#### Properties of sets

- $I. \quad \varnothing \subseteq X$
- 2. If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$
- 3.  $X \cap Y \subseteq X, X \cap Y \subseteq Y$
- 4.  $X \subseteq X \cup Y, Y \subseteq X \cup Y$
- 5. If  $X_1 \subseteq Y_1$  and  $X_2 \subseteq Y_2$ , then  $X_1 \cap X_2 \subseteq Y_1 \cap Y_2$
- 6. If  $X_1 \subseteq Y_1$  and  $X_2 \subseteq Y_2$ , then  $X_1 \cup X_2 \subseteq Y_1 \cup Y_2$
- 7.  $X \cap Y = X$  iff  $X \subseteq Y$
- 8.  $X \cap X = X$  (idempotence)
- 9.  $X \cup X = X$  (idempotence)

 $10. X \cap \emptyset = \emptyset$ 

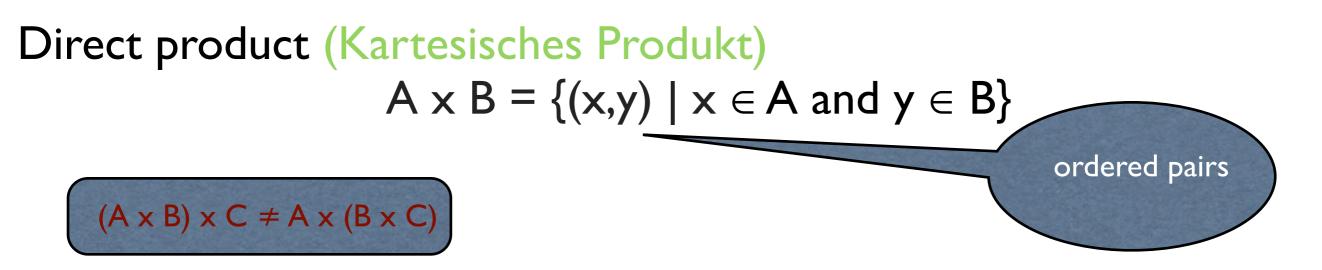
### Properties of sets

11. 
$$X \cup \emptyset = X$$
12.  $X \cap Y = Y \cap X$  (commutativity)13.  $X \cup Y = Y \cup X$  (commutativity)14.  $X \cap (Y \cap Z) = (X \cap Y) \cap Z$  (associativity)15.  $X \cup (Y \cup Z) = (X \cup Y) \cup Z$  (associativity)16.  $X \cap (X \cup Y) = X$  (absorption)17.  $X \cup (X \cap Y) = X$  (absorption)18.  $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$  (distributivity)19.  $X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$  (distributivity)20.  $X \setminus Y \subseteq X$ 

## Properties of sets

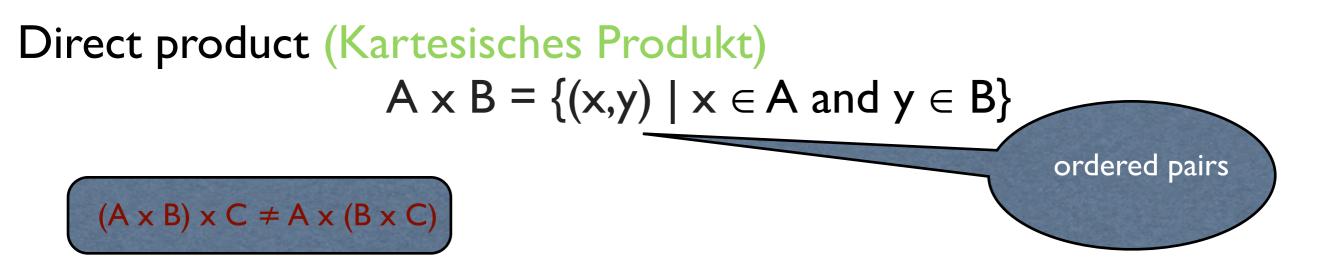
21.	$(X \setminus Y) \cap Y = \emptyset$
	$X \cup Y = X \cup (Y \setminus X)$
23.	$X \setminus X = \emptyset$
24.	$X \setminus \emptyset = X$
25.	$\emptyset \setminus X = \emptyset$
26.	If $X \subseteq Y$ , then $X \setminus Y = \emptyset$
27.	$(X^c)^c = X$
28.	$(X \cap Y)^c = X^c \cup Y^c$ (De Morgan)
29.	$(X \cup Y)^c = X^c \cap Y^c$ (De Morgan)
30.	$X \times \emptyset = \emptyset$
31.	$\varnothing \mathbf{X} \mathbf{X} = \varnothing$
32.	If $X \subseteq Y$ , then $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$

#### Direct product (Kartesisches Produkt) $A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$ ordered pairs (A × B) × C ≠ A × (B × C)



