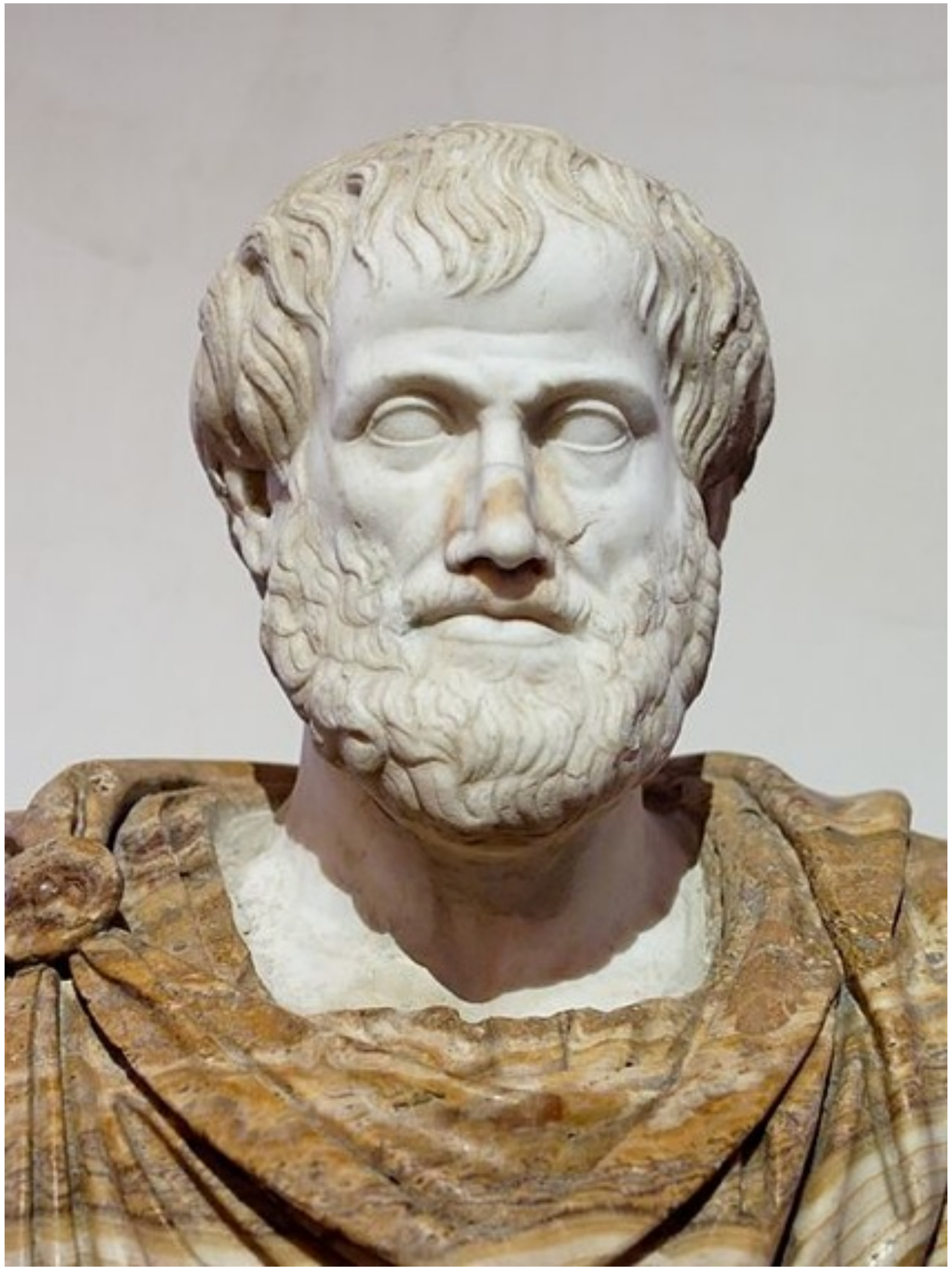
Logic

In the beginning



Aristotle   +/- 350 B.C.

