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Generic Trace Theory

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Abstract

Trace semantics has been defined for various non-deterministic systems with different input/output types, or with different types of "non-determinism" such as classical non-determinism (with a set of possible choices) vs. probabilistic non-determinism. In this paper we claim that these various forms of "trace semantics" are instances of a single categorical construction, namely coinduction in a Kleisli category. This claim is based on our main technical result that an initial algebra in the category of sets and functions yields a final coalgebra in the Kleisli category, for monads with a suitable order structure. The proof relies on coincidence of limits and colimits, like in the work of Smyth and Plotkin.

Key words: coalgebra, trace semantics, linear time semantics, monad, Kleisli category, non-determinism, probability

1 Introduction

Trace semantics is a commonly used semantic relation for reasoning about non-deterministic[1] systems [24]. The notion of traces has been defined for various kinds of systems: for different input/output types, and more fundamentally for different types of "non-determinism" such as classical non-determinism or probabilistic non-determinism. Our claim in this paper is that those various forms of "trace semantics" are instances of a general construction, namely coinduction in a Kleisli category. Our point of view here is categorical, coalgebraic in particular: see [12,19] for preliminaries. Hence this paper demonstrates the abstraction power of categorical/coalgebraic methods in computer science, uncovering basic mathematical structures underlying various concrete examples.

[1] In this paper we use the terminology *non-determinism* in its broader sense. It includes: *classical non-determinism* where one has a set of possible choices; *probabilistic non-determinism* where one has a probability distribution over possible choices; also systems with *non-termination*.

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The first observation in the coalgebraic exploration in computer science was that a system is modelled as a coalgebra $X \to FX$ in **Sets**, and that the principle of coinduction captures bisimilarity. In contrast, when we consider trace semantics for non-deterministic systems, it is appropriate to model a system as a coalgebra $X \to TFX$ in **Sets**, where

- a monad T on **Sets** specifies the type of non-determinism, with the help of its monad structure;
- a functor F on **Sets** specifies the input/output type;
- a distributive law $\pi : FT \Rightarrow TF$ distributes the effect of T over F.

Via the distributive law π the functor F is lifted to a functor $\mathcal{K}\ell(F)$ on the Kleisli category $\mathcal{K}\ell(T)$: this allows us to move our base category from **Sets** to $\mathcal{K}\ell(T)$. In $\mathcal{K}\ell(T)$ the system is just a (functor-)coalgebra $X \to \mathcal{K}\ell(F)X$. The following diagram of coinduction, now in $\mathcal{K}\ell(T)$ for $\mathcal{K}\ell(F)$-coalgebras, captures trace semantics.

$$\begin{tikzcd}
\mathcal{K}\ell(F)X \arrow[r, dashed, "\mathcal{K}\ell(F)(\mathsf{tr}_c)"] & \mathcal{K}\ell(F)A \\
X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{tr}_c"'] & A \arrow[u, "\cong"']
\end{tikzcd}$$

It is standard (see e.g. [7,17]) that in such a situation—where we have a distributive law $FT \Rightarrow TF$—an initial F-algebra in **Sets** yields an initial $\mathcal{K}\ell(F)$-algebra in $\mathcal{K}\ell(T)$. Our interest is in a final $\mathcal{K}\ell(F)$-coalgebra: in fact it coincides with an initial $\mathcal{K}\ell(F)$-algebra for a wide variety of a functor F and a monad T equipped with a suitable order structure. This is our main result. A special case of this result for the powerset monad has been presented in [9] and preliminary investigations for the probability subdistribution monad have been reported in [8]. Here we generalize those results to monads with an order structure. The coincidence of initial algebra and final coalgebra—surprising at first sight—follows from the classic work [22] on limit-colimit coincidence. Here it is adapted to the setting of **DCpo**-enriched Kleisli categories.

Many known non-deterministic systems are actually modelled as TF-coalgebras in **Sets**, with such T and F that our main result applies to. Then our finality result assigns to a system $X \to TFX$ a function $X \to TA$ where A is an initial F-algebra: we call this function the *finite trace* of the system. We present several examples where this categorically characterized finite traces coincide with a standard, concrete definition of (finite) traces.

As a monad $T : \mathbf{Sets} \to \mathbf{Sets}$ we have three examples.

- The *lift monad* $\mathcal{L} = 1 + \_$ where $1 = \{\bot\}$. It models systems with non-termination (such as exceptions or deadlocks). Its monad structure is a standard one induced by coproduct. For each set X, the set $\mathcal{L}X$ has a flat order with bottom $\bot$: for $u, v \in X$, $u \leq v$ if either $u = v$ or $u = \bot$.