Therefore, we define

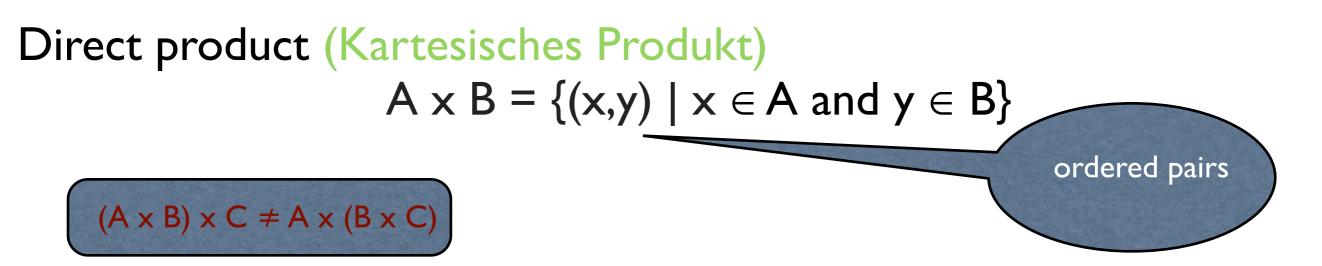
 $A \times B \times C = \{(x,y,z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$ 



Therefore, we define  $A \times B \times C = \{(x,y,z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$ 

In general, for sets  $A_1$ ,  $A_2$ , ...,  $A_n$  with  $n \ge I$ ,

 $A_{I} \times A_{2} \times ... \times A_{n} = \prod_{1 \leq i \leq n} A_{i} = \{(x_{1}, x_{2}, ..., x_{n}) \mid x_{i} \in A_{i} \text{ for } I \leq i \leq n\}$ 

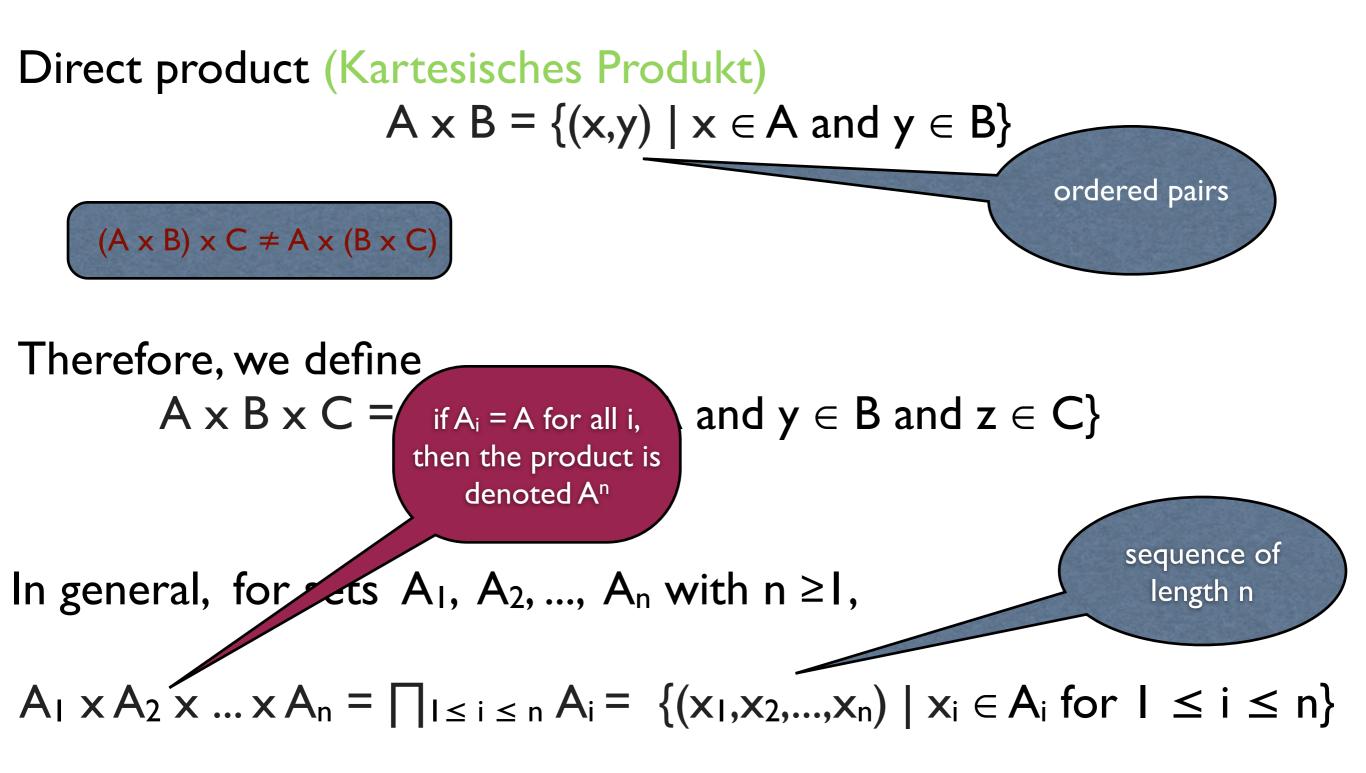


Therefore, we define  $A \times B \times C = \{(x,y,z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$ 

