Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

 $\forall x [x \in \mathbb{Z} \land x \geq 2 : x^2 - 2x \geq 0]$

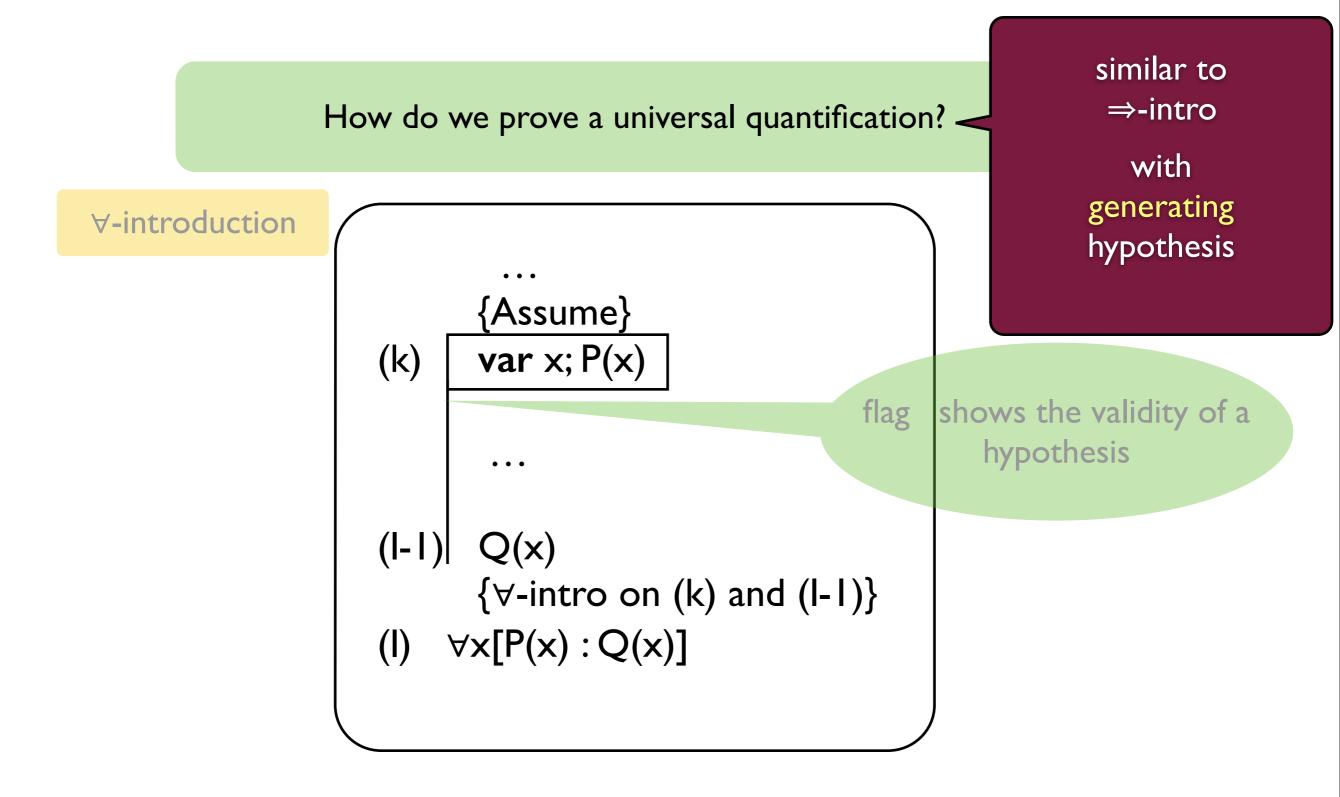
Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \ge 2$.

Then, for this particular x, it holds that $x^2 - 2x = x(x-2) \ge 0$ (Why?)

