Introduction to the Theory of Computation

Automata

1VO + 1PS

Lecturer: Dr. Ana Sokolova

http://cs.uni-salzburg.at/~anas/

Setup and Dates

- Lectures and Instructions

23.10.	3.11.	17.11.	24.11.	1.12.	11.12.	18.12.

- Books

Introduction to the Theory of Computation by M. Sipser

Introduction to Automata Theory, Languages, and Computation by J. E. Hopcroft, R. Motwani, and J.D. Ullman

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	automata	grammars	push-down automata	Turing machines		

- Boo

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automata

grammars

push-down automata

Turing machines

computability

- Bool

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The Rules... Instructions

- Instruction exercises on the web
  http://cs.uni-salzburg.at/~anas/Ana_Sokolova/Automata2015.html
  after class
- To be solved by you, the students, (in groups of at most 3 people) and handed in as homework at the next meeting.
- In class I will present a sample solution and you, the students, will be asked to present solutions/discuss the exercises

The Rules... Instructions

- One randomly chosen exercise will be graded each week
- The graded exercise will be returned at the next meeting.
- Grade based on
  (1) exam
  (2) the grades of the corrected exercise and
  (3) activity in class (ability to present solutions)
- All information about the course / rules / exams / grading is / will be on the course webpage

The Rules... Grading

- Written exam on December 18, 10 am - 12:30 pm
- Grade based on the number of points on the written exam (80%), homework grades, and activity in class (20%)
- For better grade oral exam after the written one upon appointment
- 55% of the maximal points are needed to pass.

Finite Automata

Alphabets and Languages

Alphabet and words

Alphabets and Languages

Alphabet and words

Σ - alphabet (finite set)

$\Sigma^n = \{a_1 a_2 .. a_n \mid a_i \in \Sigma\}$ is the set of words of length n

$\Sigma^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\ a_1, a_2, .. , a_n \in \Sigma.\ w = a_1 a_2 .. a_n\}$ is the set of all words over Σ

Alphabets and Languages

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$\Sigma^0 = \{\varepsilon\}$ contains only the empty word

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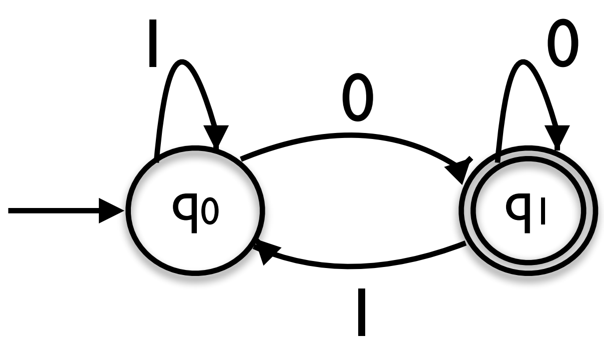
A language L over Σ is a subset $L \subseteq \Sigma^*$

Deterministic Automata (DFA)

Informal example

$\Sigma = \{0,1\}$

M_1:



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alphabet

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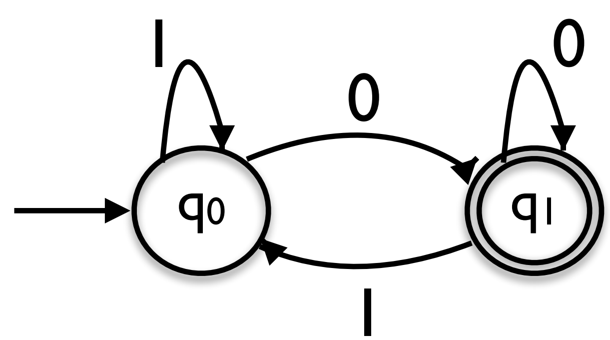

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q0, q1 are states

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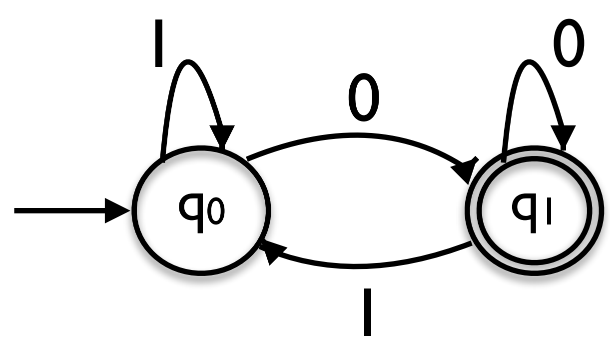