In general, for sets  $A_1$ ,  $A_2$ , ...,  $A_n$  with  $n \ge 1$ ,

sequence of length n

 $A_{I} \times A_{2} \times ... \times A_{n} = \prod_{1 \leq i \leq n} A_{i} = \{(x_{1}, x_{2}, ..., x_{n}) \mid x_{i} \in A_{i} \text{ for } I \leq i \leq n\}$ 



#### Finite sequences, words

#### Finite sequences, words

Let A be a set, an aphabet

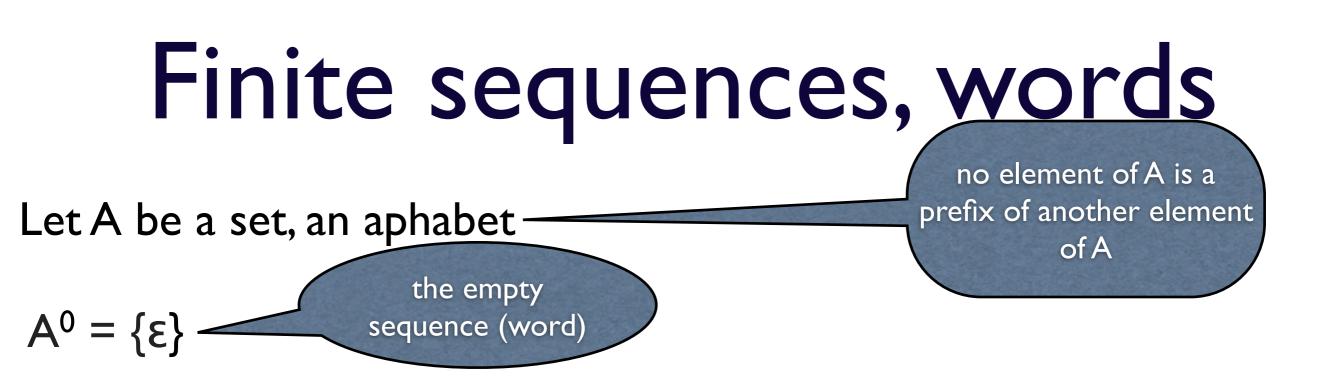
 $A^0 = \{\epsilon\}$ 

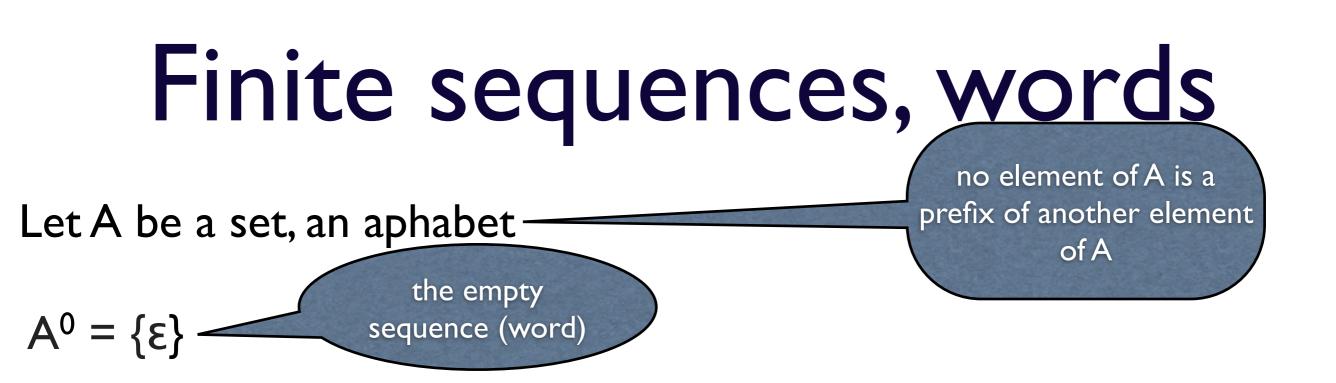


Let A be a set, an aphabet

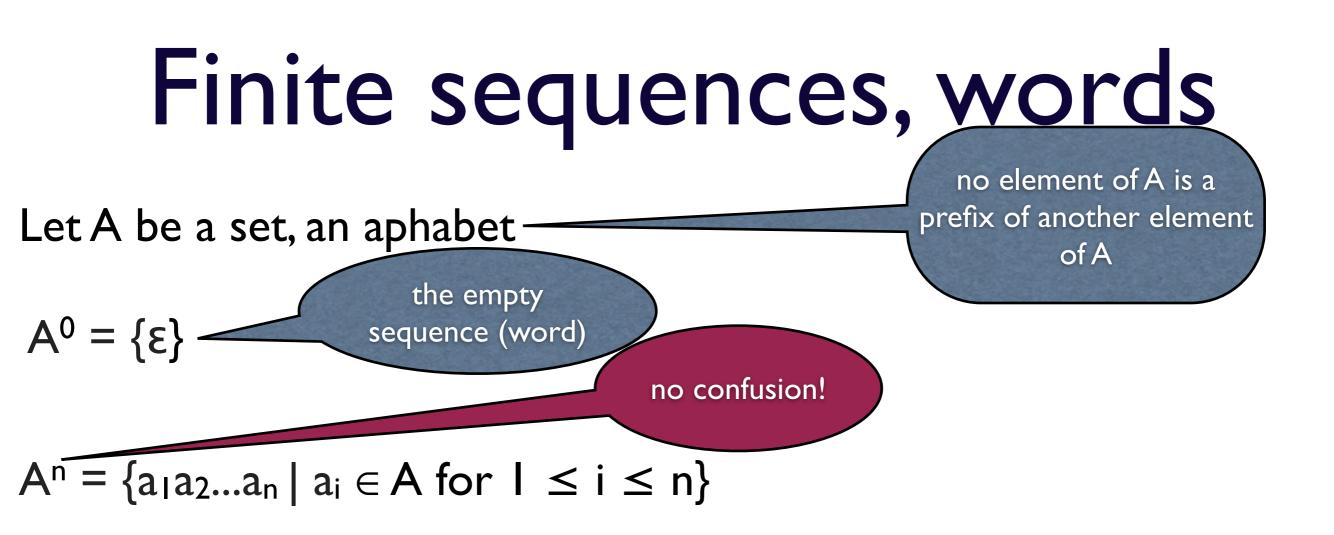
no element of A is a prefix of another element of A

