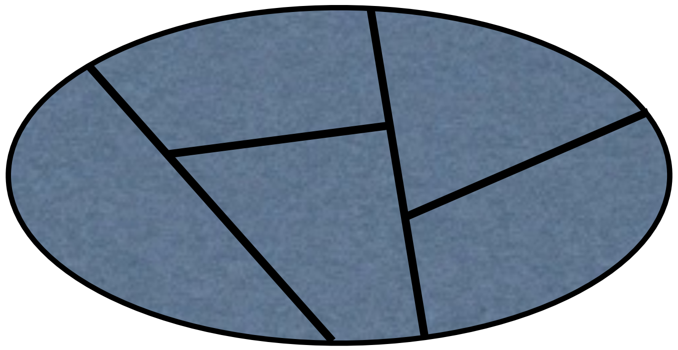
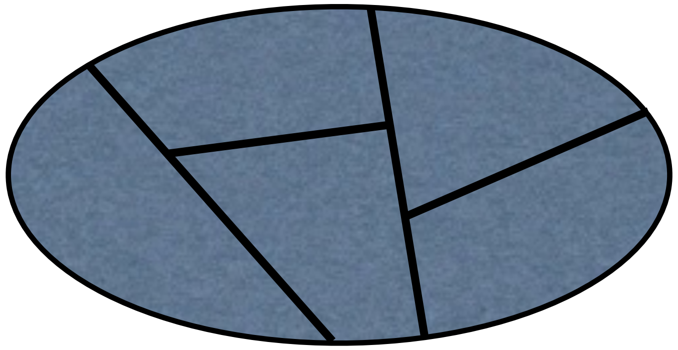


# Partitions = Equivalences



# Partitions = Equivalences

Theorem PE: Let X be a set.

(1) If R is an equivalence on X, then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of X.

(2) If P is a partition of X, then the relation

$$R(P) = \{(x,y) \in X \times X \mid \text{there is } A \in P \text{ such that } x,y \in A\}$$

is an equivalence relation.

Moreover, the assignments $R \mapsto P(R)$ and $P \mapsto R(P)$ are inverse to each other, i.e., $R(P(R)) = R$ and $P(R(P)) = P$ .

# Transitive closure

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Let R be a relation on a set X. The transitive closure (transitive Hülle) of R, notation $R^+$, is the relation

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Proposition TC: Let R be a relation on X. The transitive closure of R is the smallest transitive relation that contains R. The reflexive and transitive closure of R is the smallest reflexive and transitive relation that contains R.

# Functions, mappings

Def. If A and B are sets, then a relation $F \subseteq A \times B$ "is" a function (mapping, Abbildung) from A to B, notation $F\colon A \longrightarrow B$ iff

for every $a \in A$, there exists a unique $b \in B$ such that $aFb$.

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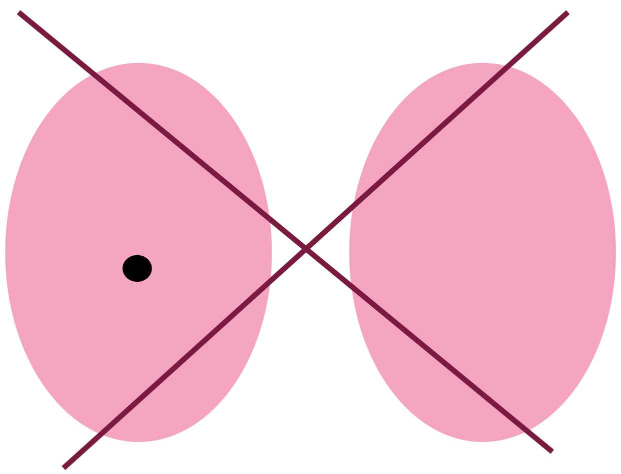
A  B

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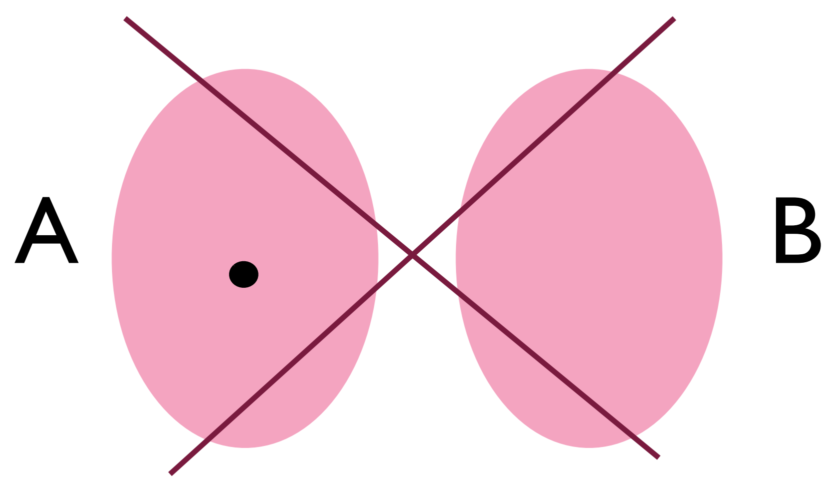