Organon

19 syllogisms

Logic = study of correct reasoning

In the beginning

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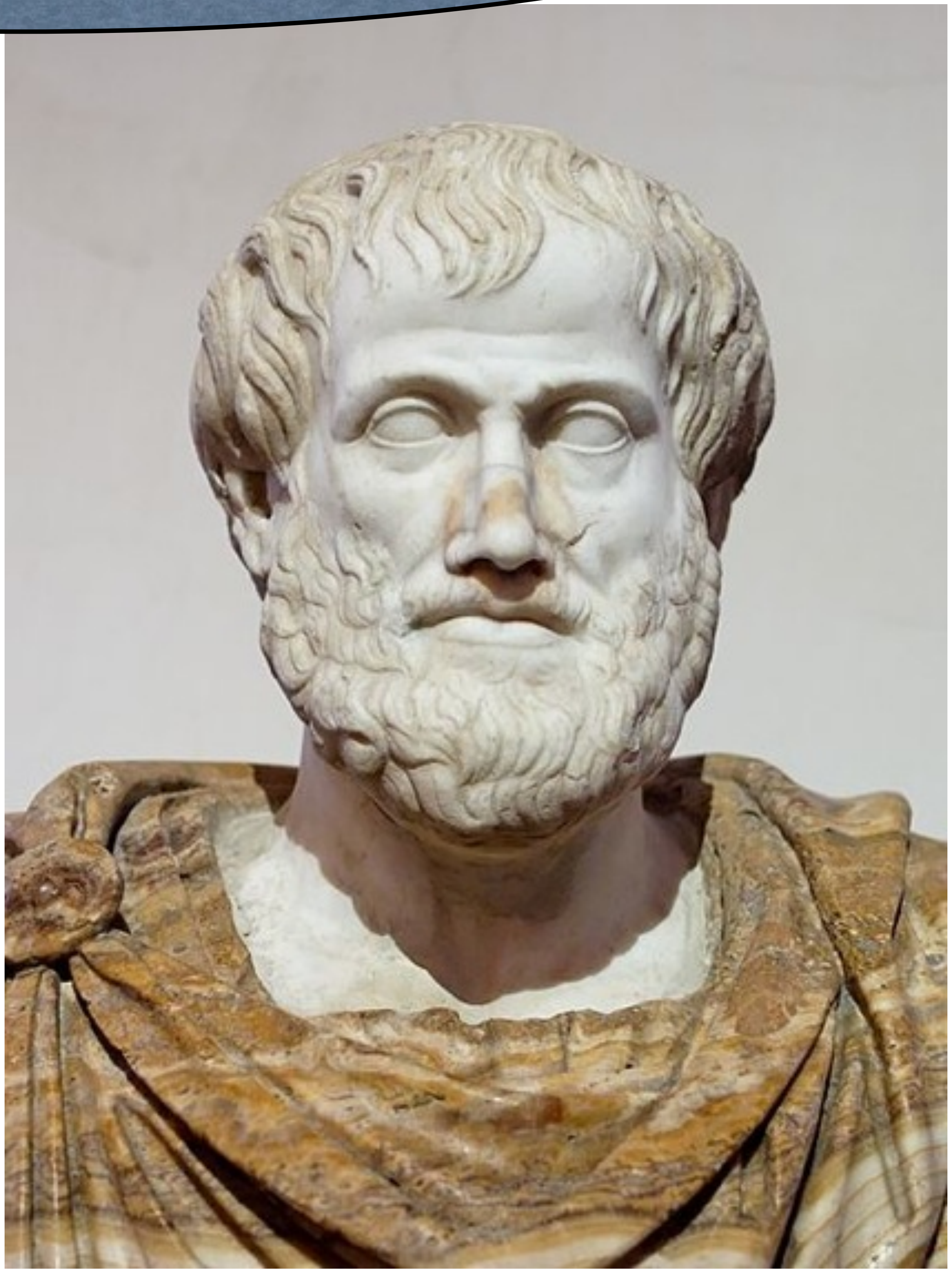