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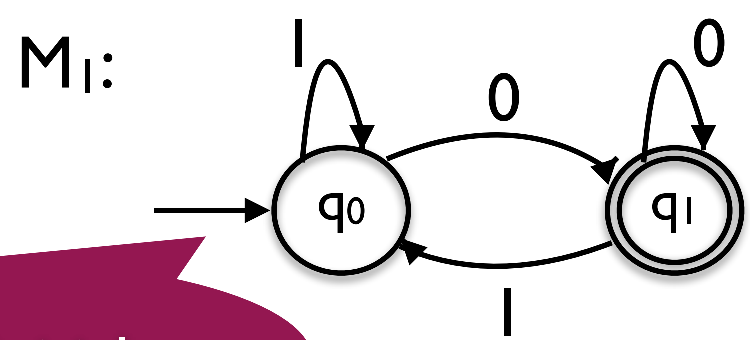

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q_0 is initial

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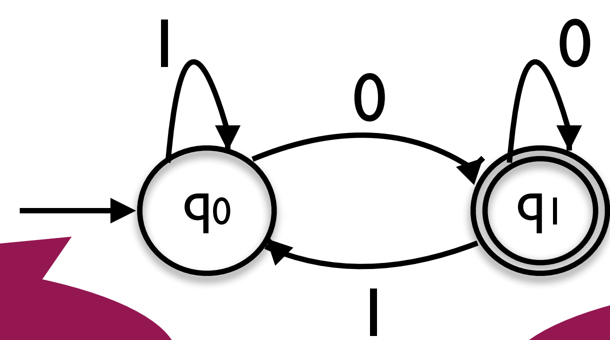

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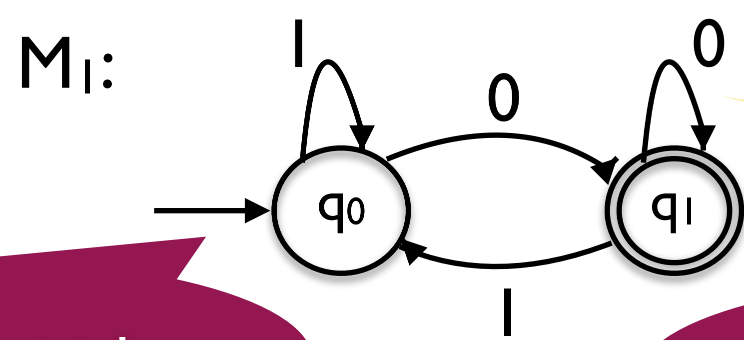