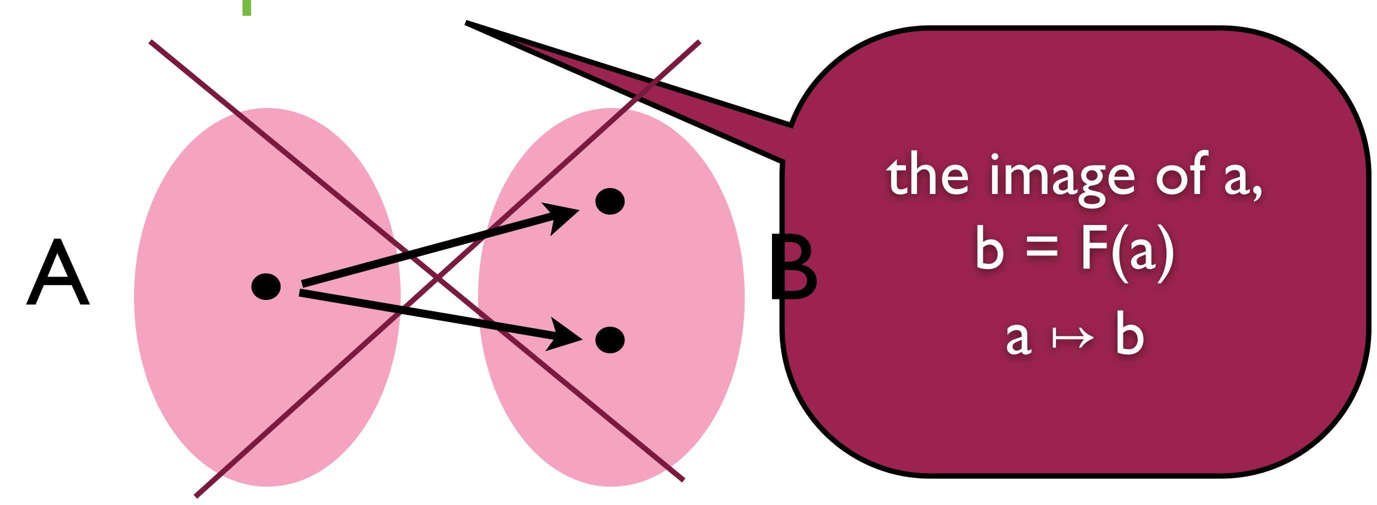

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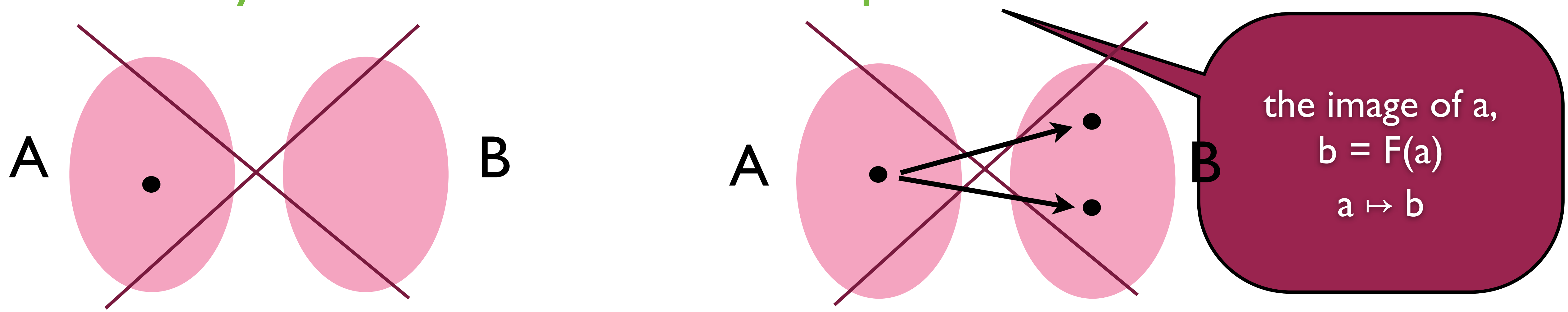