Barbara syllogism

All K's are L's
All L's are M's

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only later called so, in the Middle Ages

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one can **always** conclude the conclusion

independent of what the parameters K,L,M are

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from the two premises

one can **always** conclude the conclusion

independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

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Connectives

- $\wedge$ for "and"
- $\vee$ for "or"
- $\neg$ for "not"
- $\Rightarrow$ for "if .. then" or "implies"
- $\Leftrightarrow$ for "if and only if"

Abstract propositions

Abstract propositions

Definition

Basis	Propositional variables are abstract propositions.
Step (Case 1)	If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2)	If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

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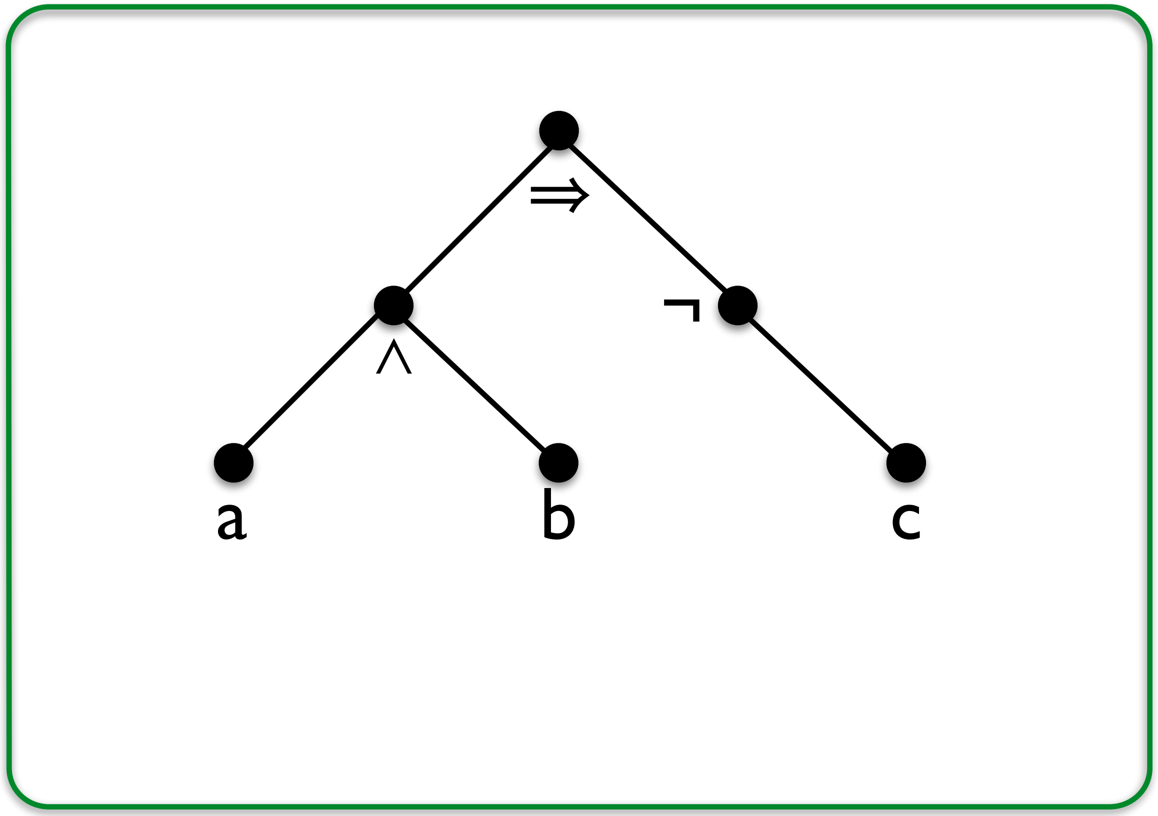
a recursive/inductive definition