q_0, q_1 are states

transitions, labelled by alphabet symbols

q_0 is initial

q_1 is final

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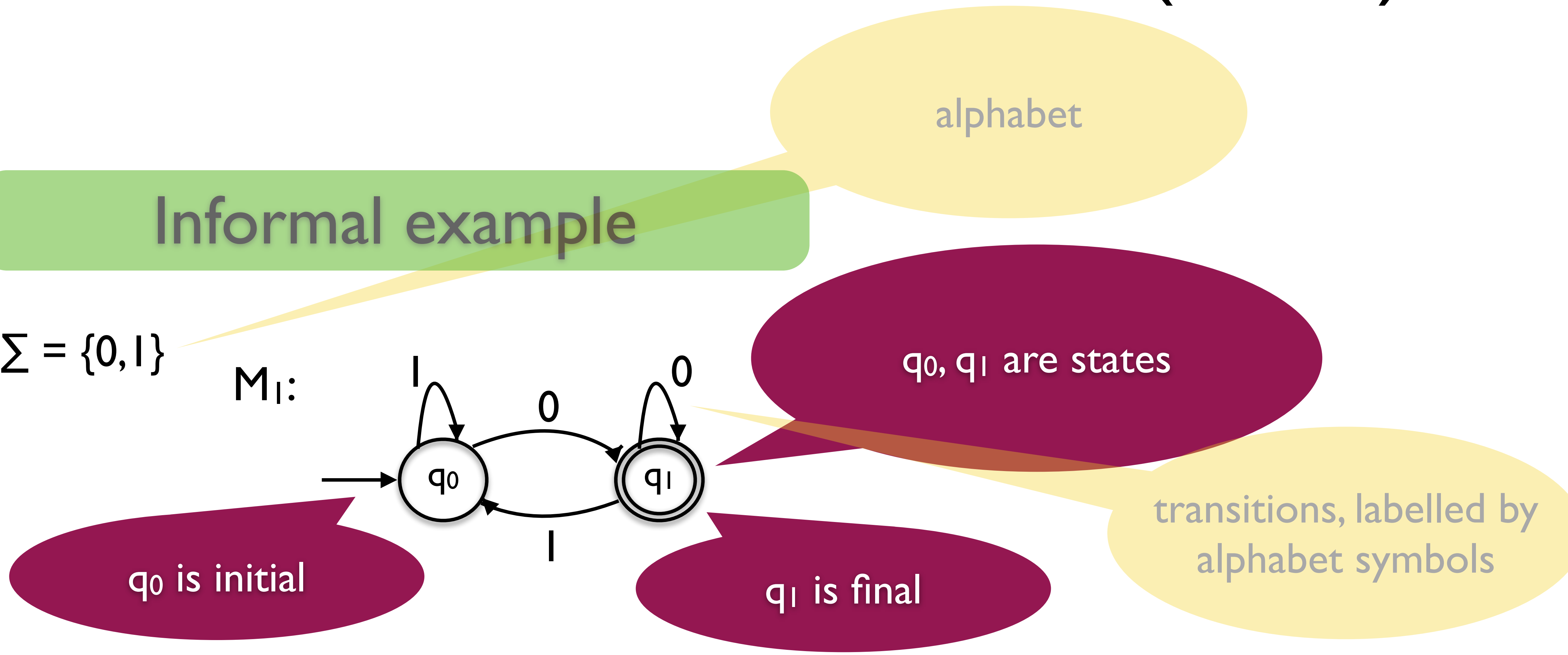

Accepts the language $L(M_1) = \{w \in \Sigma^* \mid w \text{ ends with a } 0\} = \Sigma^* 0$

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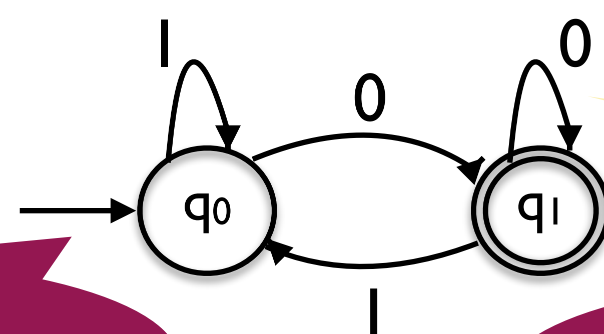

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regular language

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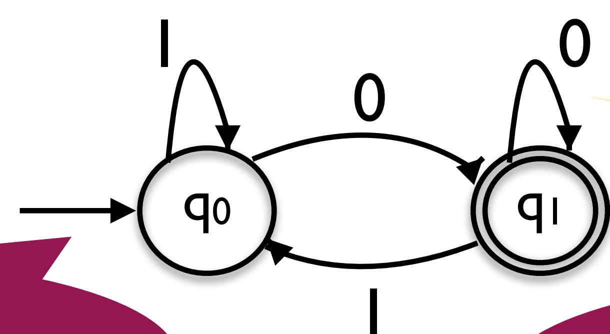

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regular language

regular expression

DFA

Definition

A deterministic finite automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
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- $\delta: Q \times \Sigma \longrightarrow Q$ is the transition function
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$\Sigma = \{0, 1\}$

$\delta(q_0, 0) = q_1$, $\delta(q_0, 1) = q_0$
$\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_0$

DFA

The extended transition function

DFA

The extended transition function

Given $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma \longrightarrow Q$ to

$\delta^*: Q \times \Sigma^* \longrightarrow Q$

inductively, by:

$\delta^*(q, \varepsilon) = q$ and $\delta^*(q, wa) = \delta(\delta^*(q, w), a)$

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The language recognised / accepted by a deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}$$

DFA

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Definition

$L(M_1) = \{w0 | w \in \{0,1\}^*\}$

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Regular languages and operations

Definition

Let Σ be an alphabet. A language L over Σ ($L \subseteq \Sigma^*$) is regular iff it is recognised by a DFA.