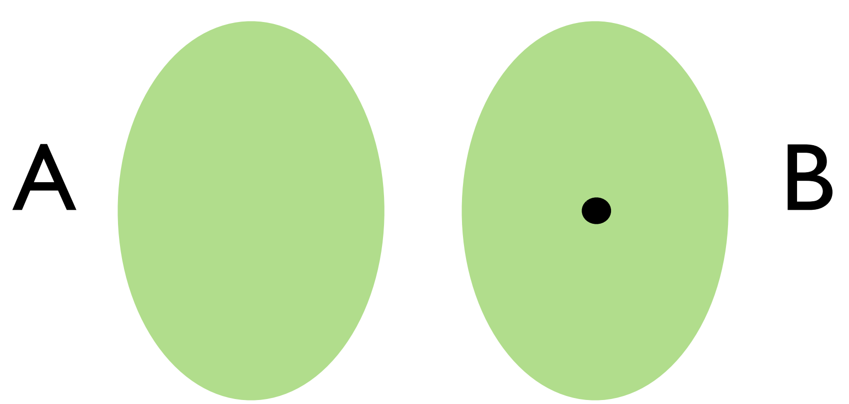

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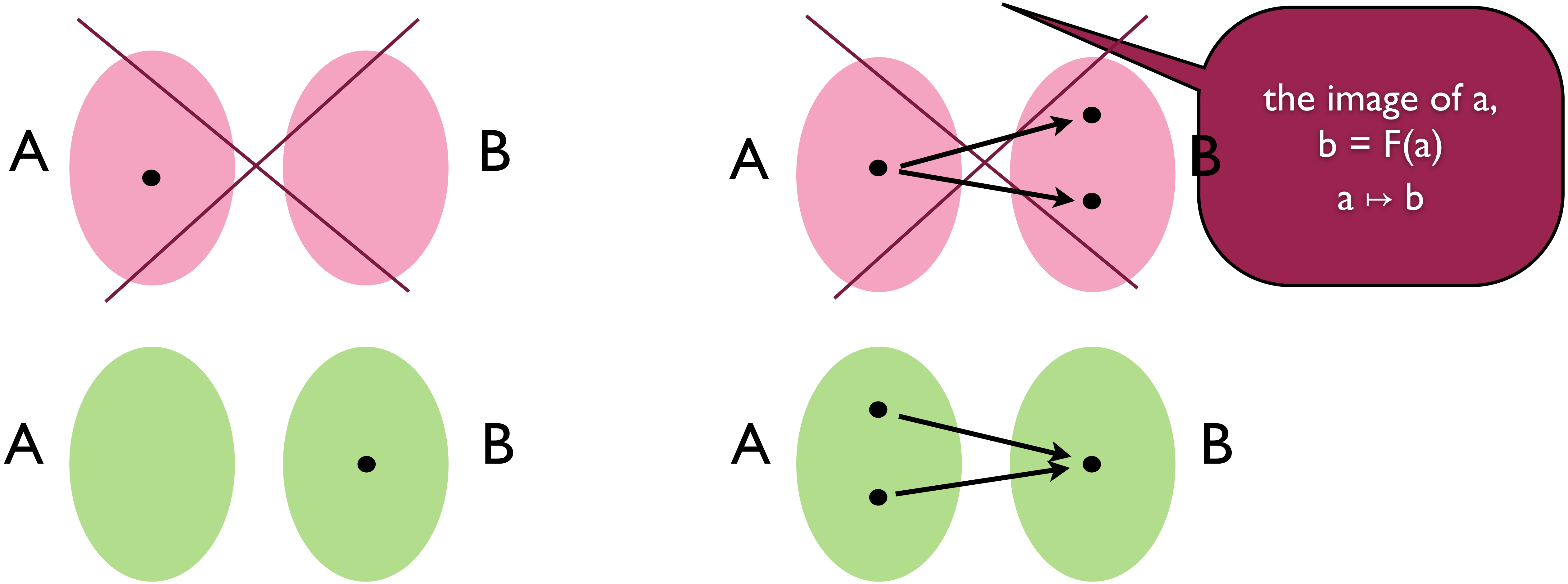

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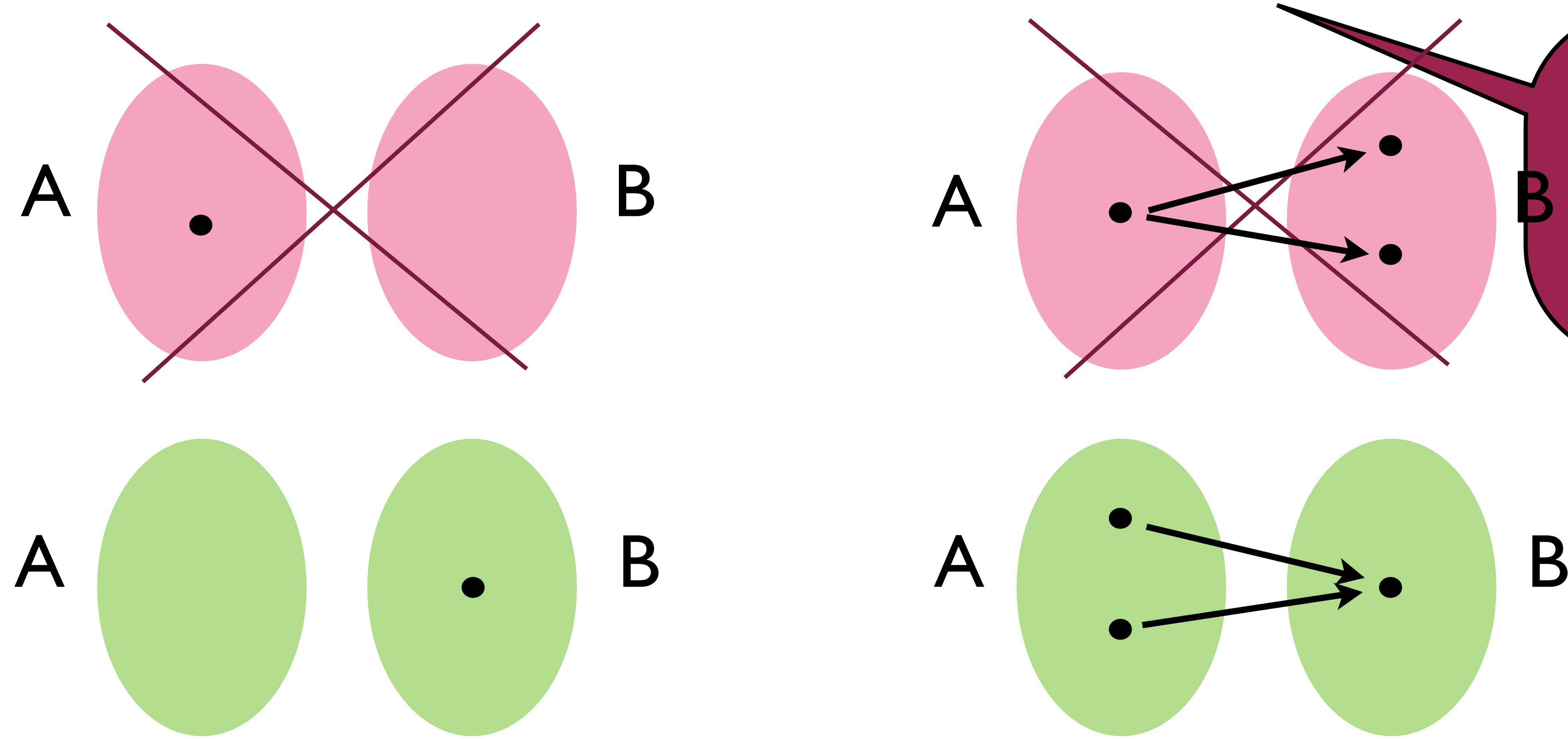


