array}$$

Excluded Middle
$$P \lor \neg P \stackrel{val}{=} T$$

Inversion
$$\neg T \stackrel{val}{=} F$$
 $\neg F \stackrel{val}{=} T$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

$$\begin{array}{c} \text{Contradiction} \\ P \land \neg P \stackrel{val}{=} F \end{array}$$

Excluded Middle
$$P \lor \neg P \stackrel{val}{=} T$$

T/F - elimination
$$P \land T \stackrel{val}{=}$$
 $P \land F \stackrel{val}{=}$ $P \lor T \stackrel{val}{=}$ $P \lor F \stackrel{val}{=}$

Inversion
$$\neg T \stackrel{val}{=} F$$
 $\neg F \stackrel{val}{=} T$

Negation
$$\neg P \stackrel{val}{=} P \Rightarrow F$$

$$\begin{array}{c} \textbf{Contradiction} \\ P \land \neg P \stackrel{val}{=} F \end{array}$$

Excluded Middle
$$P \lor \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \land T \stackrel{val}{=} P$$

$$P \land F \stackrel{val}{=} F$$

$$P \lor T \stackrel{val}{=} T$$

$$P \lor F \stackrel{val}{=} P$$

Distributivity, De Morgan

Distributivity

 $P \land (Q \lor R) \stackrel{val}{=} (P \land Q) \lor (P \land R)$ $P \lor (Q \land R) \stackrel{val}{=} (P \lor Q) \land (P \lor R)$

Distributivity, De Morgan

Distributivity

 $P \land (Q \lor R) \stackrel{val}{=} (P \land Q) \lor (P \land R)$ $P \lor (Q \land R) \stackrel{val}{=} (P \lor Q) \land (P \lor R)$



De Morgan $\left[\neg (P \land Q) \stackrel{val}{=} \neg P \lor \neg Q \\ \neg (P \lor Q) \stackrel{val}{=} \neg P \land \neg Q \right]$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \lor Q$$
 $P \lor Q \stackrel{val}{=} \neg P \Rightarrow Q$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \lor Q$$
 $P \lor Q \stackrel{val}{=} \neg P \Rightarrow Q$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \lor Q$$
 $P \lor Q \stackrel{val}{=} \neg P \Rightarrow Q$

Contraposition
$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$$

$$\land$$

$$common$$

$$mistake!$$

Bi-implication and Selfequivalence

Bi-implication
$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \land (Q \Rightarrow P)$$

Bi-implication and Selfequivalence

Bi-implication
$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \land (Q \Rightarrow P)$$

Self-equivalence
$$P \Leftrightarrow P \stackrel{val}{=}$$

Bi-implication and Selfequivalence

Bi-implication
$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \land (Q \Rightarrow P)$$

Self-equivalence
$$P \Leftrightarrow P \stackrel{val}{=} T$$

Calculating with equivalent propositions (the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{\text{\tiny sl}}{=} Q$, iff they induce the same truth-function.

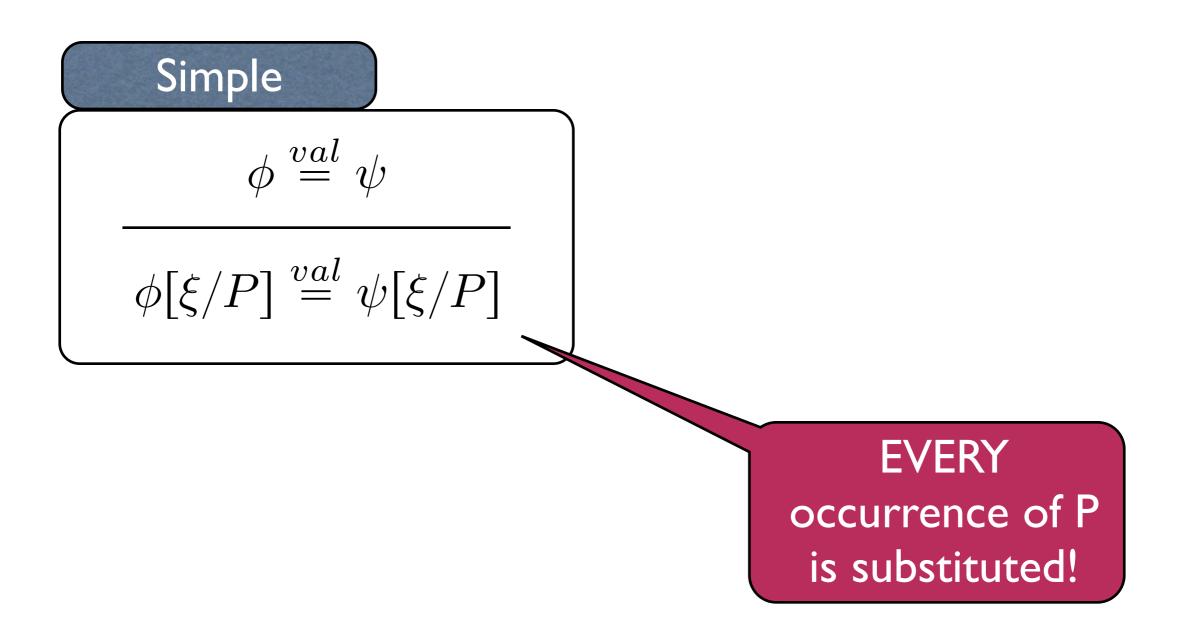
on any sequence containing their common variables