- The *powerset monad* $\mathcal{P}$. It models systems with *classical non-determinism*. Its unit takes a singleton, and its multiplication takes a union. A set $\mathcal{P}X$ is ordered by inclusion.
- The *subdistribution* monad $\mathcal{D}$. It models *probabilistic* systems, or systems with *probabilistic non-determinism*: see Example 5.3. Its action is: for a set X and a function $f : X \to Y$,

$$\mathcal{D}X = \{d : X \to [0,1] \mid \sum_{x \in X} d(x) \le 1\} \ , \qquad (\mathcal{D}f)(d) = \lambda y. \sum_{x \in f^{-1}(\{y\})} d(x) \ ,$$

where $d \in \mathcal{D}X$. Hence the set $\mathcal{D}X$ consists of probability distributions on X, with sum ≤ 1, instead of $= 1$. Its unit and multiplication is as follows.

$$\eta_X(x) = \lambda x'. \begin{cases} 1 & \text{if } x' = x, \\ 0 & \text{otherwise.} \end{cases} \qquad \mu_X(\xi) = \lambda x. \sum_{d \in \mathcal{D}X} \xi(d) \cdot d(x)$$

A set $\mathcal{D}X$ has a pointwise order: $d \le e$ if for each $x \in X$ we have $d(x) \le e(x)$.

The *distribution monad* $\mathcal{D}_{=1}$ is such that $\mathcal{D}_{=1}X$ consists of distributions whose sum is equal to 1. We are not interested in it because it only carries a trivial order structure: for $d, e \in \mathcal{D}_{=1}X$, $d \le e$ only if $d = e$.

The paper is organized as follows. In Section 2 we present some preliminaries: construction of initial algebra (final coalgebra) via initial (final) sequence, distributive laws which allow us to work in a Kleisli category, and the basic result in [22] on limit-colimit coincidence. We prove our main technical result in Section 3. To get an intuition about a finite trace map induced by finality, in Section 4 we take a closer look at its construction. Finally, in Section 5 we instantiate the general result and present concrete examples.

2 Preliminaries

2.1 Initial/final sequence

Here we recall the standard construction [2] of initial algebras (or final coalgebras) via the initial (or final) sequence. The construction will be heavily utilized throughout the paper: notice that the base category need not be **Sets**.

Let $\mathbb{C}$ be a category with initial object 0, and $F : \mathbb{C} \to \mathbb{C}$ an endofunctor. The *initial sequence*[2] of F is a diagram

$$0 \xrightarrow{\ ¡\ } F0 \xrightarrow{F\,¡} \cdots \xrightarrow{F^{n-1}\,¡} F^n 0 \xrightarrow{F^n\,¡} \cdots$$

where $¡ : 0 \to X$ is the unique arrow.

Now assume that:

[2] In this paper we consider only initial/final sequences of length ω.

- the initial sequence has an ω-colimit[3] $(\alpha_n : F^n 0 \to A)_{n<\omega}$;
- the functor F preserves that ω-colimit.

Then we have two cocones $(\alpha_n)_{n<\omega}$ and $(F\alpha_{n-1})_{n<\omega}$ over the initial sequence. Moreover, the latter is again a colimit: hence we have mediating isomorphisms between these cones.

$$
\begin{tikzcd}
 & & & & A \arrow[dd, bend left, "\alpha^{-1}"] \\
\cdots \arrow[r] & F^n 0 \arrow[r, "F^n \text{¡}"] \arrow[urrr, bend left, "\alpha_n"] \arrow[drrr, bend right, "F\alpha_{n-1}"'] & F^{n+1} 0 \arrow[r] \arrow[urr, "\alpha_{n+1}"] \arrow[drr, "F\alpha_n"'] & \cdots & \\
 & & & & FA \arrow[uu, bend left, "\alpha"] \arrow[uu, phantom, "\cong"]
\end{tikzcd}
$$

Proposition 2.1 *The F-algebra $\alpha : FA \xrightarrow{\cong} A$ is initial.*

Proof. For future reference we prove the dual result: see Proposition 2.2. $\square$

The dual of this construction yields a final F-coalgebra. Assume that the base category $\mathbb{C}$ has a terminal object 1. The *final sequence* of F is

$$
1 \xleftarrow{\;!\;} F1 \xleftarrow{\;F\,!\;} \quad \cdots \quad \xleftarrow{\;F^{n-1}\,!\;} F^n 1 \xleftarrow{\;F^n\,!\;} \quad \cdots ,
$$

where $!: X \to 1$ is the unique arrow. Assume that it has a ω^{op}-limit $(\zeta_n : Z \to F^n 1)_{n<\omega}$, and also that F preserves that ω^{op}-limit. We have the following situation.

$$
\begin{tikzcd}
 & & & & Z \arrow[dd, bend left, "\zeta"] \arrow[dlll, "\zeta_{n+1}"'] \arrow[dllll, bend right, "\zeta_n"'] \\
\cdots & F^n 1 \arrow[l] & F^{n+1} 1 \arrow[l, "F^n\,!"'] & \cdots \arrow[l] & \\
 & & & & FZ \arrow[uu, bend left, "\zeta^{-1}"] \arrow[uu, phantom, "\cong"] \arrow[ulll, "F\zeta_n"] \arrow[ullll, bend left, "F\zeta_{n-1}"]
\end{tikzcd}
$$

Proposition 2.2 *The coalgebra $\zeta : Z \xrightarrow{\cong} FZ$ is final.*

Proof. Any F-coalgebra $c : X \to FX$ induces a cone $(\beta_n : X \to F^n 1)_{n<\omega}$ over the final sequence in the following way.

$$
\beta_0 = \; ! : X \to 1 \;, \qquad \beta_{n+1} = F\beta_n \circ c \;.
$$

Now we can prove the following: for an arrow $f : X \to Z$, f is a morphism of coalgebras from c to ζ if and only if f is a mediating arrow from the cone $(\beta_n)_{n<\omega}$ to the limit $(\zeta_n)_{n<\omega}$. Hence such a morphism of coalgebras uniquely exists. $\square$

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[3] An ω-colimit is a colimit of a diagram whose shape is the ordinal ω.