$\{(a, F(a)) \mid a \in A\}$ is the graph of the function F

# Functions, mappings

When $f\colon A \longrightarrow B$ then $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$

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domain of F
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The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

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if $a \in A'$, then $f(a) \in f(A')$

So f extends to a function f: $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

# Functions, mappings

Let $f\colon A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of $B'$ is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

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$a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

# Recall…

Def. If A and B are sets, then a relation $F \subseteq A \times B$ "is" a function (mapping, Abbildung) from A to B, notation $F\colon A \longrightarrow B$ iff

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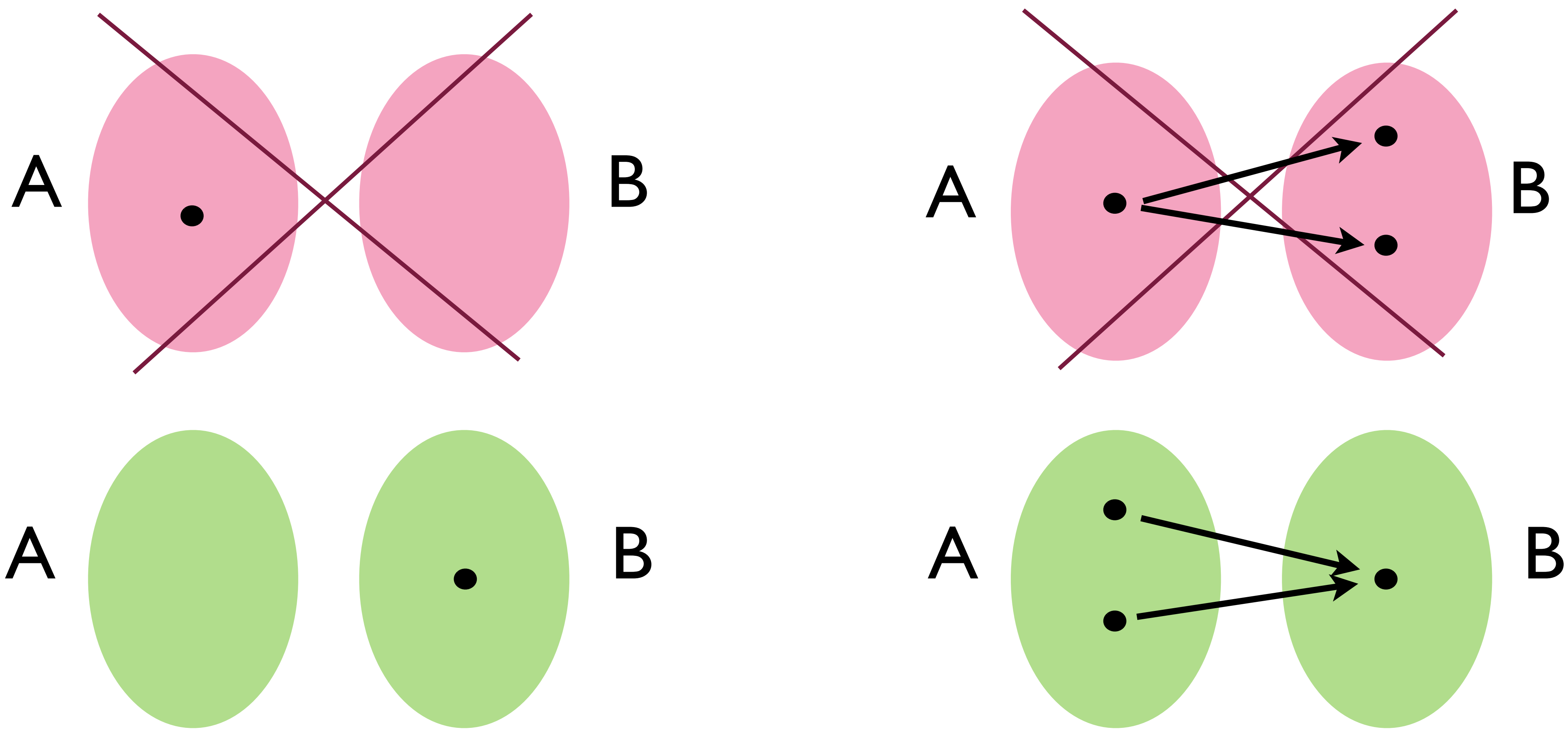