Regular languages and operations

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Regular operations

Let L, L_1, L_2 be languages over Σ. Then
$L_1 \cup L_2$, $L_1 \cdot L_2$, and L^* are languages, where

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}$$

$$L^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\, w_1, w_2, .., w_n \in L.\ w = w_1 w_2 .. w_n\}$$

Regular languages and operations

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$\varepsilon \in L^*$ always

Regular expressions

Definition

finite representation of infinite languages

Regular expressions

Definition

finite representation of infinite languages

Regular expressions

Definition

Let Σ be an alphabet. The following are regular expressions

1. a for $a \in \Sigma$
2. ε
3. $\emptyset$
4. $(R_1 \cup R_2)$ for R_1, R_2 regular expressions
5. $(R_1 \cdot R_2)$ for R_1, R_2 regular expressions
6. $(R_1)^*$ for R_1 regular expression

finite representation of infinite languages

Regular expressions

inductive

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finite representation of infinite languages

Regular expressions

inductive

example: $(ab \cup a)^*$

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finite representation of infinite languages

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corresponding languages

$$L(a) = \{a\}$$
$$L(\varepsilon) = \{\varepsilon\}$$
$$L(\varnothing) = \varnothing$$
$$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$$
$$L(R_1 \cdot R_2) = L(R_1) \cdot L(R_2)$$
$$L(R_1^*) = L(R_1)^*$$

Equivalence of regular expressions and regular languages

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Theorem (Kleene)

A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

Equivalence of regular expressions and regular languages

Theorem (Kleene)

A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

Proof $\Leftarrow$ easy, via the closure properties discussed next,
$\Rightarrow$ not so easy, we'll skip it for now…

Closure under regular operations

Closure under regular operations

Theorem C1

The class of regular languages is closed under union

Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

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Theorem C1

The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Closure under regular operations

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Theorem C1

The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

Theorem C2

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Theorem C3

The class of regular languages is closed under concatenation

Theorem C4

The class of regular languages is closed under Kleene star

Closure under regular operations

also under intersection

We can already prove these!

Theorem C1

The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

Theorem C4

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Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

We can already prove these!

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

But not yet these two…

Theorem C4

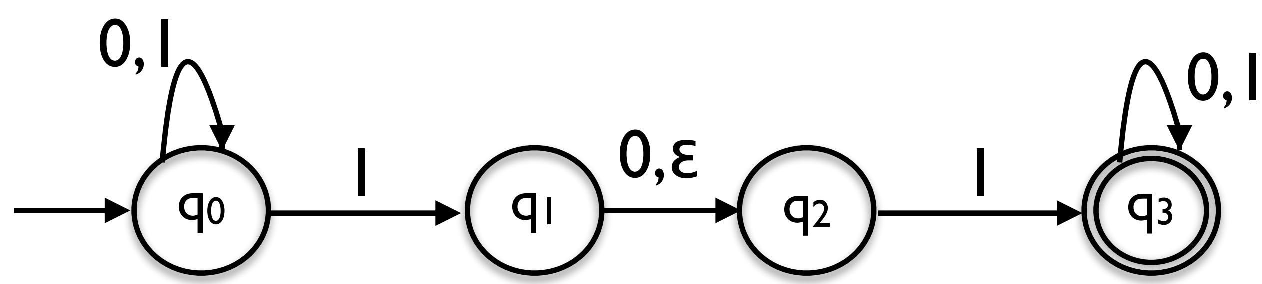
The class of regular languages is closed under Kleene star

Nondeterministic Automata (NFA)

Informal example

$\Sigma = \{0,1\}$

M_2:



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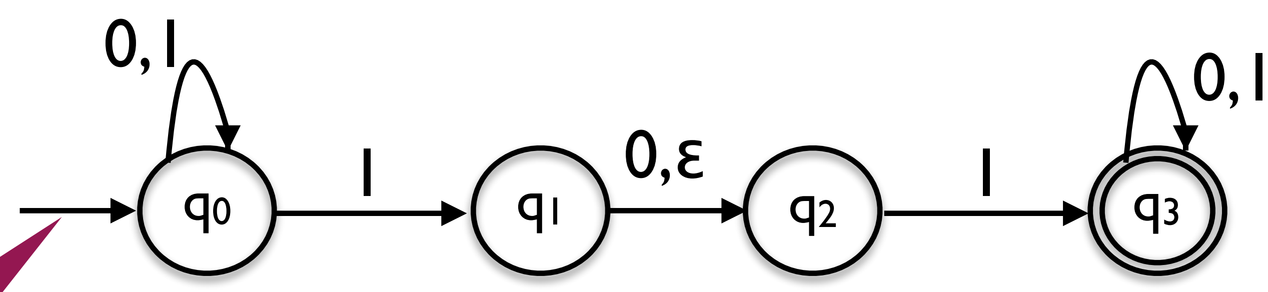