Conclusion: $\forall x [x \in \mathbb{Z} \land x \ge 2 : x^2 - 2x \ge 0].$

∀ introduction



Using a universal quantification

We know

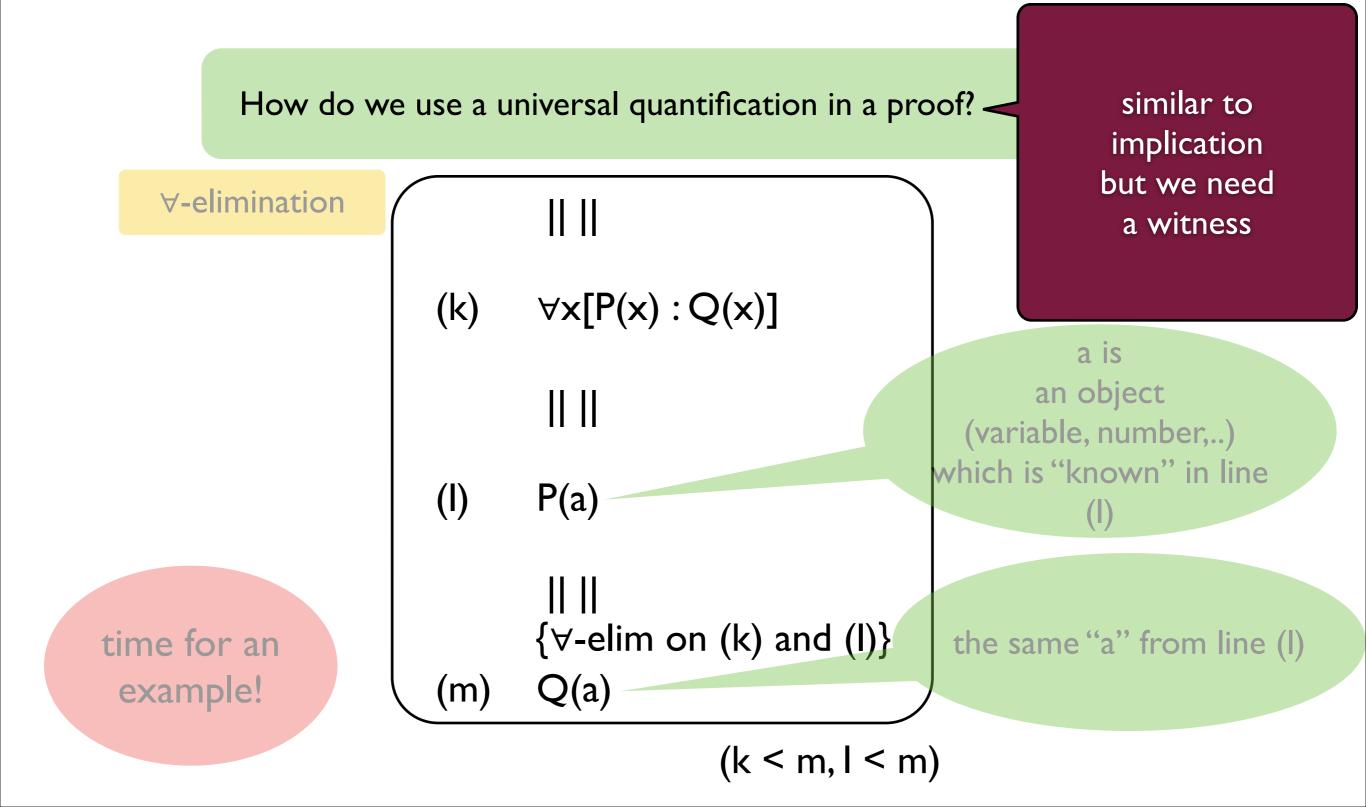
 $\forall x [x \in \mathbb{Z} \land x \geq 2 : x^2 - 2x \geq 0]$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \ge 2$,

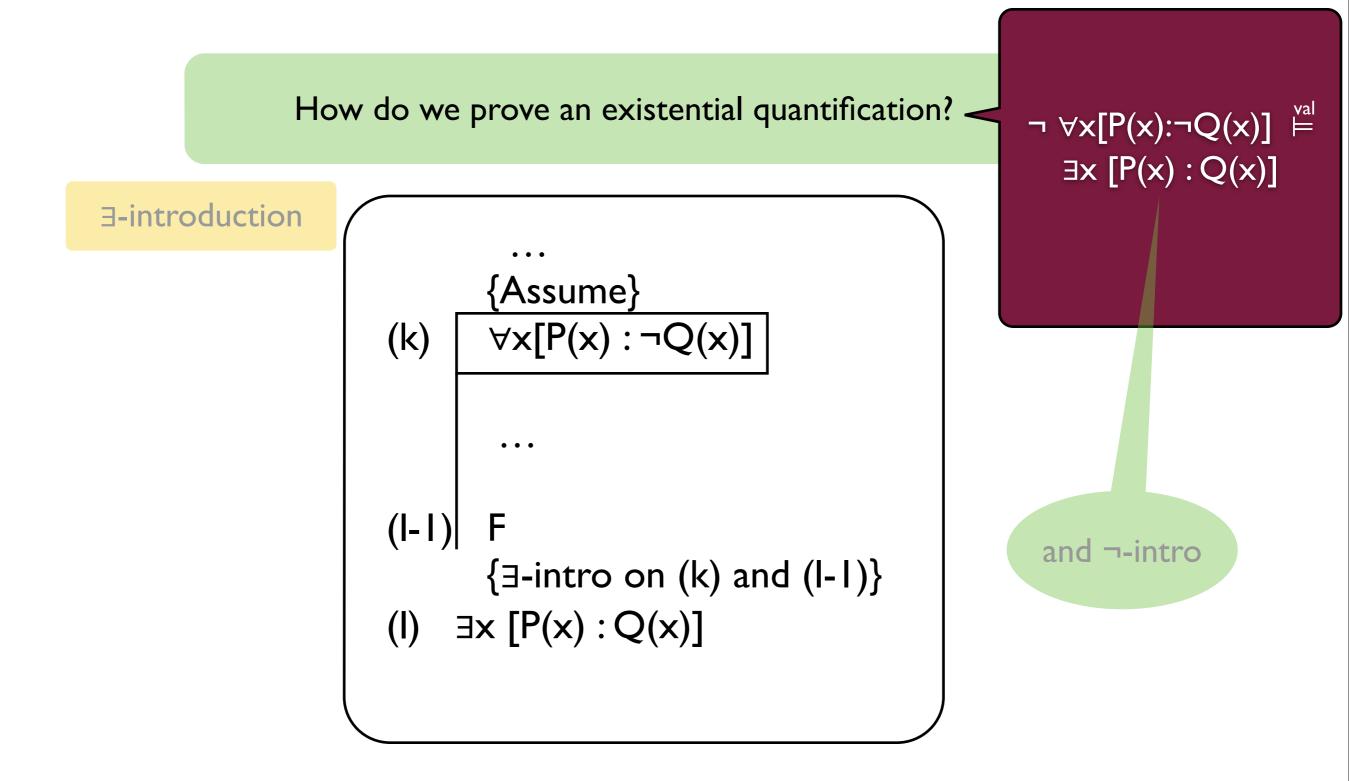
we can conclude that $a^2 - 2a \ge 0$.