# Special functions

The number of ingoing arrows for a function can be 0,1, or more.
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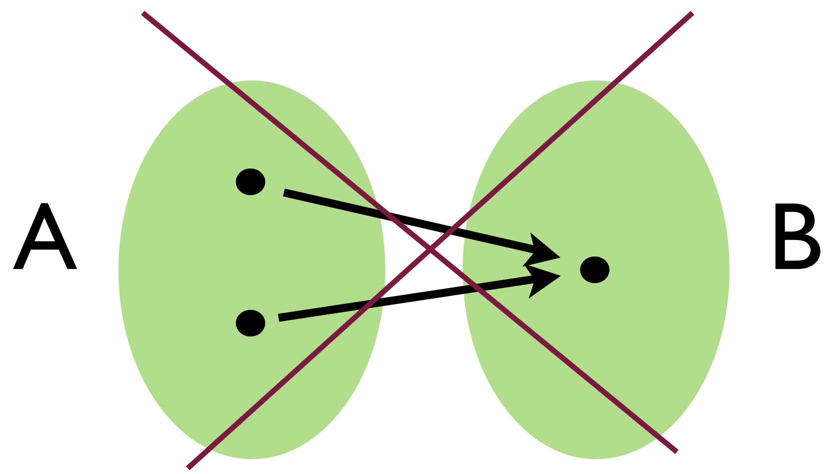

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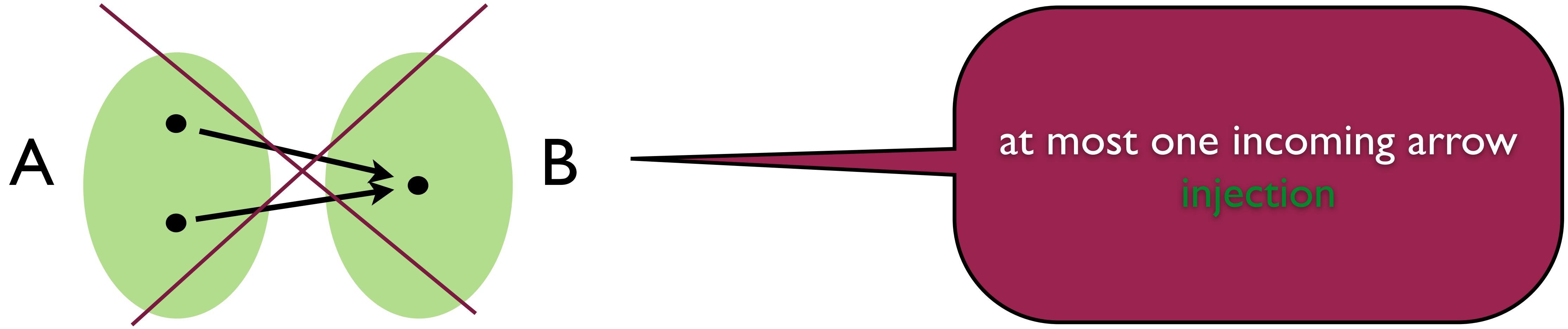

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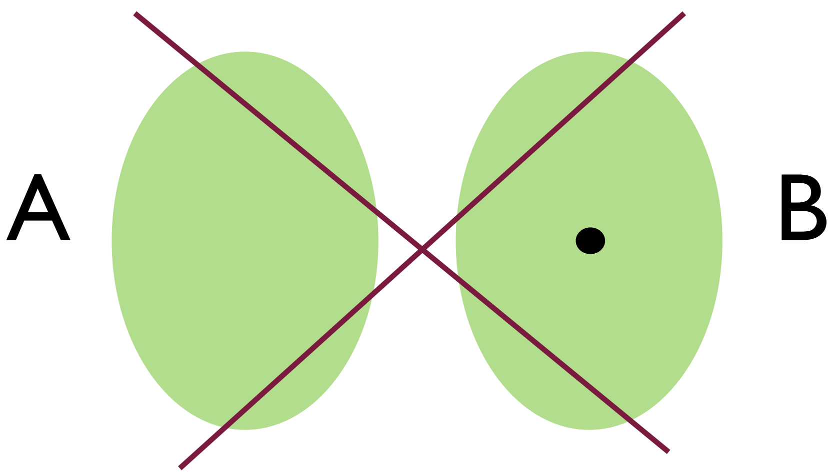