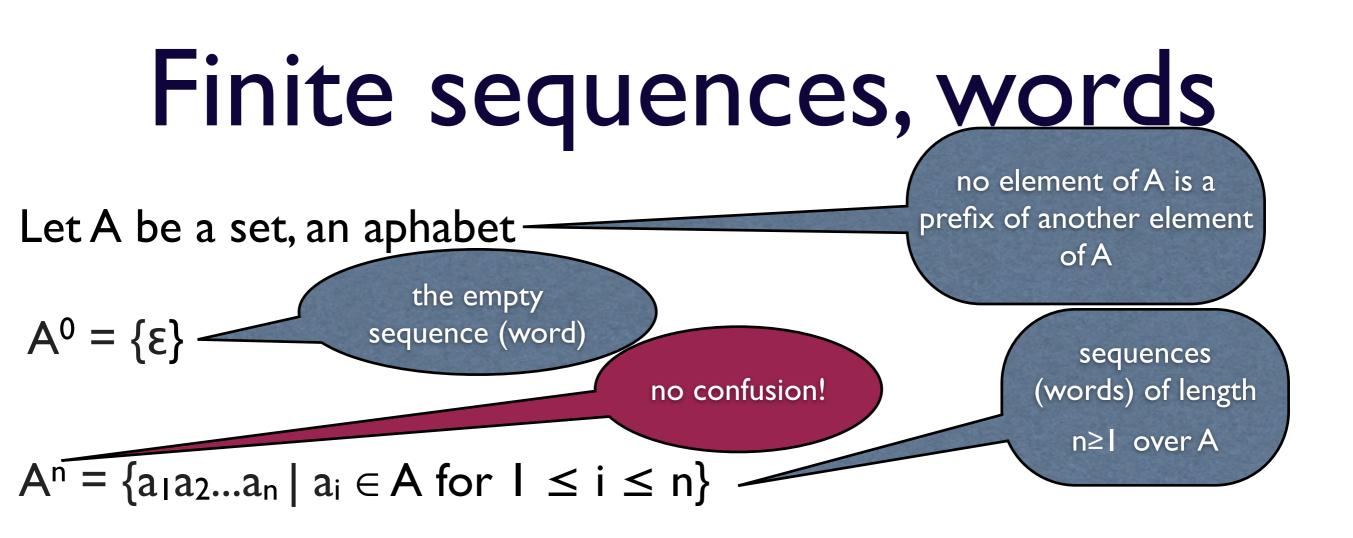
 $A^0 = \{\epsilon\}$ 

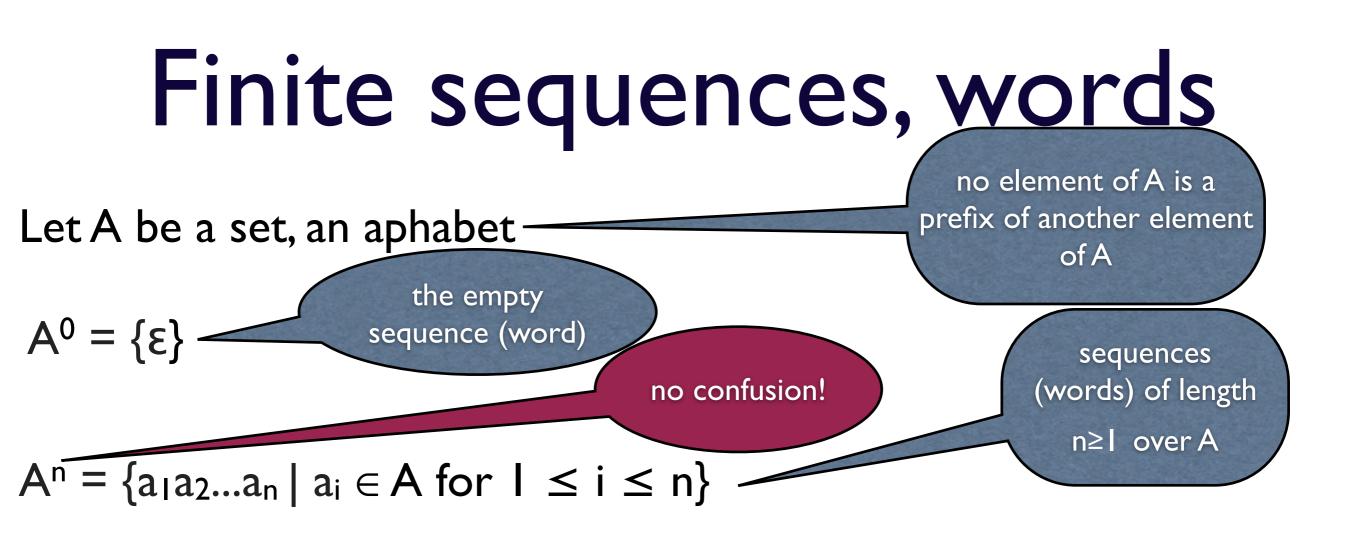