sources of nondeterminism

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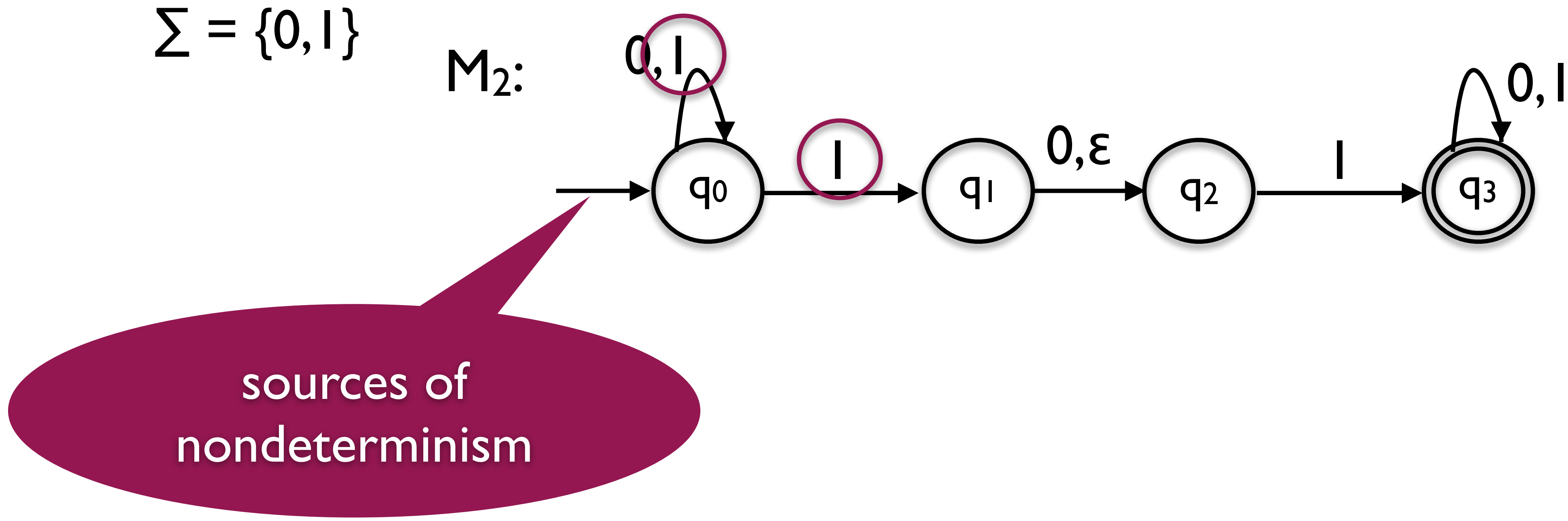