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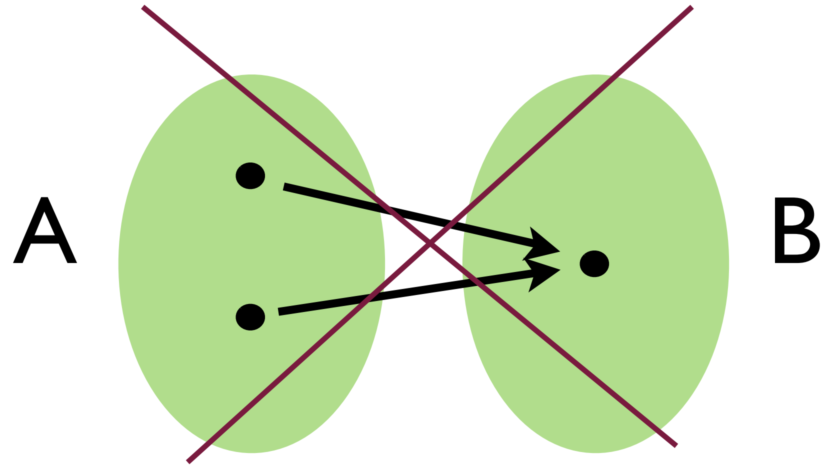
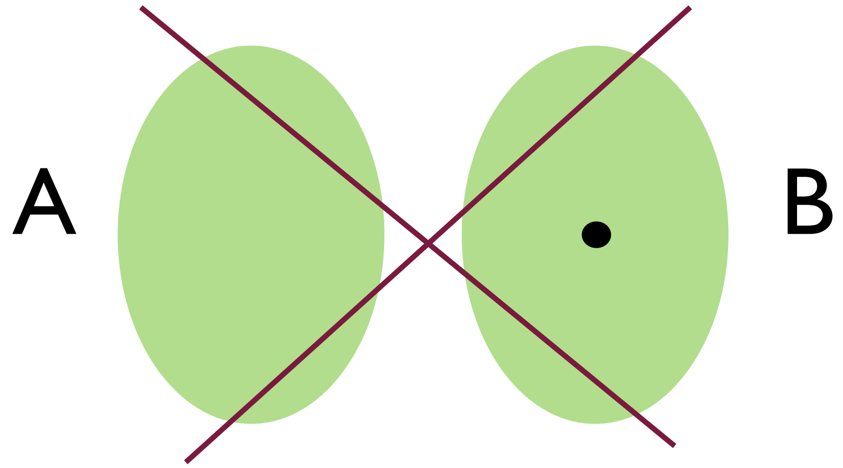



at most one incoming arrow
injection

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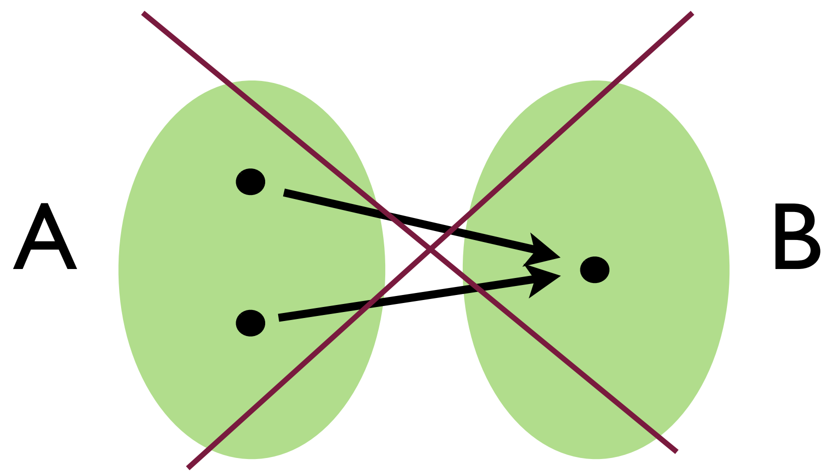


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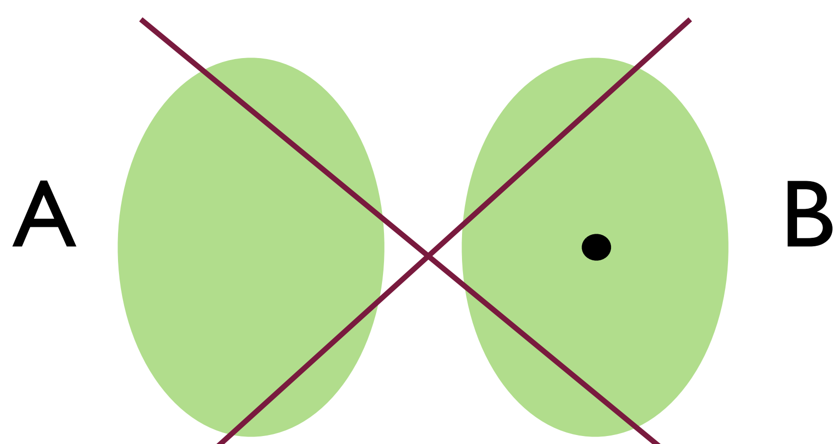


at least one incoming arrow
surjection

exactly one incoming arrow (injection + surjection) bijection

# Special functions

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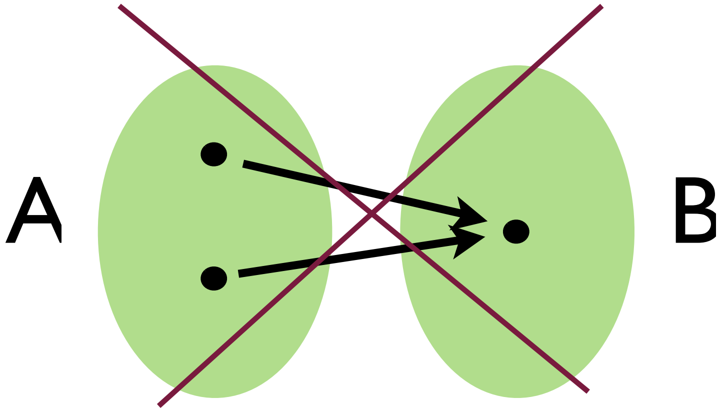
Def. A function $f: A \longrightarrow B$ is injective iff