For example, $(52387^2 - 2 \cdot 52387) \ge 0$ since 52387 $\in \mathbb{Z}$ and 52387 ≥ 2 .

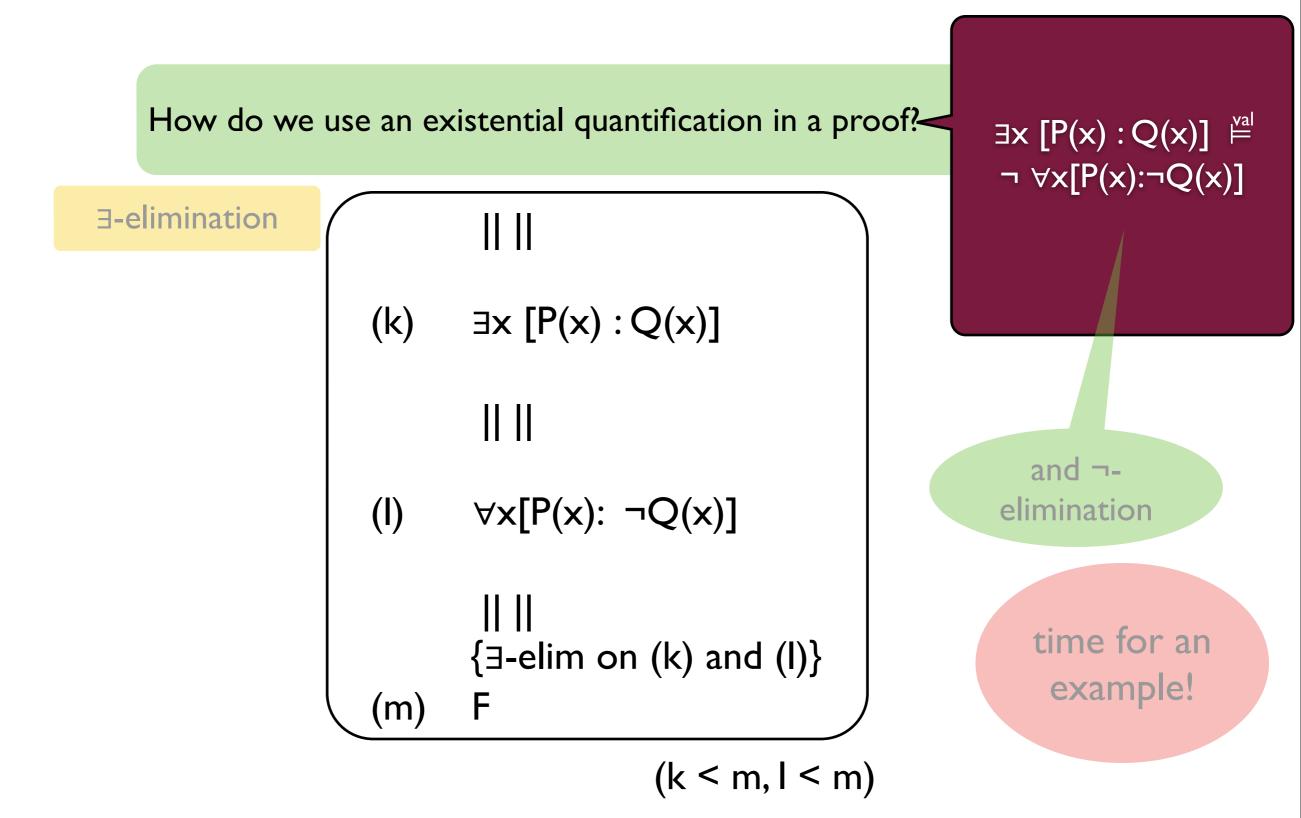
∀ elimination



∃ introduction



elimination



Proofs with 3-introduction and 3elimination are unnecessarily long and cumbersome...