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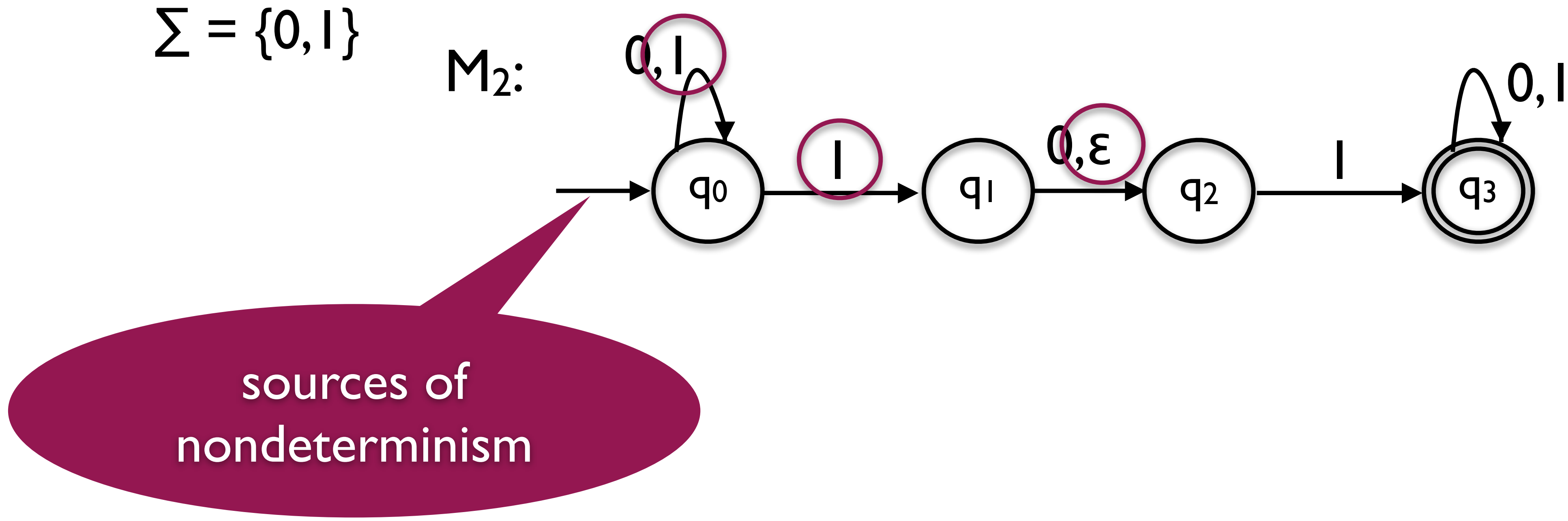

Nondeterministic Automata (NFA)

no 1 transition

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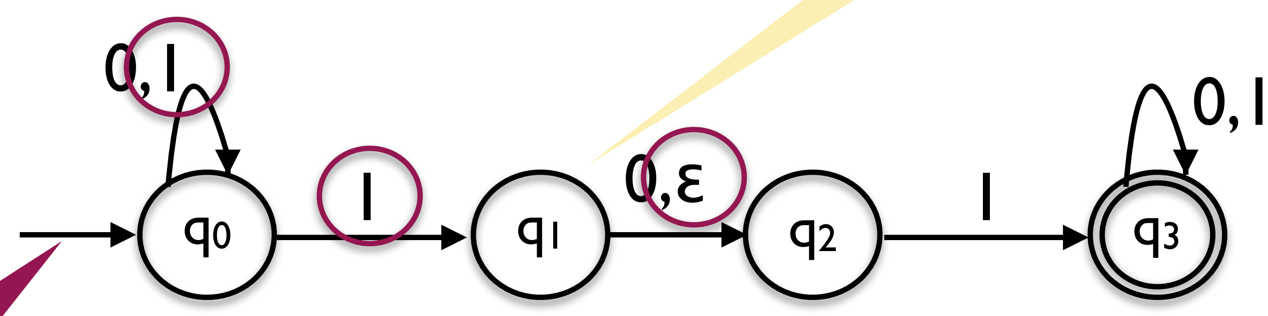