for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



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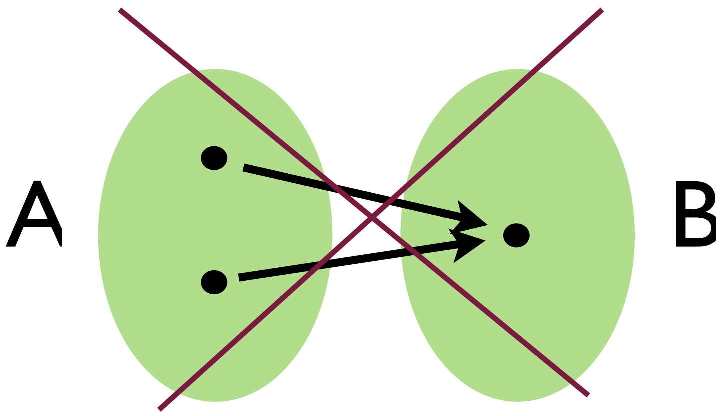


Def. A function $f: A \longrightarrow B$ is surjective iff

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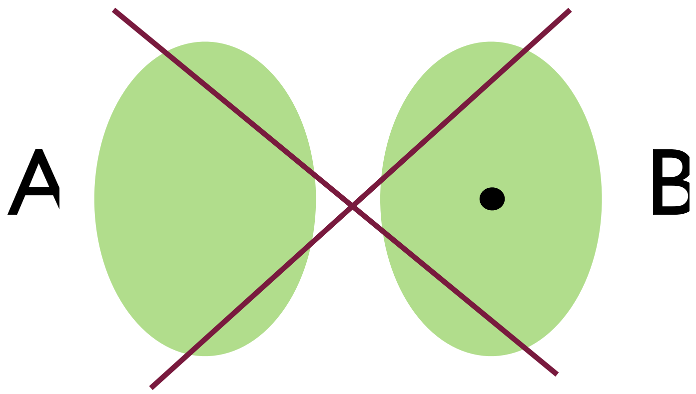

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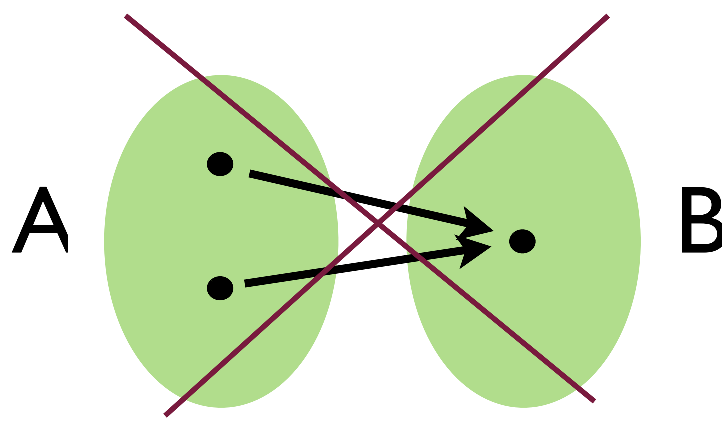


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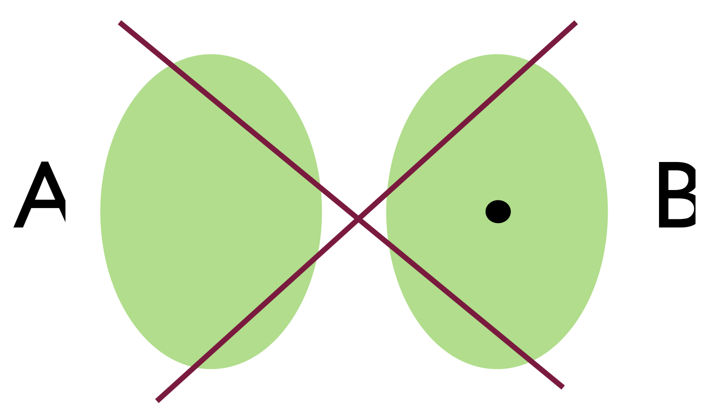


Def. A function $f: A \longrightarrow B$ is bijective iff

for all $b \in B$, there exists unique $a \in A$ with $f(a) = b$.

# Simple characterisations

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Lemma 11: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

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at most one incoming arrow

injection

# Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow

injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff

$f(A) = B$.

# Simple characterisations

Lemma I1: A function $f{:}A \longrightarrow B$ is injective iff

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at most one incoming arrow
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at least one incoming arrow
surjection

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at least one incoming arrow
surjection

Lemma B1: A function $f{:}A \longrightarrow B$ is bijective iff

$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
$f$ is both injective and surjective.

# Simple characterisations

Lemma I1: A function $f:A \longrightarrow B$ is injective iff

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at most one incoming arrow

injection

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at least one incoming arrow

surjection

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f is both injective and surjective.

exactly one incoming arrow

bijection

# Some properties

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Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

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if holds always!

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if holds always!

Prop. 13: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

# Some properties

Lemma I2: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then $f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. I3: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then $f^{-1}(f(A')) = A'$.

Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then $f(f^{-1}(B')) = B'$.