2.2 Distributive laws and Kleisli categories

In this section we recall some basic facts on monads, Kleisli categories and distributive laws. A distributive law allows us to move our base category from **Sets** to $\mathcal{K}\ell(T)$, by lifting a functor F. This shift, first exploited in [18], plays a central role in this paper's study about trace semantics for non-deterministic systems.

Although some material applies to more general settings, here we restrict our base category to **Sets** for the sake of simplicity.

Let F be an endofunctor and T be a monad, both on **Sets**. A *distributive law* $\pi : FT \Rightarrow TF$ is a natural transformation which is compatible with the structure of the monad $\langle T, \eta, \mu \rangle$. That is, $\pi \circ F\eta = \eta F$ and $\pi \circ F\mu = \mu F \circ T\pi \circ \pi T$.

Such a distributive law induces a lifting of the functor $F : \mathbf{Sets} \to \mathbf{Sets}$ to a functor $\mathcal{K}\ell(F) : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ on the Kleisli category of the monad T by:

$$\mathcal{K}\ell(F)(X) = FX \quad \text{and} \quad \mathcal{K}\ell(F)\left(X \xrightarrow{f} TY\right) = \left(FX \xrightarrow{\pi \circ Ff} TFY\right) \quad .$$

We thus have a situation:

$$\begin{tikzcd} \mathbf{Sets} \arrow[loop left, "F"] \arrow[r, bend left, "J"] \arrow[r, phantom, "\perp"] & \mathcal{K}\ell(T) \arrow[l, bend left, "K"] \arrow[loop right, "\mathcal{K}\ell(F)"] \end{tikzcd}$$

where $J \dashv K$ is the standard adjunction associated with a Kleisli construction. For further reference we explicitly note that $JX = X$ for any set X and $Jf = \eta_Y \circ f$ for a map $f : X \to Y$.

The functor $\mathcal{K}\ell(F)$ is indeed a "lifting" of F, in the following sense.

Lemma 2.3 *The following diagram commutes.*

$$\begin{tikzcd} \mathcal{K}\ell(T) \arrow[r, "\mathcal{K}\ell(F)"] & \mathcal{K}\ell(T) \\ \mathbf{Sets} \arrow[u, "J"] \arrow[r, "F"'] & \mathbf{Sets} \arrow[u, "J"'] \end{tikzcd}$$

□

We shall now investigate the condition under which this distributive law $\pi : FT \Rightarrow TF$ is available. For the case $T = \mathcal{P}$, we have the following construction via relation lifting.

Lemma 2.4 (From [11]) *Let* $F\colon \mathbf{Sets} \to \mathbf{Sets}$ *be a functor that preserves weak pullbacks. Then there exists a "power law"* $\pi\colon F\mathcal{P} \Rightarrow \mathcal{P}F$ *that forms a distributive law between* F *and the powerset monad* $\mathcal{P}$*.*

The map $\pi_X\colon F(\mathcal{P}X) \to \mathcal{P}(FX)$ is defined as

$$\pi_X(u) = \{v \in FX \mid (v, u) \in \mathrm{Rel}(F)(\in)\},$$

where $\mathrm{Rel}(F)(R) \subseteq FX \times FY$, for a relation $R \subseteq X \times Y$, is the relation lifting associated with F. In the above definition of π_X it is applied to the membership relation $\in \hookrightarrow X \times \mathcal{P}X$.

We would like to generalize this result to other monads than powerset. If the monad is *commutative* and the functor is in the inductively defined family of *shapely functors*, we can construct a distributive law in an inductive manner. These classes of monads and functors are so wide that all the examples in this paper fall in there. However our main result may still hold for monads that are not commutative and functors that are not shapely—we just require existence of a distributive law.

Definition 2.5 (Shapely functors, [13]) The family of *shapely functors* on **Sets** is defined inductively by the following BNF notation:

$$F, G, F_i ::= \mathrm{id} \mid \Sigma \mid F \times G \mid \textstyle\coprod_{i \in I} F_i \ ,$$

where Σ denotes the constant functor into an arbitrary set Σ.

Notice that we do not allow taking infinite products—hence exponentials F^{Σ} with Σ infinite—in an inductive construction. Due to this choice every shapely functor preserves ω-colimits and ω^{op}-limits: hence we can use the construction in Propositions 2.1 and 2.2.