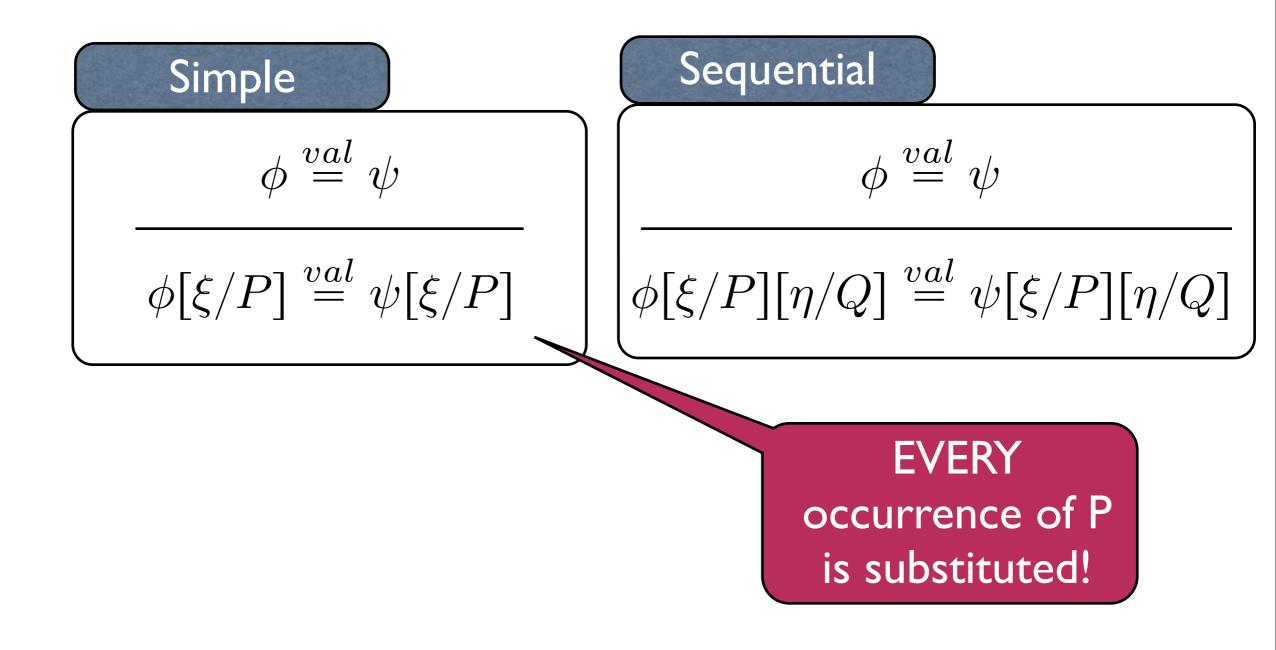
Property: The relation $\stackrel{\text{\tiny M}}{=}$ is an equivalence on the set of all abstract propositions.