# Inverse function

Let $f: A \longrightarrow B$ be a bijection

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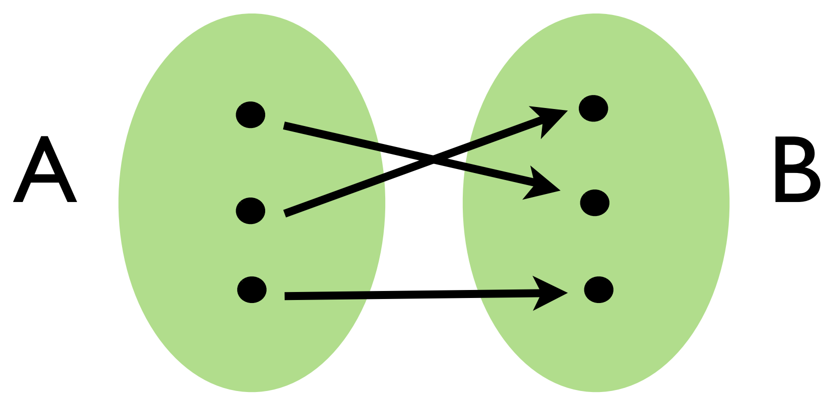
Let f:A $\longrightarrow$ B be a bijection



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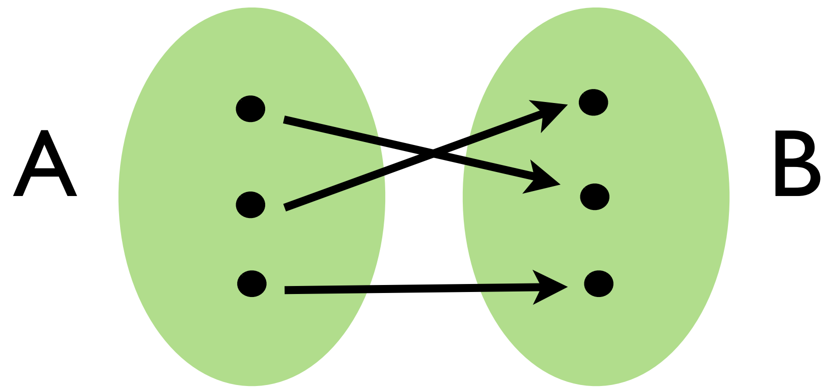


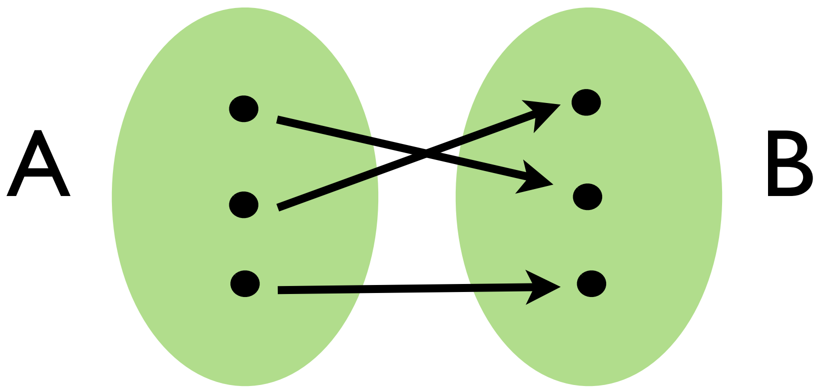

Def. The inverse function $f^{-1}$: B $\longrightarrow$ A is defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b, \quad b \in B.$$

# Inverse function

Let $f: A \longrightarrow B$ be a bijection

well defined only if f is bijective!

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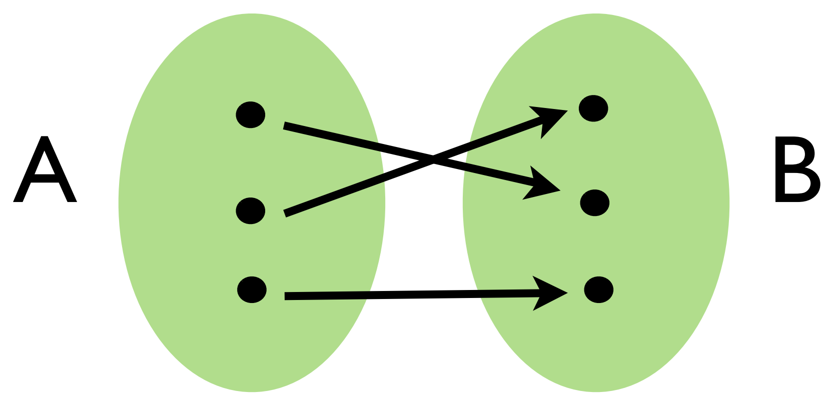


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$$f^{-1}(b) = a \;\text{ iff }\; f(a) = b,\quad b \in B.$$

Lemma B2: The inverse function $f^{-1}$ for a bijection f is bijective.

# Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

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Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

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Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

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$g \circ f : A \longrightarrow B \longrightarrow C$

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Lemma 14: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

# Function composition

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Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then

$g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then

$g \circ f$ is surjective.

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**Def.** The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

**Lemma I4:** Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

**Lemma S3:** Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then $g \circ f$ is surjective.

Corollary B2: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be bijective. Then so is $g \circ f$.

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