…and their structure



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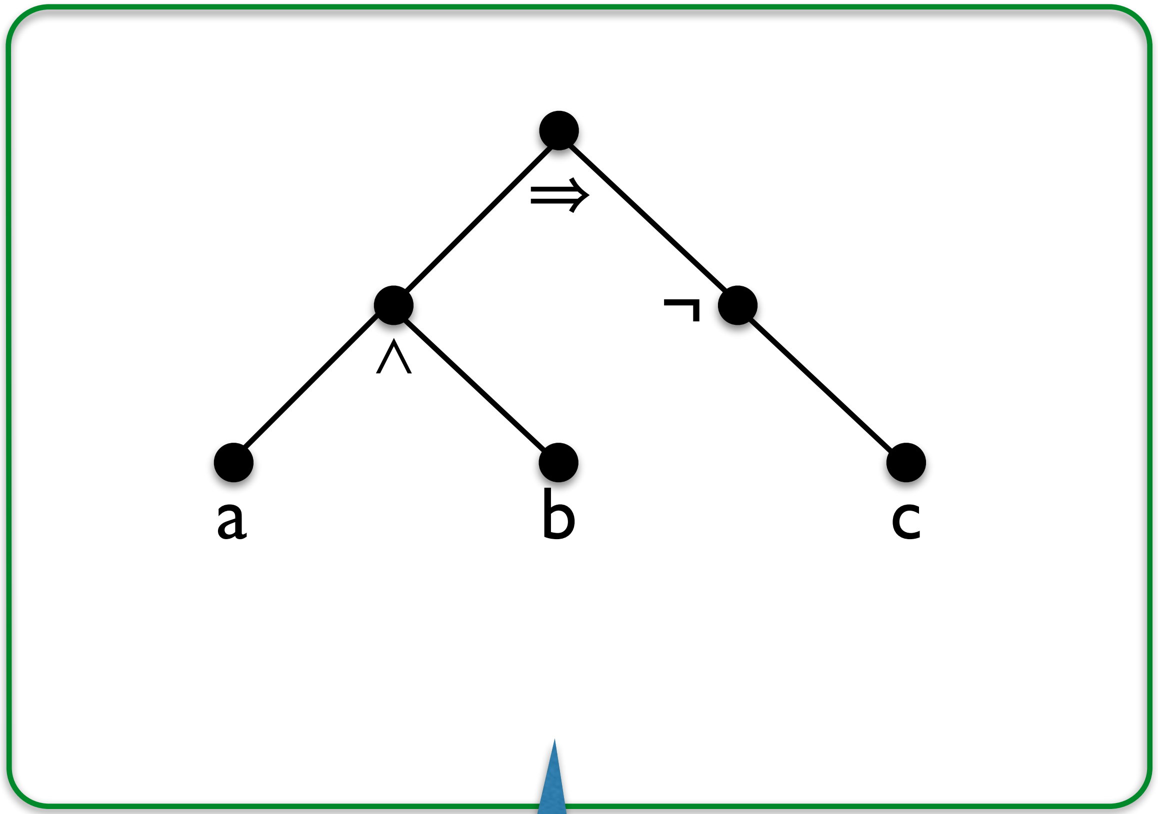