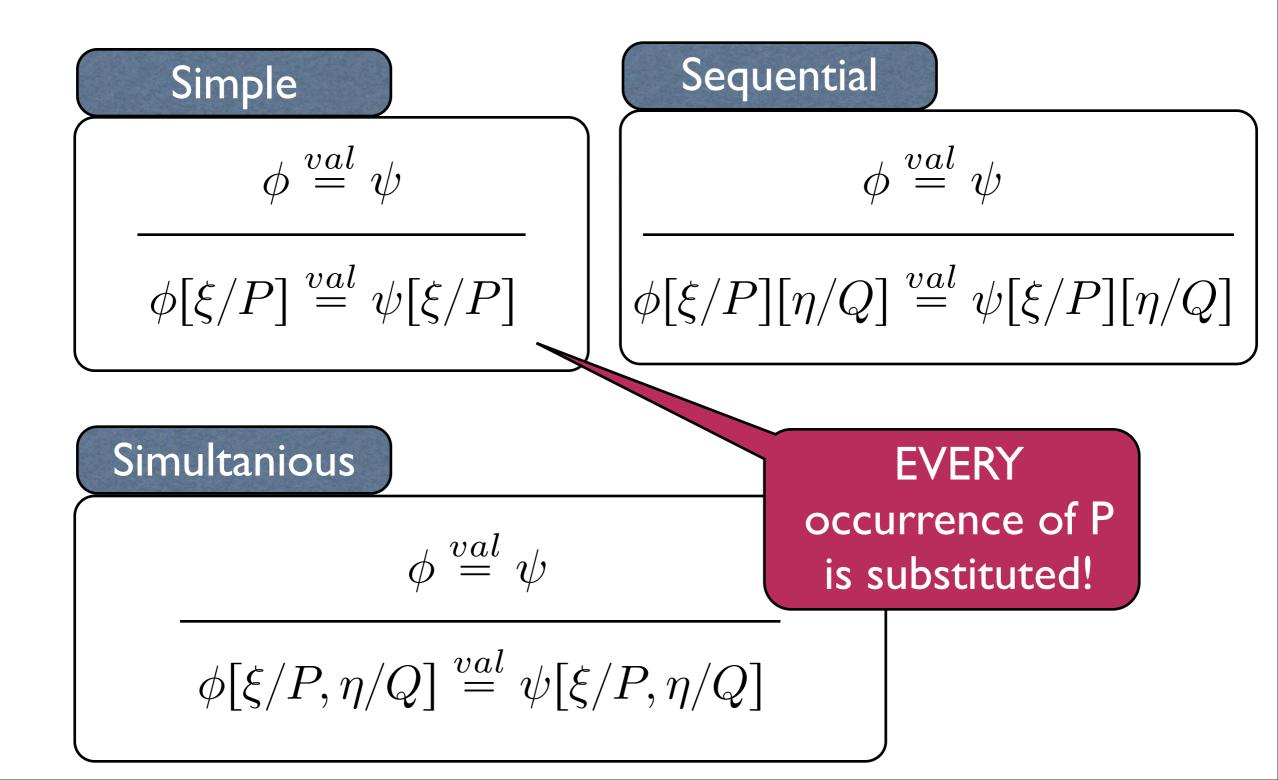
Simple

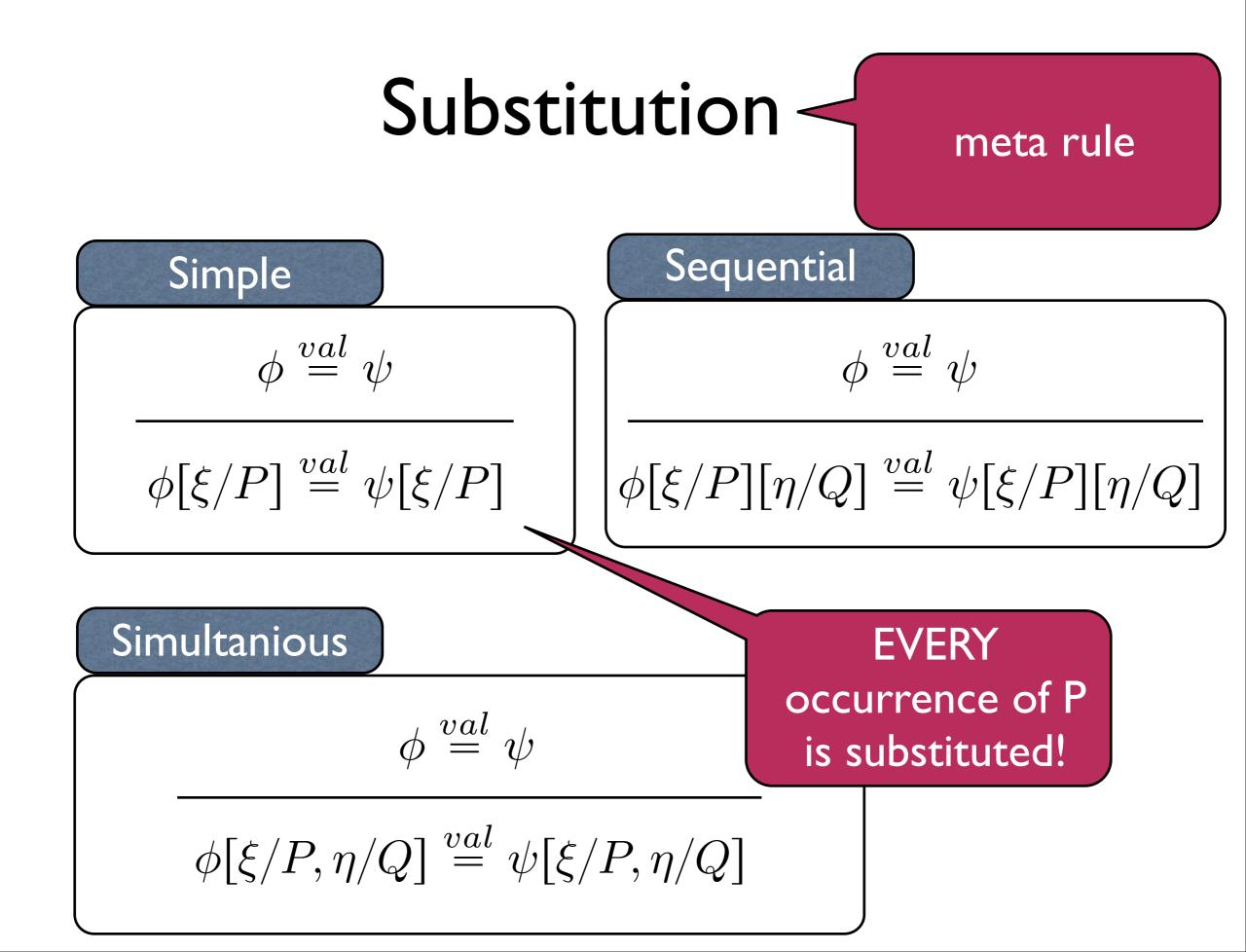
$$\phi \stackrel{val}{=} \psi$$

$$\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]$$

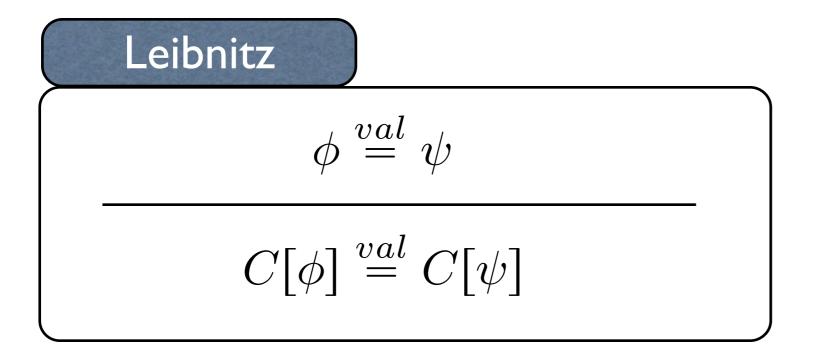




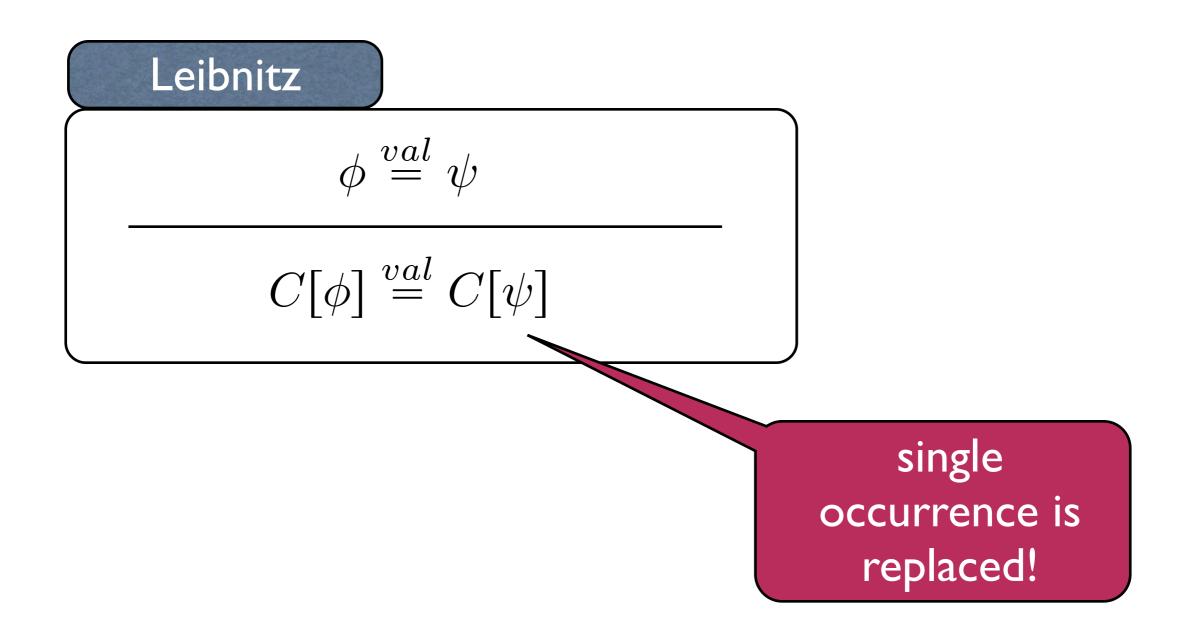




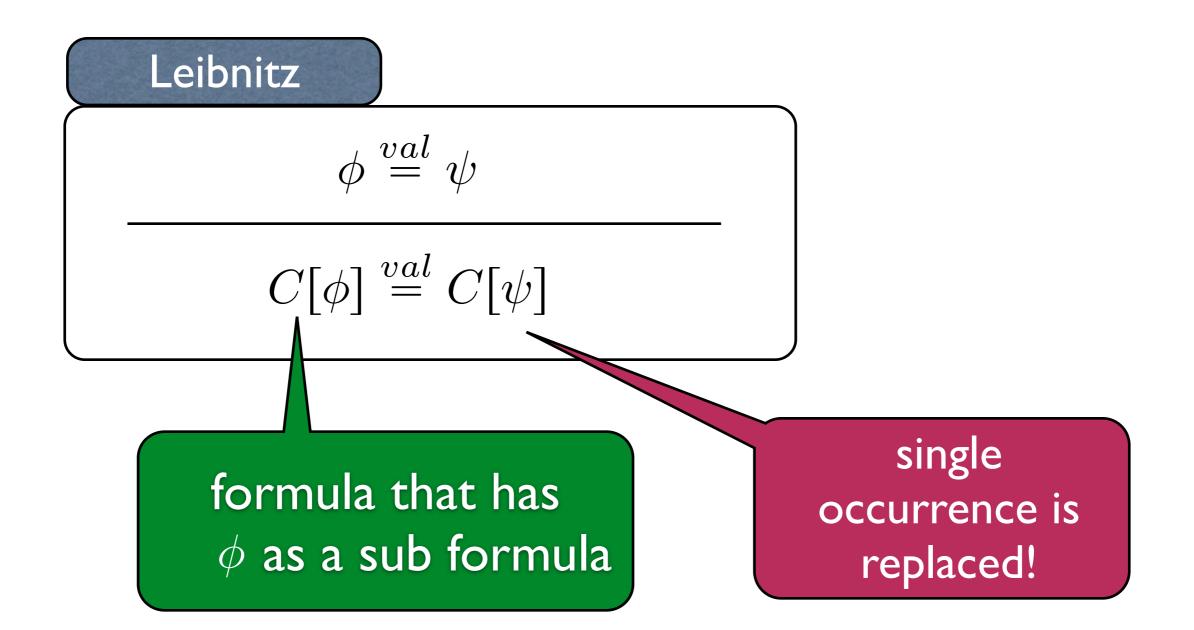
The rule of Leibnitz

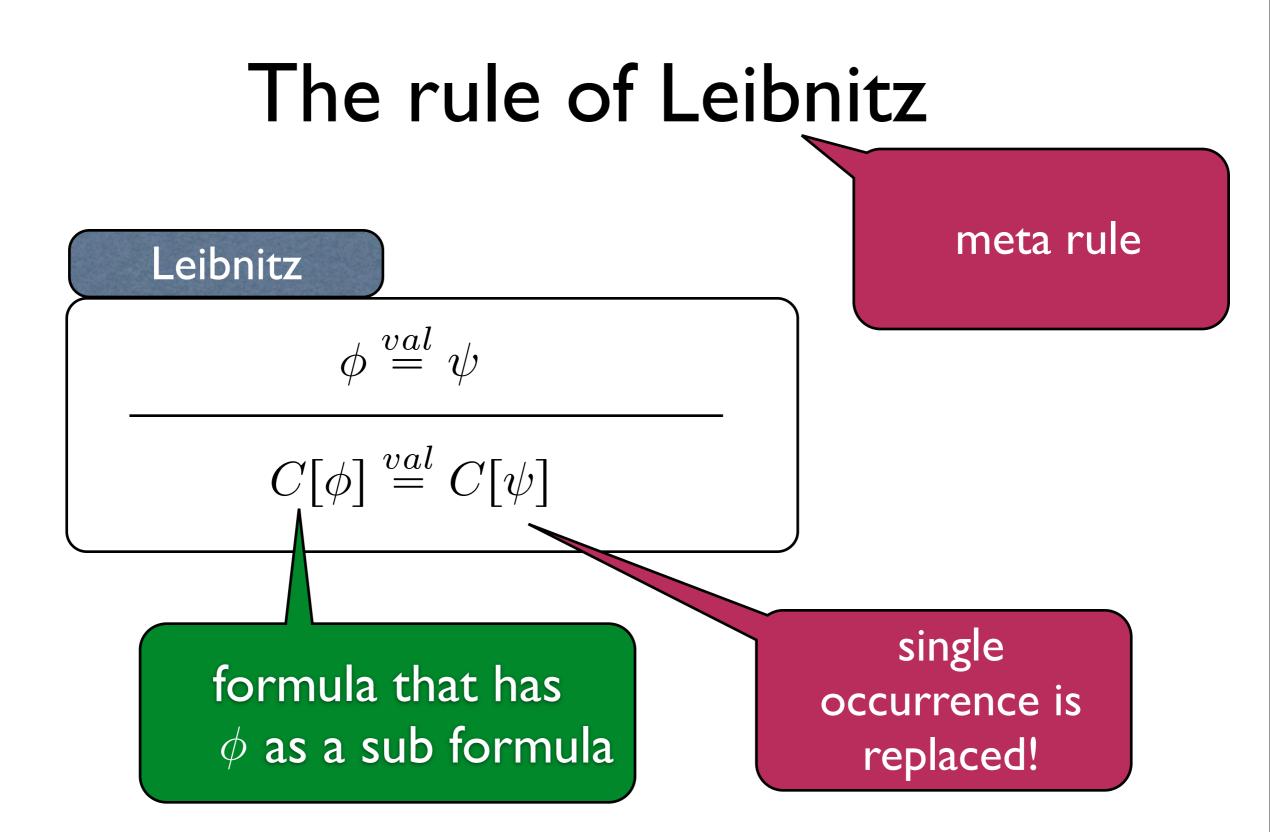


The rule of Leibnitz



The rule of Leibnitz





Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation P [→] Q, iff
(1) Always when P has truth value I, also Q has truth value I, and
(2) Always when Q has truth value I, also P has truth value I.

We had