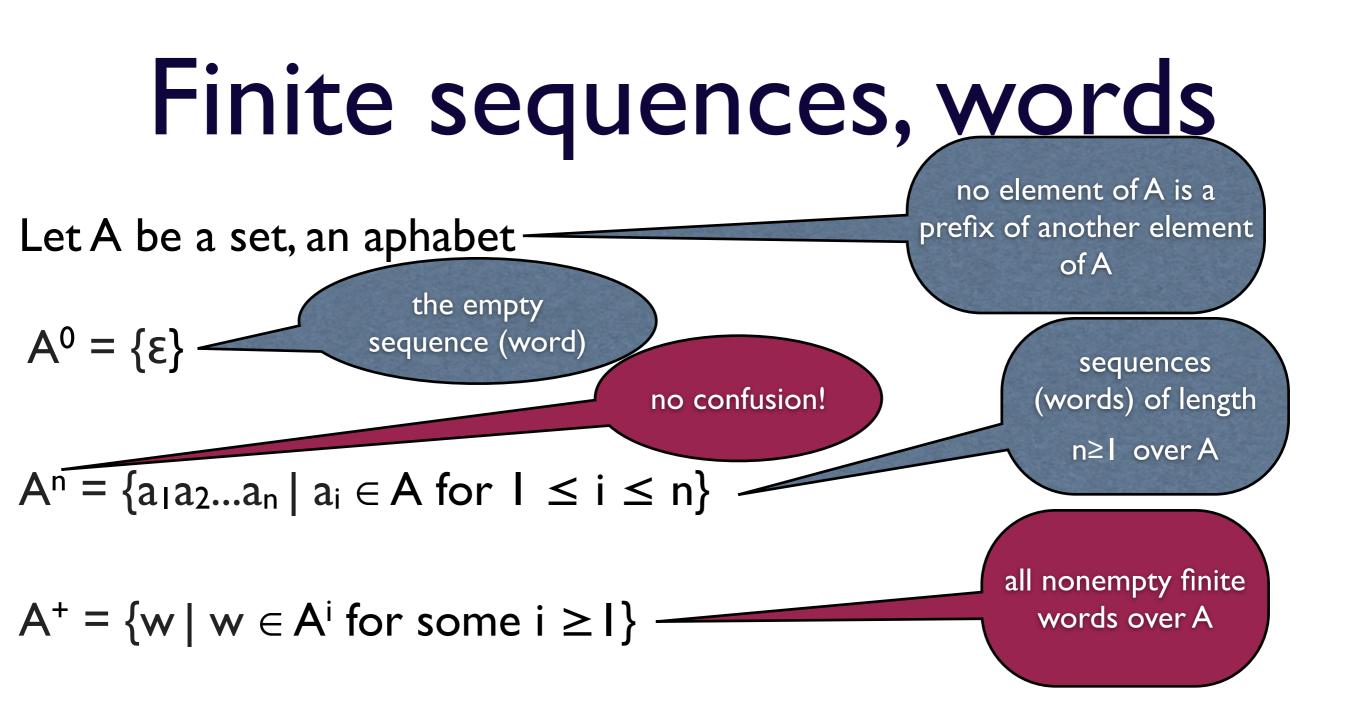
#### $A^n = \{a_1a_2...a_n \mid a_i \in A \text{ for } I \leq i \leq n\}$