Lemma 2.6 *Every shapely functor $F : \mathbf{Sets} \to \mathbf{Sets}$ has both an initial algebra and a final coalgebra.* $\square$

We recall (see e.g. [10]) that each monad T on **Sets** is *strong*, *i.e.* it comes with a natural transformation $\mathsf{st}\colon X \times TY \to T(X \times Y)$ that commutes appropriately with the monad's unit and multiplication.

Then there are two "obvious" maps $TX \times TY \rightrightarrows T(X \times Y)$:

$$\begin{array}{ccccc}
TX \times TY & \xrightarrow{\mathsf{st}_{TX,Y}} & T(TX \times Y) & \xrightarrow{T\mathsf{st}_{Y,X}} & T^2(X \times Y) \\
\downarrow{\scriptstyle \mathsf{st}_{TY,X}} & & & & \downarrow{\scriptstyle \mu_{X \times Y}} \\
T(X \times TY) & \xrightarrow[T\mathsf{st}_{X,Y}]{} & T^2(X \times Y) & \xrightarrow[\mu_{X \times Y}]{} & T(X \times Y)
\end{array}$$

where isomorphisms $X \times Y \xrightarrow{\cong} Y \times X$ are used implicitly. The monad T is called *commutative* if these two maps are identical. In that case we call the resulting map the *double strength* of T and denote by $\mathsf{dst}_{X,Y} : TX \times TY \to T(X \times Y)$. This definition is due to [15].

Lemma 2.7 *Let $T\colon \mathbf{Sets} \to \mathbf{Sets}$ be a commutative monad, and $F\colon \mathbf{Sets} \to \mathbf{Sets}$ a shapely functor. Then there is a distributive law $\pi\colon FT \Rightarrow TF$.*

Proof. By induction on the structure of F.

- If F is the identity functor, then the π is simply the identity natural transformation $T \Rightarrow T$.

- If F is a constant functor, say $X \mapsto \Sigma$, then π is the unit $\eta_\Sigma \colon \Sigma \to T\Sigma$ at $\Sigma \in \mathbf{Sets}$.
- If $F = F_1 \times F_2$ we use induction in the form of distributive laws $\pi^{F_i} \colon F_i T \Rightarrow TF_i$ for $i \in \{1, 2\}$ to form the composite:

$$F_1(TX) \times F_2(TX) \xrightarrow{\pi^{F_1} \times \pi^{F_2}} T(F_1X) \times T(F_2X) \xrightarrow{\mathsf{dst}} T(F_1X \times F_2X) \ .$$

- If F is a coproduct $\coprod_{i \in I} F_i$ then we use laws $\pi^{F_i} \colon F_i T \Rightarrow TF_i$ for $i \in I$ in:

$$\textstyle\coprod_{i \in I} F_i(TX) \xrightarrow{[T(\kappa_i) \circ \pi^{F_i}]_{i \in I}} T(\coprod_{i \in I} F_i X) \ .$$

$\square$

It is not hard to see that a distributive law $F\mathcal{P} \Rightarrow \mathcal{P}F$ arising from this inductive construction is a power law as described in Lemma 2.4.

Example 2.8 The three monads $\mathcal{L}, \mathcal{P}$ and $\mathcal{D}$ mentioned in the introduction are easily shown to be commutative. Their double strengths are as follows.

$$\begin{array}{lll}
\mathsf{dst}^{\mathcal{L}}_{X,Y}(u, v) & = \begin{cases} (u, v) & \text{if } u \in X, v \in Y, \\ \bot & \text{if } u = \bot \text{ or } v = \bot, \end{cases} & \text{for } u \in \mathcal{L}X \text{ and } v \in \mathcal{L}Y, \\
\mathsf{dst}^{\mathcal{P}}_{X,Y}(X', Y') & = X' \times Y' \ , & \text{for } X' \in \mathcal{P}X \text{ and } Y' \in \mathcal{P}Y, \\
\mathsf{dst}^{\mathcal{D}}_{X,Y}(d, e) & = \lambda(x, y).\ d(x) \cdot e(y) \ , & \text{for } d \in \mathcal{D}X \text{ and } e \in \mathcal{D}Y.
\end{array}$$

2.3 Limit-colimit coincidence

We review the relevant notions and results from [22]. The idea is that in a suitable order-enriched setting, a (co)limit is equivalently described in the order-theoretic terms. Due to the duality inherent in those alternative order-theoretic notions, we obtain also the duality between limits and colimits. This yields so-called *limit-colimit coincidence*.

We denote by **DCpo** the category which has as objects directed cpo's (dcpo's in short), and (Scott-)continuous maps as arrows. For more details the reader is referred to [1].