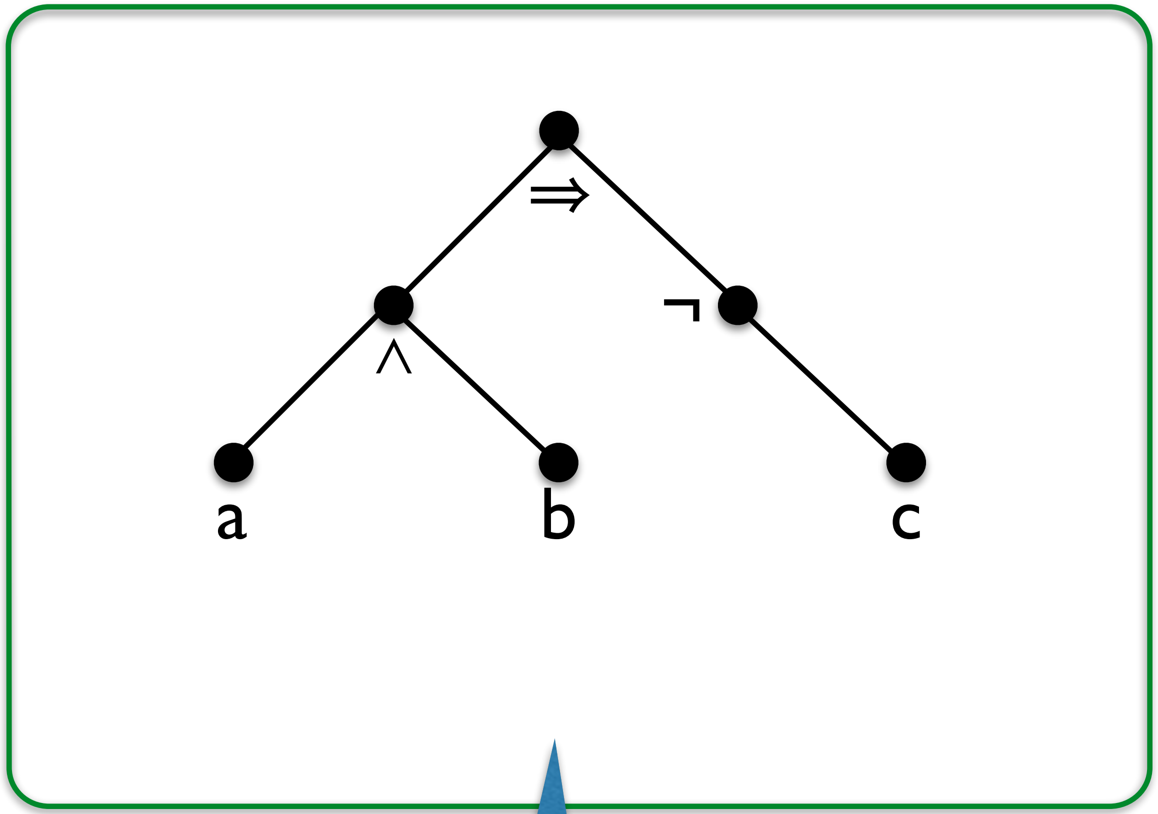
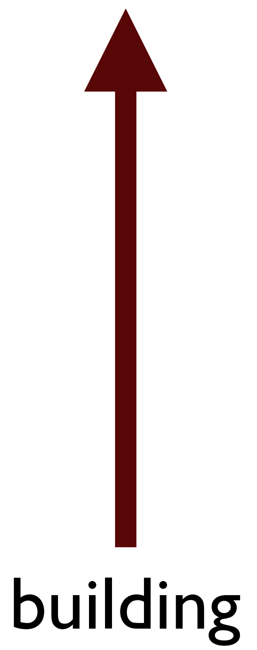