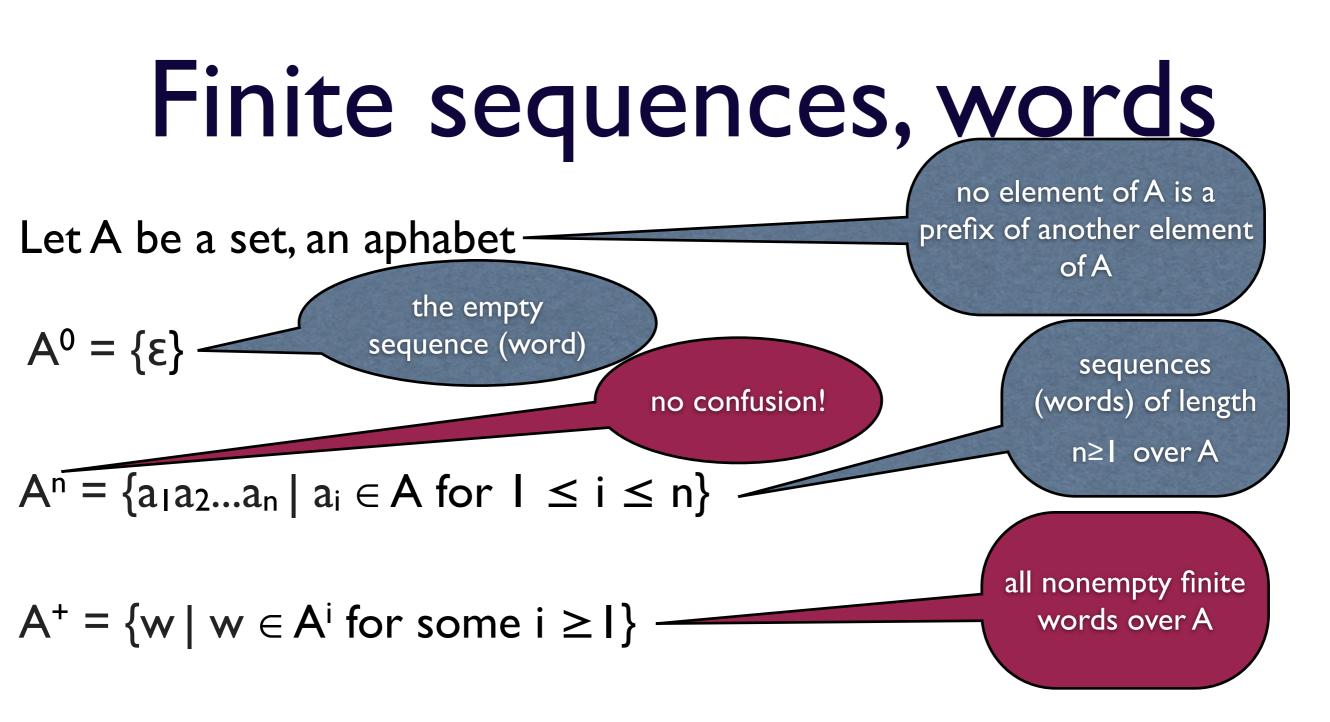




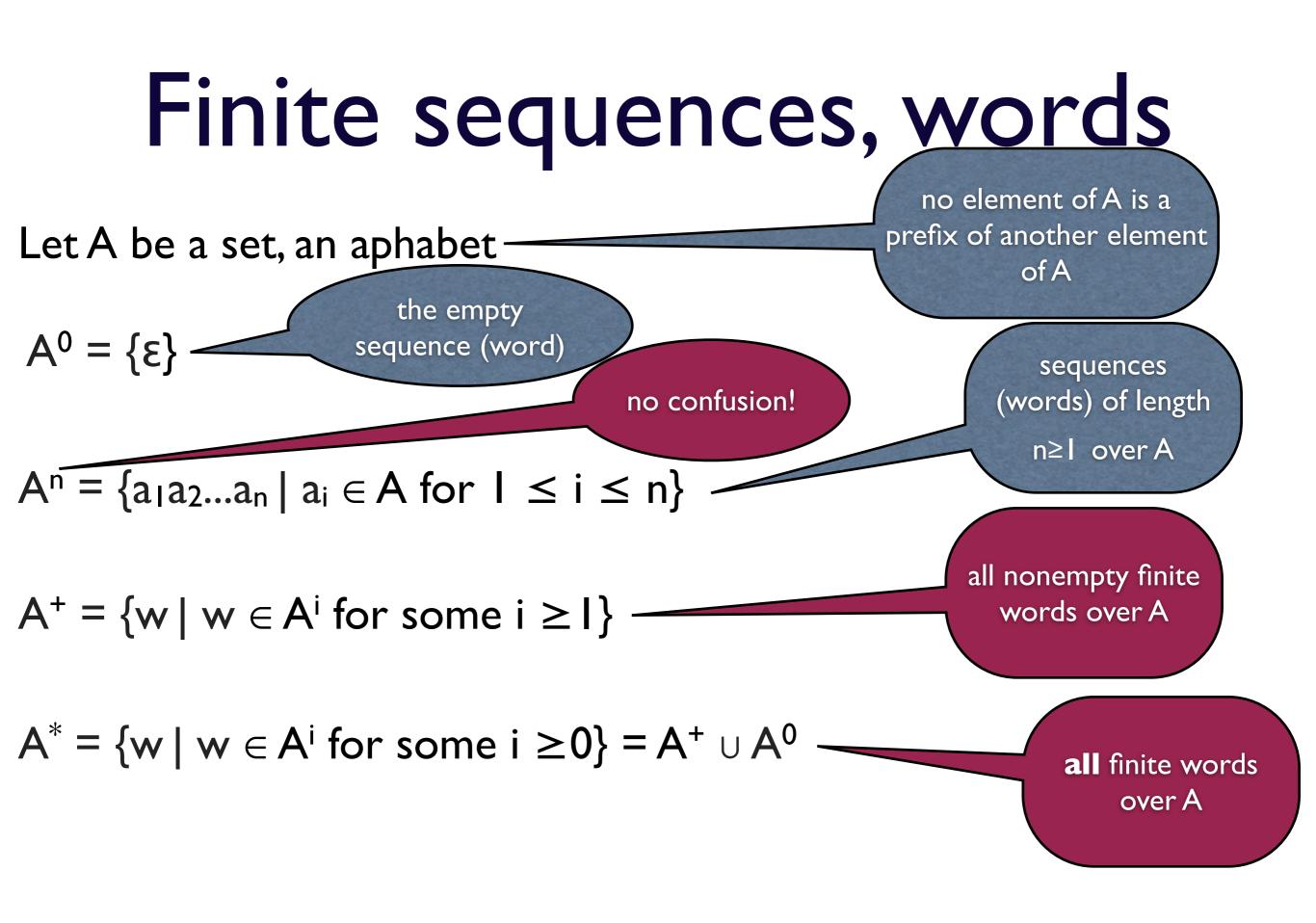


$$A^+ = \{w \mid w \in A^i \text{ for some } i \ge I\}$$





 $A^* = \{w \mid w \in A^i \text{ for some } i \ge 0\} = A^+ \cup A^0$ 



#### Relations

**Def.** If A and B are sets, then any subset  $R \subseteq A \times B$  is a (binary) relation between A and B

**Def.** R is a relation on A if  $R \subseteq A \times A$ 

some relations are special

#### Relations

**Def.** If A and B are sets, then any subset  $R \subseteq A \times B$  is a (binary) relation between A and B

similarly, unary relation (subset), n-ary relation...

**Def.** R is a relation on A if  $R \subseteq A \times A^{\vee}$ 

some relations are special

#### Special relations

#### A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$ , (a,a) $\in R$
symmetric	iff	for all $a, b \in A$ , if $(a, b) \in R$ , then $(b, a) \in R$
transitive	iff	for all a,b,c $\in$ A, if (a,b) $\in$ R and (b,c) $\in$ R,
		then $(a,c) \in R$
irreflexive	iff	for all $a \in A$ , (a,a) $\not\in R$
antisymmetric	iff	for all $a, b \in A$ , if $(a, b) \in R$ and $(b, a) \in R$
		then a = b
asymmetric	iff	for all a,b $\in$ A, if (a,b) $\in$ R, then (b,a) $\notin$ R
total	iff	for all $a, b \in A$ , $(a, b) \in R$ or $(b, a) \in R$