Throughout this section we assume that our base category $\mathbb{C}$—later instantiated with $\mathcal{K}\ell(T)$—is **DCpo**-enriched. Spelling out the definition of enriched categories (see e.g. [14,5]), this means that each homset $\mathbb{C}(X, Y)$ carries a partial order $\leq$ in such a way that each directed collection $(f_i)_{i \in I}$ of maps $f_i \colon X \to Y$ in $\mathbb{C}$ has a join $\bigvee_{i \in I} f_i \colon X \to Y$. Additionally, composition preserves such joins:

$$g \circ \left(\textstyle\bigvee_{i \in I} f_i\right) = \textstyle\bigvee_{i \in I}(g \circ f_i) \quad \text{and} \quad \left(\textstyle\bigvee_{i \in I} f_i\right) \circ h = \textstyle\bigvee_{i \in I}(f_i \circ h).$$

Definition 2.9 (Embedding-projection pairs) A pair $(e : X \to Y, p : Y \to X)$ of arrows in $\mathbb{C}$ is said to be an *embedding-projection pair* if both

$p \circ e = \mathrm{id}$ and $e \circ p \leq \mathrm{id}$ hold. Here $\leq$ is the order in $\mathbb{C}(Y, Y)$ which is available due to **DCpo**-enrichedness. Diagramatically presented,

$$\begin{tikzcd}
X \arrow[r, tail, "e"] \arrow[dr, "\mathrm{id}"'] & Y \arrow[d, two heads, "p"] \arrow[dr, "\mathrm{id}"] & \\
& X \arrow[r, tail, "e"'] & Y.
\end{tikzcd}$$

Proposition 2.10 *Let $(e, p), (e', p') : X \rightleftarrows Y$ be two embedding-projection pairs with the same (co)domains. Then $e \leq e'$ holds if and only if $p' \leq p$.*

As a consequence, one component of an embedding-projection pair determines the other. $\square$

This proposition justifies the notation e^P for the projection corresponding to a given embedding e, and p^E for the embedding corresponding to a given projection p. It is easy to check: $(e \circ f)^P = f^P \circ e^P$ and $(p \circ q)^E = q^E \circ p^E$.

Definition 2.11 (O-(co)limits) Let $X_0 \stackrel{f_0}{\to} X_1 \stackrel{f_1}{\to} \cdots$ be an ω-chain in $\mathbb{C}$. A cocone $(\sigma_n : X_n \to C)_{n<\omega}$ over this chain is said to be an **O**-*colimit* if:

- each σ_n is an embedding;
- the sequence of arrows $(\, C \xrightarrow{\sigma_n^P} X_n \xrightarrow{\sigma_n} C\,)_{n<\omega}$ is increasing. Moreover its join taken in the dcpo $\mathbb{C}(C, C)$ is id_C.

Dually, a cone $(\gamma_n : C \to Y_n)_{n<\omega}$ over an ω^{op}-chain $Y_0 \stackrel{g_0}{\leftarrow} Y_1 \stackrel{g_1}{\leftarrow} \cdots$ is an **O**-*limit* if: each γ_n is a projection, and the sequence $(\gamma_n^E \circ \gamma_n : C \to C)_{n<\omega}$ is increasing and its join is id_C.

The following proposition establishes the equivalence between (co)limits and **O**-(co)limits. For its full proof the reader is referred to [22].

Proposition 2.12 (Proposition A, B, C, D in [22]) *Let $X_0 \stackrel{e_0}{\to} X_1 \stackrel{e_1}{\to} \cdots$ be an ω-chain where each e_n is an embedding.*

(i) *Let $(\sigma_n : X_n \to C)_{n<\omega}$ be a colimit over the chain. Then each σ_n is also an embedding. Moreover, $(\sigma_n)_{n<\omega}$ is an* **O**-*colimit.*

(ii) *Conversely, an* **O**-*colimit $(\sigma_n : X_n \to C)_{n<\omega}$ over the chain is a colimit.*

Dually, let $X_0 \stackrel{p_0}{\leftarrow} X_1 \stackrel{p_1}{\leftarrow} \cdots$ be an ω^{op}-chain where each p_n is a projection.

(i) *Let $(\tau_n : D \to X_n)_{n<\omega}$ be a limit over the chain. Then each τ_n is also a projection. Moreover $(\tau_n)_{n<\omega}$ is an* **O**-*limit.*

(ii) *Conversely, an* **O**-*limit $(\tau_n : D \to X_n)_{n<\omega}$ over the chain is a limit.*

Proof. For later reference we present the proof of the dual statement of (ii). Let $(\beta_n : B \to X_n)_{n<\omega}$ be an arbitrary cone over the chain $X_0 \stackrel{p_0}{\leftarrow} X_1 \stackrel{p_1}{\leftarrow} \cdots$.