Definition: Two abstract propositions P and Q are equivalent, notation P [™] Q, iff
(1) Always when P has truth value I, also Q has truth value I, and
(2) Always when Q has truth value I, also P has truth value I.

if we relax this, we get strengthening

Definition: The abstract proposition P is stronger than Q, notation P ⊨ Q, iff (1) Always when P has truth value I, also Q has truth value I,and (2) Always when Q has truth value I, also P has truth value I.

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(1) Always when P has truth value I, also Q has truth value I, and
(2) Always when Q has truth value I, also P has truth value I.

Q is weaker than P

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> always when P is true, Q is also true

Definition: The abstract proposition P is stronger than Q, notation P ⊨ Q, iff always when P has truth value I, also Q has truth value I.

> always when P is true, Q is also true

Q is weaker than P

Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology. Lemma EWI: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

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Lemma W2: Weakening is a reflexive relation on abstract propositions.

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Lemma W2: Weakening is a reflexive relation on abstract propositions.

Lemma W3: Weakening is a transitive relation on abstract propositions.

Lemma EI: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology. Lemma EWI: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

Lemma W2: Weakening is a reflexive relation on abstract propositions.

Lemma W3: Weakening is a transitive relation on abstract propositions.

val

Lemma W4: $P \models Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \land Q \models P$$

 val
 $P \models P \lor Q$

$$\begin{array}{c} \text{val} \\ F \models P \\ P \models T \end{array}$$

Calculating with weakenings (the use of standard weakenings)

Substitution