#### Special relations

#### A relation $R \subseteq A \times A$ is:

iff	for all $a \in A$ , (a,a) $\in R$
iff	for all $a, b \in A$ , if $(a, b) \in R$ , then $(b, a) \in R$
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	then $(a,c) \in R$
iff	for all $a \in A$ , (a,a) $\not\in R$
iff	for all $a, b \in A$ , if $(a, b) \in R$ and $(b, a) \in R$
	then a = b
iff	for all $a, b \in A$ , if $(a, b) \in R$ , then $(b, a) \not\in R$
iff	for all $a, b \in A$ , $(a, b) \in R$ or $(b, a) \in R$
	ff ff ff

(infix) notation aRb for  $(a,b) \in R$ 

#### Special relations

A relation R on A, i.e.,  $R \subseteq A \times A$  is:

- equivalence iff R is reflexive, symmetric, and transitive
- partial order iff R is reflexive, antisymmetric, and transitive
- strict order iff R is irreflexive and transitive
- preorder iff R is reflexive and transitive

total (linear) order

iff R is a total partial order

## Obvious properties

- I. Every partial order is a preorder.
- 2. Every total order is a partial order.
- 3. Every total order is a preorder.
- 4. If  $R \subseteq A \times A$  is a relation that contains cycles, i.e. there are  $a, b \in A$  such that a = b,  $(a,b) \in R$  and  $(b,a) \in R$ , then R is not a preorder, nor a partial order, nor a total order.