the tree of
$((a \wedge b) \Rightarrow (\neg c))$

…and their structure





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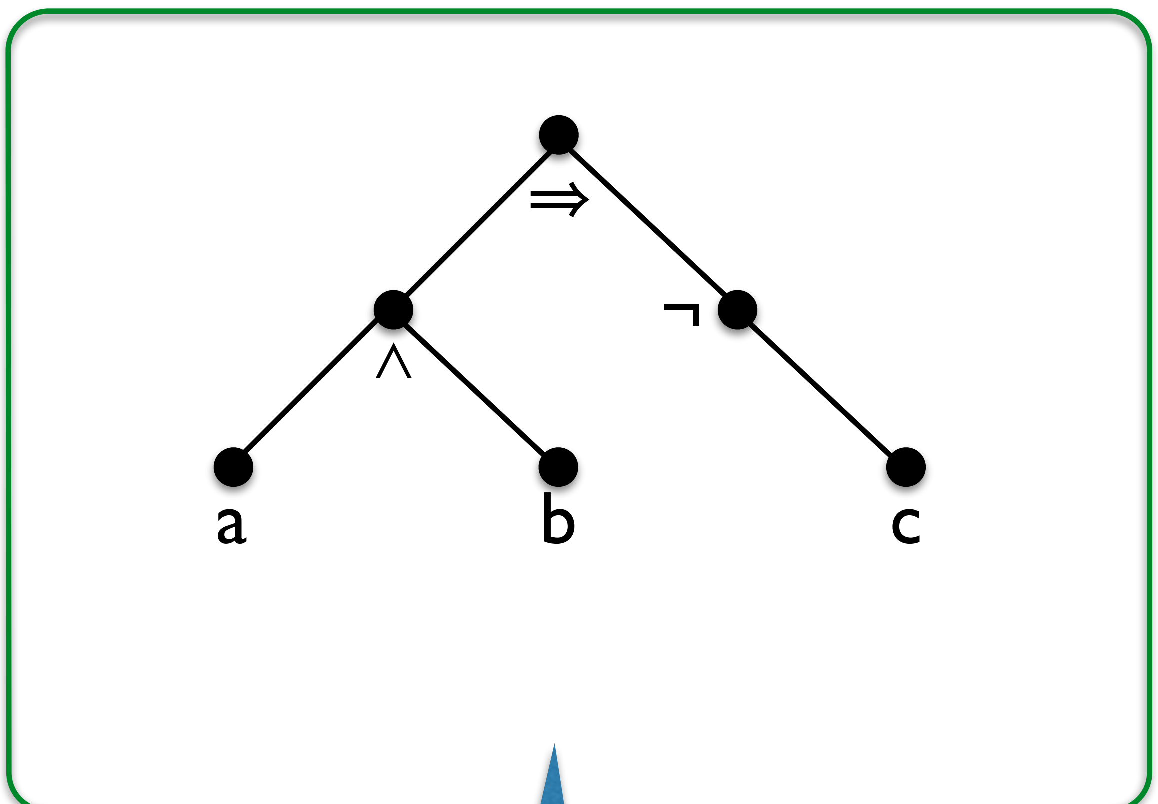

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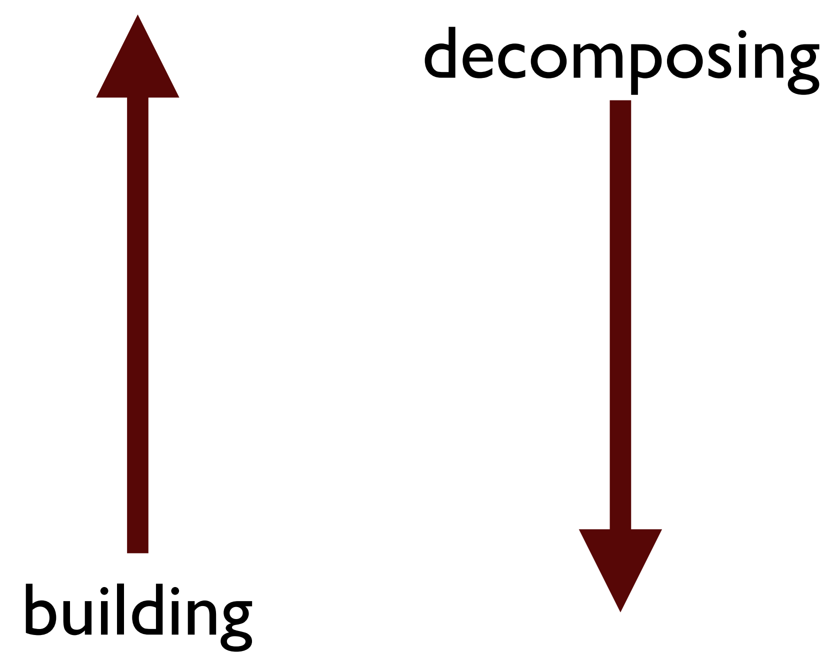