There are alternatives!

Proving an existential quantification

To prove

∃x[x ∈ ℤ : x³ - 2x - 8 ≥0]

Proof

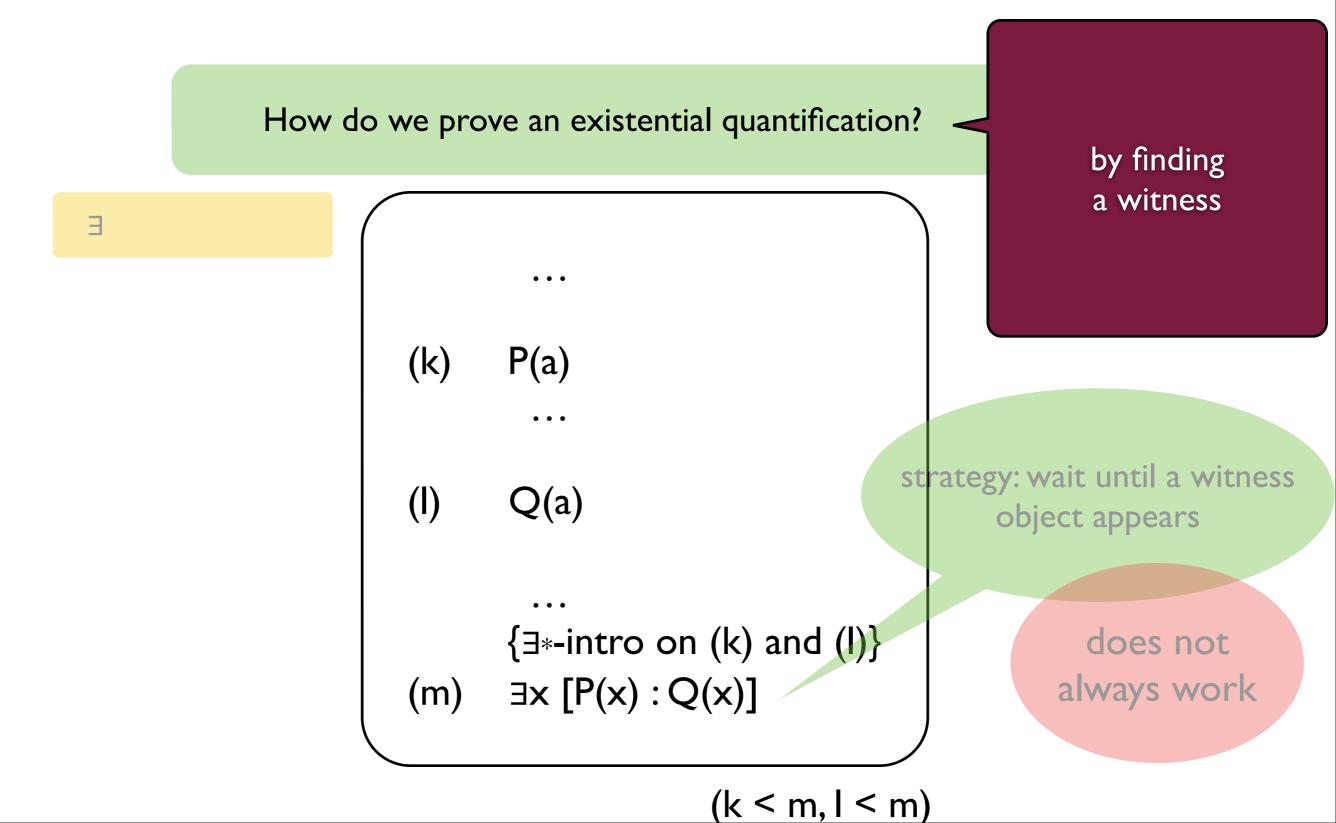
It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying $x^3 - 2x - 8 \ge 0$.

x = 3 is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \ge 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \ge 0].$

also x = 5 is a witness...

Alternative 3 introduction



Using an existential quantification

We know

$$\exists x [x \in \mathbb{R} : a - x < 0 < b - x]$$

We can declare an $x \in \mathbb{Z}$ (a witness) such that a - x < 0 < b - xand use it further in the proof. For example: From a - x < 0, we get a < x. From b - x > 0, we get x < b. Hence, a < b.

Alternative 3 elimination