First we prove the uniqueness of a mediating map $f : B \to D$.

$$\begin{aligned} f = \mathrm{id}_D \circ f &= \big(\textstyle\bigvee_{n<\omega}(\tau_n^E \circ \tau_n)\big) \circ f && ((\tau_n)_{n<\omega} \text{ is an } \mathbf{O}\text{-limit}) \\ &= \textstyle\bigvee_{n<\omega}(\tau_n^E \circ \tau_n \circ f) && (\text{Composition is continuous}) \\ &= \textstyle\bigvee_{n<\omega}(\tau_n^E \circ \beta_n) \ . && (f \text{ is mediating}) \end{aligned}$$

We conclude the proof by showing that the sequence $(\tau_n^E \circ \beta_n)_{n<\omega}$ is increasing, hence such f indeed exists.

$$\tau_n^E \circ \beta_n \;=\; \tau_n^E \circ p_n \circ \beta_{n+1} \;=\; \tau_{n+1}^E \circ p_n^E \circ p_n \circ \beta_{n+1} \;\le\; \tau_{n+1}^E \circ \beta_{n+1} \ . \quad \square$$

Theorem 2.13 (Limit-colimit coincidence) *Let $X_0 \stackrel{e_0}{\to} X_1 \stackrel{e_1}{\to} \cdots$ be an ω-chain where each e_n is an embedding, and $(\sigma_n : X_n \to C)_{n<\omega}$ be a colimit over the chain. Then each σ_n is an embedding, and the cone $(\sigma_n^P : C \to X_n)_{n<\omega}$ is a limit over the ω^{op}-chain $X_0 \stackrel{e_0^P}{\leftarrow} X_1 \stackrel{e_1^P}{\leftarrow} \cdots$.*

Dually, a limit of an ω^{op}-chain of projections consists of projections. By taking the corresponding embeddings we obtain a colimit of an ω-chain of embeddings.

Proof. We prove the first statement. By Proposition 2.12 each σ_n is an embedding, and moreover $(\sigma_n)_{n<\omega}$ is an **O**-colimit. Now obviously $(\sigma_n^P)_{n<\omega}$ is a cone over $X_0 \stackrel{e_0^P}{\leftarrow} X_1 \stackrel{e_1^P}{\leftarrow} \cdots$. The condition that $(\sigma_n)_{n<\omega}$ is an **O**-colimit is exactly the same as that $(\sigma_n^P)_{n<\omega}$ is an **O**-limit. We use Proposition 2.12 to conclude the proof. $\square$

3 Final coalgebra in the Kleisli category

In this section we present our main technical result: for a monad T with a suitable order structure, an initial algebra in **Sets** yields a final coalgebra in $\mathcal{K}\ell(T)$.

In the remainder of this paper we assume the following.

(i) A monad $\langle T, \eta, \mu\rangle$ on **Sets** is such that the associated Kleisli category $\mathcal{K}\ell(T)$ is $\mathbf{DCpo}_\perp$-enriched with composition being left-strict. This means that $\mathcal{K}\ell(T)$ is **DCpo**-enriched (the same condition as in the previous section), plus the following conditions about bottom elements:
- each homset $\mathcal{K}\ell(T)(X, Y)$—that is, $\mathbf{Sets}(X, TY)$—is a dcpo with the bottom element $\perp_{X,Y}$;
- composition of arrows is *left-strict*, i.e., for each arrow $f : X \to Y$ in $\mathcal{K}\ell(T)$, $\perp_{Y,Z} \circ f = \perp_{X,Z}$. In particular this implies that composition preserves bottoms: $\perp_{Y,Z} \circ \perp_{X,Y} = \perp_{X,Z}$ in $\mathcal{K}\ell(T)$.

(ii) A functor $F : \mathbf{Sets} \to \mathbf{Sets}$ that comes with a distributive law $\pi : FT \Rightarrow TF$. Hence we have a lifting $\mathcal{K}\ell(F)$ of F, as in Section 2.2.

(iii) The lifted functor $\mathcal{K}\ell(F) : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$ is locally monotone. More precisely, $\mathcal{K}\ell(F)$'s action on arrows is a monotone map of dcpo's: for

$f, g : X \rightrightarrows Y$ in $\mathcal{K}\ell(T)$ with $f \leq g$, we have $\mathcal{K}\ell(F)(f) \leq \mathcal{K}\ell(F)(g)$. We do not need local continuity of $\mathcal{K}\ell(F)$: see Remark 3.6.

(iv) The functor $F\colon \mathbf{Sets} \to \mathbf{Sets}$ preserves ω-colimits. By Proposition 2.1 we construct an initial F-algebra $\alpha : FA \xrightarrow{\cong} A$ in **Sets**, via the initial sequence.

In order to emphasize that certain property holds under these global assumptions, we mark the lemmas and the theorems that depend on them by $*$.

We start by the main line of the proof of our main result. The details are provided in the form of subsequent lemmas.

Theorem 3.1 (Main theorem) * *An initial F-algebra $\alpha : FA \xrightarrow{\cong} A$ yields in $\mathcal{K}\ell(T)$ both an initial $\mathcal{K}\ell(F)$-algebra and a final $\mathcal{K}\ell(F)$-coalgebra as follows.*

$$\begin{tikzcd} \mathcal{K}\ell(F)A \arrow[d, "\cong", "J\alpha = \eta_A \circ \alpha"'] \\ A \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathcal{K}\ell(F)A \\ A \arrow[u, "\cong"', "(J\alpha)^{-1} = J(\alpha^{-1}) = \eta_{FA} \circ \alpha^{-1}"] \end{tikzcd}$$

Here $J : \mathbf{Sets} \to \mathcal{K}\ell(T)$ is the standard left-adjoint in a Kleisli construction.