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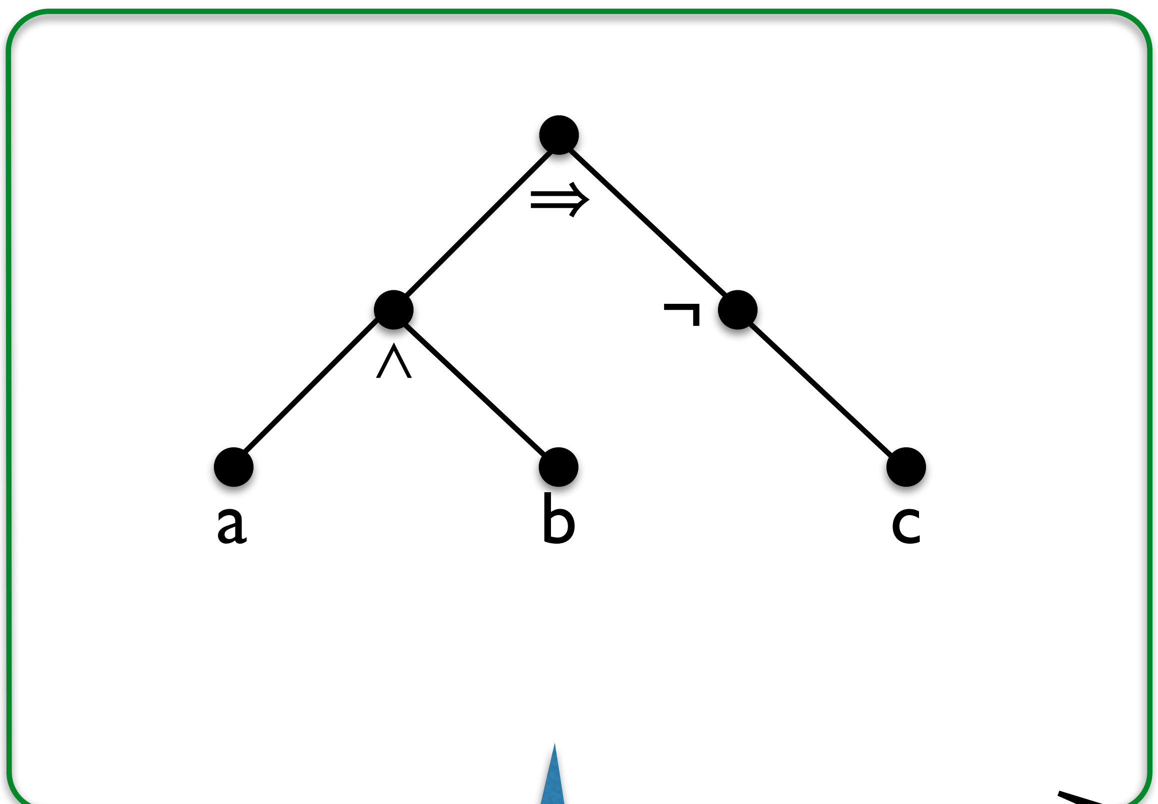