Let  $R \subseteq A \ge B$  and  $S \subseteq B \ge C$  be two relations. Their composition is the relation

 $R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$ 

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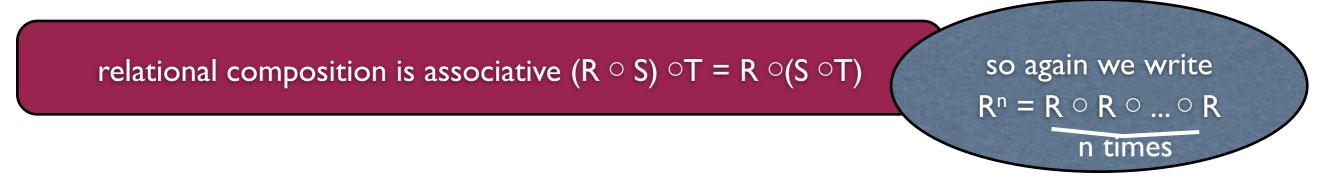
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relational composition is associative  $(R \circ S) \circ T = R \circ (S \circ T)$ 

so again we write  $R^n = R \circ R \circ ... \circ R$ n times

Let  $R \subseteq A \ge B$  be a relation. The inverse relation of R is the relation

$$\mathsf{R}^{\mathsf{-I}} = \{(\mathsf{b},\mathsf{a}) \in \mathsf{B} \times \mathsf{A} \mid (\mathsf{a},\mathsf{b}) \in \mathsf{R}\}$$

#### Characterizations

Lemma: Let R be a relation over the set A. Then

- I. R is reflexive iff  $\Delta_A \subseteq R$
- 2. R is symmetric iff  $R \subseteq R^{-1}$
- 3. R is transitive iff  $R^2 \subseteq R$

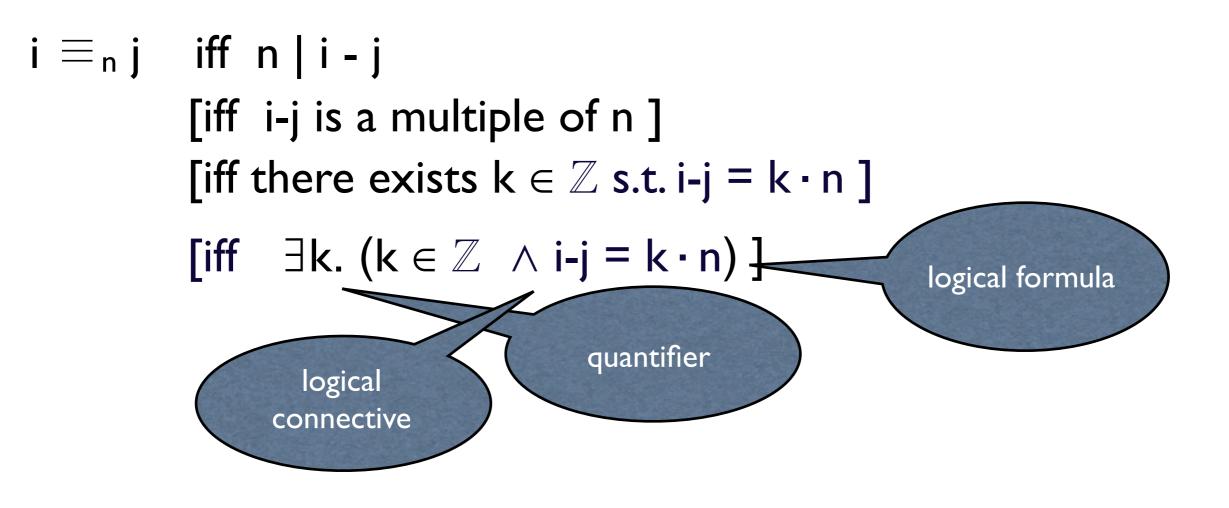
**Def.** For a natural number n, the relation  $\equiv_n$  is defined as

 $i \equiv_n j$  iff  $n \mid i - j$ 

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```
\begin{split} i &\equiv_n j \quad \text{iff } n \mid i - j \\ & [\text{iff } i\text{-}j \text{ is a multiple of } n ] \\ & [\text{iff there exists } k \in \mathbb{Z} \text{ s.t. } i\text{-}j = k \cdot n ] \\ & [\text{iff } \exists k. \ (k \in \mathbb{Z} \ \land i\text{-}j = k \cdot n) ] \end{split}
```

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