Proof. By the global assumption (iv) we obtain the initial algebra via the initial sequence in **Sets**.

In **Sets**

$$\begin{tikzcd} & & & & A \arrow[dd, bend left, "\alpha^{-1}"] \\ \cdots \arrow[r, "F^{n-1}\text{¡}"] & F^n 0 \arrow[r] \arrow[urrr, bend left, "\alpha_n"] \arrow[drrr, bend right, "F\alpha_{n-1}"'] & F^{n+1} 0 \arrow[r] \arrow[urr, "\alpha_{n+1}"] \arrow[drr, "F\alpha_n"'] & \cdots & \\ & & & & FA \arrow[uu, bend left, "\alpha"] \arrow[uu, phantom, "\cong"] \end{tikzcd} \tag{1}$$

We apply the functor $J : \mathbf{Sets} \to \mathcal{K}\ell(T)$ to the whole diagram. Since J is a left adjoint it preserves colimits: hence the two cocones in the following diagram are both colimits again.

In $\mathcal{K}\ell(T)$

$$\begin{tikzcd} & & & & A \arrow[dd, bend left, "J\alpha^{-1}"] \\ \cdots \arrow[r, "JF^{n-1}\text{¡}"] & F^n 0 \arrow[r] \arrow[urrr, bend left, "J\alpha_n"] \arrow[drrr, bend right, "JF\alpha_{n-1}"'] & F^{n+1} 0 \arrow[r] \arrow[urr, "J\alpha_{n+1}"] \arrow[drr, "JF\alpha_n"'] & \cdots & \\ & & & & FA \arrow[uu, bend left, "J\alpha"] \arrow[uu, phantom, "\cong"] \end{tikzcd} \tag{2}$$

The ω-chain in the diagram is the initial sequence for the functor $\mathcal{K}\ell(F)$ (Lemma 3.2): note for example that a left adjoint J preserves initial objects. Moreover the lower cone is the image of the upper cone under $\mathcal{K}\ell(F)$ (Lemma 2.3).

Hence Diagram (2) is equal to the following one, where ¡ denotes the unique arrow $¡ : 0 \to F0$ in $\mathcal{K}\ell(T)$.

$$\boxed{\text{In } \mathcal{K}\ell(T)}$$

$$\begin{tikzcd}
 & & & & A \arrow[d, bend left=40, "J\alpha^{-1}"] \\
\cdots \arrow[r, "\mathcal{K}\ell(F)^{n-1}¡"] & F^n0 \arrow[r] \arrow[urrr, bend left, "J\alpha_n"] \arrow[drrr, bend right, "\mathcal{K}\ell(F)J\alpha_{n-1}"'] & F^{n+1}0 \arrow[r] \arrow[urr, "J\alpha_{n+1}"] \arrow[drr, "\mathcal{K}\ell(F)J\alpha_n"'] & \cdots & \\
 & & & & FA \arrow[u, bend left=20, "J\alpha"] \arrow[u, phantom, shift left=3, "\cong"]
\end{tikzcd} \tag{3}$$

Thus Proposition 2.1 yields that $J\alpha : FA \xrightarrow{\cong} A$ is an initial $\mathcal{K}\ell(F)$-algebra.

To prove the second statement of the theorem, we shall transform the diagram (3) to a diagram of final sequence and its limits.

We notice (Lemma 3.4) that each arrow $\mathcal{K}\ell(F)^n¡$ in the initial sequence is an embedding. Hence the limit-colimit coincidence Theorem 2.13 says that every arrow in the diagram is an embedding. Note that $J\alpha$ and $J\alpha^{-1}$, inverse to each other, form an embedding-projection pair.

By taking the corresponding projections we obtain the following diagram: the limit-colimit coincidence Theorem 2.13 says that the two resulting cones are both limits. It is also obvious that the whole diagram commutes.

$$\boxed{\text{In } \mathcal{K}\ell(T)}$$

$$\begin{tikzcd}
 & & & & A \arrow[dlll, bend right, "(J\alpha_n)^P"'] \arrow[dll, "(J\alpha_{n+1})^P"'] \arrow[d, bend left=20, "(J\alpha)^P"'] \\
\cdots & F^n0 \arrow[l, "(\mathcal{K}\ell(F)^{n-1}¡)^P"'] & F^{n+1}0 \arrow[l] & \cdots \arrow[l] & \\
 & & & & FA \arrow[ulll, bend left, "(\mathcal{K}\ell(F)J\alpha_{n-1})^P"] \arrow[ull, "(\mathcal{K}\ell(F)J\alpha_n)^P"] \arrow[u, bend right=40, "(J\alpha^{-1})^P"'] \arrow[u, phantom, shift left=3, "\cong"]
\end{tikzcd} \tag{4}$$

The ω^{op}-chain here is indeed a final sequence: Lemma 3.3 shows—using our global assumption (i) on left-strictness—that 0 is also final in $\mathcal{K}\ell(T)$, and according to Lemma 3.4 we have $(\mathcal{K}\ell(F)^n¡)^P = \mathcal{K}\ell(F)^n!$. As to the lower cone we have $\big(\mathcal{K}\ell(F)J\alpha_n\big)^P = \mathcal{K}\ell(F)\big((J\alpha_n)^P\big)$ by Lemma 3.5.