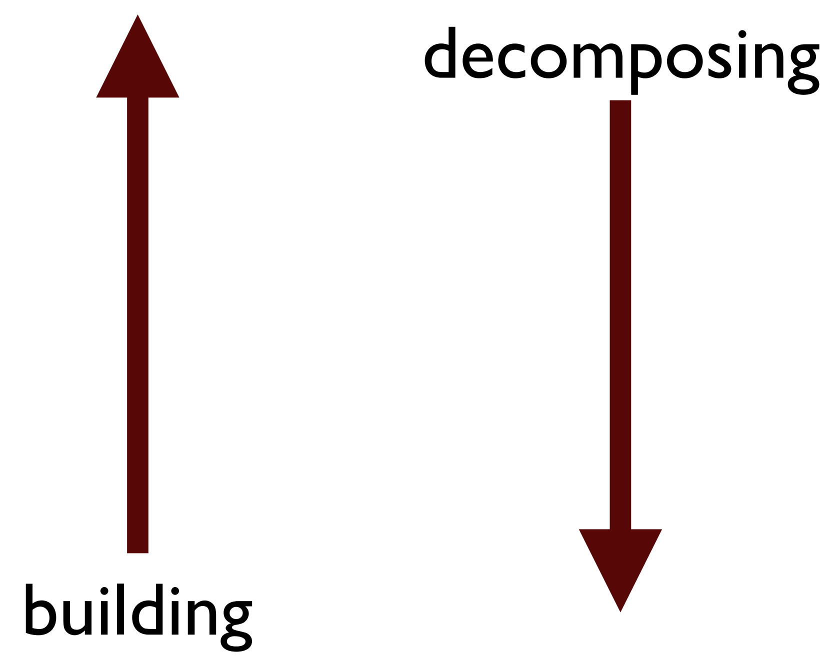

the tree of
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tree representation
(no need of
parenthesis)

Dropping parenthesis

$\neg$
$\wedge \quad \vee$
$\Rightarrow$
$\Leftrightarrow$

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priority schema
(top binds the most)

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Dropping parenthesis

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priority schema
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decreasing priority

incresing priority

Dropping parenthesis

$\neg$

$\wedge \quad \vee$

$\Rightarrow$

$\Leftrightarrow$

priority schema
(top binds the most)

increasing priority

decreasing priority

Example: $((a \wedge b) \Rightarrow (\neg c))$

becomes

$a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	P∧Q
0	0	0
0	1	0
1	0	0
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Conjunction

P	Q	$P \wedge Q$
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only true when both P and Q are true

Truth tables

Disjunction

P	Q	P∨Q
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Truth tables

Negation

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unary connective

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true when P
is false

Truth tables

Implication

Truth tables

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needs more attention

Truth tables

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P	Q	$P \Rightarrow Q$
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0	1	1
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Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
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Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
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0	1	1
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only false when P is true and Q is false

Truth tables

Bi-implication

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$P \Leftrightarrow Q$

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Truth tables

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true when P and Q have the same truth value

Truth-functions

Def. A truth-function or Boolean function is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

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Notation in the book…

$$\begin{array}{l} \quad a, b \\ \left\{ \begin{array}{l} (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array} \right. \end{array}$$

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$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

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Truth-functions

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c)$: $(a \wedge b) \vee b$

induces

$$\left.\begin{array}{lcl} \text{a, b, c} & & \\ (0,0,0) & \longmapsto & 0 \\ (0,0,1) & \longmapsto & 0 \\ (0,1,0) & \longmapsto & 1 \\ (0,1,1) & \longmapsto & 1 \\ (1,0,0) & \longmapsto & 0 \\ (1,0,1) & \longmapsto & 0 \\ (1,1,0) & \longmapsto & 1 \\ (1,1,1) & \longmapsto & 1 \end{array}\right.$$

Equivalence of propositions

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on any sequence containing their common variables

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Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
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Their truth values are the same, so they are equivalent

$$b \wedge \neg b \overset{val}{=} c \wedge \neg c$$

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Tautologies and contradictions

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all contradictions are equivalent

but not all contingencies!

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