sources of nondeterminism

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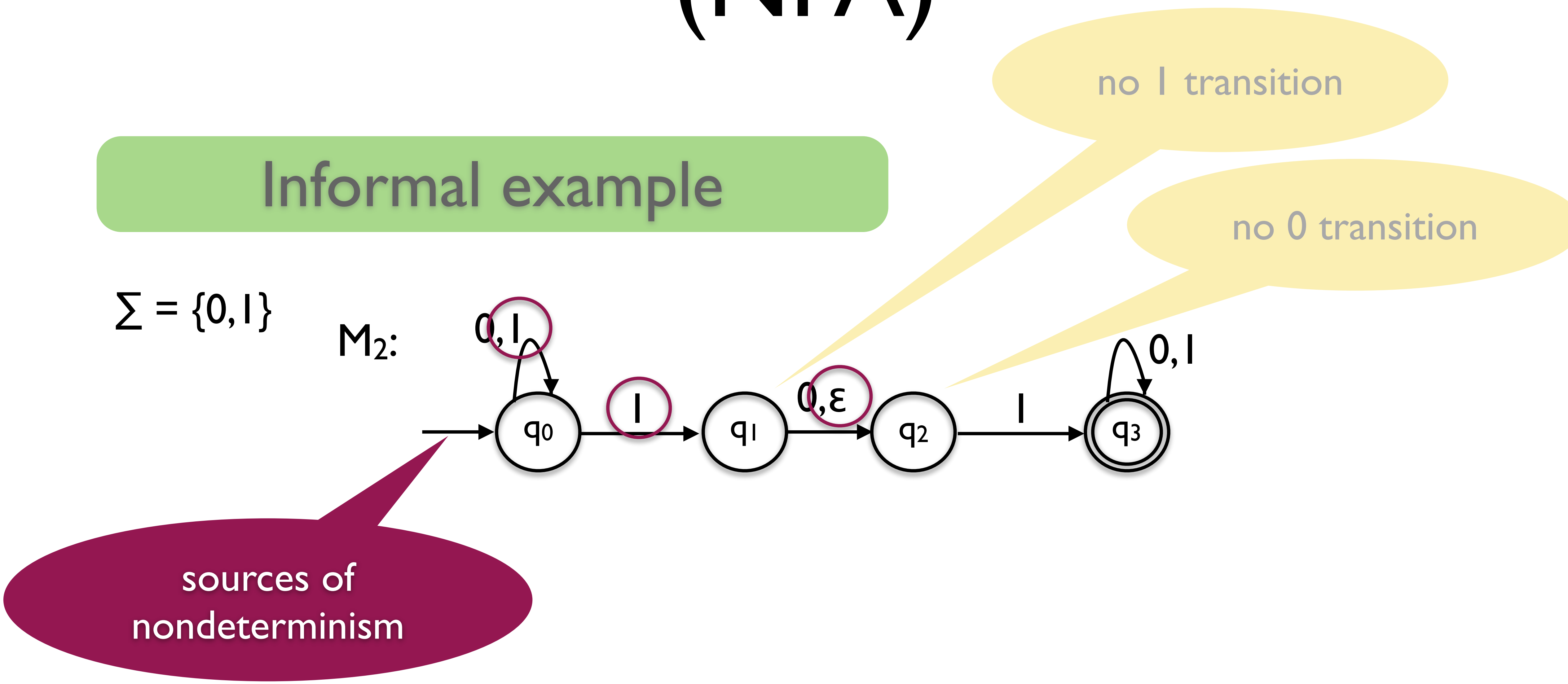

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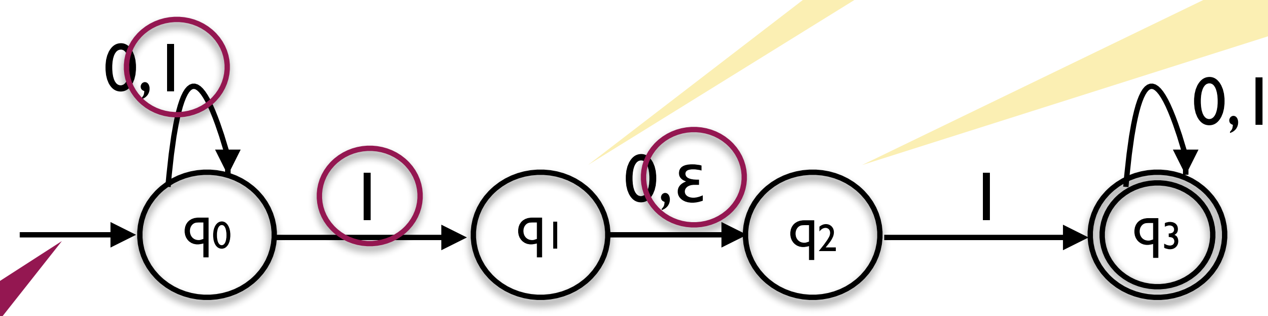

no 1 transition

no 0 transition

sources of nondeterminism

Accepts a word iff there exists an accepting run

NFA

Definition

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$M_2 = (Q, \Sigma, \delta, q_0, F)$ for

$Q = \{q_0, q_1, q_2, q_3\}$

NFA

Definition

A nondeterministic automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

Q is a finite set of states
Σ is a finite alphabet
$\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function
q_0 is the initial state, $q_0 \in Q$
F is a set of final states, $F \subseteq Q$

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In the example M

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$\Sigma = \{0, 1\}$ $\quad$ $F = \{q_3\}$

$\delta(q_0, 0) = \{q_0\}$
$\delta(q_0, 1) = \{q_0, q_1\}$
$\delta(q_0, \varepsilon) = \varnothing$
…..