Hence Diagram (4) is equal to the following one, showing the final sequence for $\mathcal{K}\ell(F)$, its limit (the upper one) and that limit mapped by $\mathcal{K}\ell(F)$ (the lower one) which is again a limit.

$$\boxed{\text{In } \mathcal{K}\ell(T)}$$

$$\begin{tikzcd}
 & & & & A \arrow[dlll, bend right, "(J\alpha_n)^P"'] \arrow[dll, "(J\alpha_{n+1})^P"'] \arrow[d, bend left=20, "J\alpha^{-1}"'] \\
\cdots & F^n0 \arrow[l, "\mathcal{K}\ell(F)^{n-1}!"'] & F^{n+1}0 \arrow[l] & \cdots \arrow[l] & \\
 & & & & FA \arrow[ulll, bend left, "\mathcal{K}\ell(F)(J\alpha_{n-1})^P"] \arrow[ull, "\mathcal{K}\ell(F)(J\alpha_n)^P"] \arrow[u, bend right=40, "J\alpha"'] \arrow[u, phantom, shift left=3, "\cong"]
\end{tikzcd} \tag{5}$$

By Proposition 2.2 we conclude that $J\alpha^{-1}$ is a final $\mathcal{K}\ell(F)$-coalgebra. □

In the remainder of this section those lemmas used in the above proof are presented.

Lemma 3.2 * *The ω-chain in Diagram (2) is indeed the initial sequence for $\mathcal{K}\ell(F)$. That is, we have for each $n < \omega$,*

$$JF^n\big(\, ¡^{\mathbf{Sets}}_{F0}\,\big) = \mathcal{K}\ell(F)^n\big(\, ¡^{\mathcal{K}\ell(T)}_{F0}\,\big) \ ,$$

where $¡^{\mathbf{Sets}}_{F0} : 0 \to F0$ in **Sets** *and $¡^{\mathcal{K}\ell(T)}_{F0} : 0 \to F0$ in $\mathcal{K}\ell(T)$ denote the unique maps.*

Proof. By induction on n. For $n = 0$ the two maps are equal due to the initiality of $J0 = 0$ in $\mathcal{K}\ell(T)$. For the step case we use Lemma 2.3. $\square$

Lemma 3.3 * *The empty set 0 is both an initial and a final object in $\mathcal{K}\ell(T)$. Therefore the object $T0$ is final in* **Sets**.

Proof. The functor $J : \mathbf{Sets} \to \mathcal{K}\ell(T)$ preserves initial objects since it is a left adjoint. Therefore $0 = J0$ is initial in $\mathcal{K}\ell(T)$. Finality follows from the left-strictness assumption. For an arbitrary set X, there always exists the bottom map $\perp_{X,0} : X \to 0$ in $\mathcal{K}\ell(T)$, which is the bottom in the poset $\mathcal{K}\ell(T)(X, 0)$. Assume there exist two arrows $f, g : X \to 0$ in $\mathcal{K}\ell(T)$. Note that the bottom map $\perp_{0,0} : 0 \to 0$ is also the identity arrow in $\mathcal{K}\ell(T)$ because of initiality. We get

$$f \;=\; \mathrm{id} \circ f \;=\; \perp_{0,0} \circ f \;\overset{(*)}{=}\; \perp_{X,0} \;\overset{(*)}{=}\; \perp_{0,0} \circ g \;=\; g$$

where the compositions are taken in $\mathcal{K}\ell(T)$ and the equalities marked by $(*)$ hold by the left-strictness of the composition.

The second point holds because the right adjoint K in the standard adjunction $J \dashv K$ preserves final objects. $\square$

Lemma 3.4 * *Each arrow $\mathcal{K}\ell(F)^n\, ¡$ in the initial sequence for $\mathcal{K}\ell(F)$, as in Diagram (3), is an embedding. Its corresponding projection is given by*

$$\big(\mathcal{K}\ell(F)^n(\, ¡\,)\big)^P = \mathcal{K}\ell(F)^n(\, !\,) \ ,$$

where $!$ denotes the unique arrow from $F0$ to the final object 0 in $\mathcal{K}\ell(T)$ (cf. Lemma 3.3).

Proof. We show that $\big(\mathcal{K}\ell(F)^n(\, ¡\,), \mathcal{K}\ell(F)^n(\, !\,)\big)$ is an embedding-projection pair for all $n < \omega$. Showing $\mathcal{K}\ell(F)^n(\, !\,) \circ \mathcal{K}\ell(F)^n(\, ¡\,) = \mathrm{id}$ is easy. For the other half we have

$$\begin{aligned}
\mathcal{K}\ell(F)^n(\, ¡\,) \circ \mathcal{K}\ell(F)^n(\, !\,) &= \mathcal{K}\ell(F)^n(\, ¡ \circ !\,) \\
&= \mathcal{K}\ell(F)^n(\perp_{0,F0} \circ\, !\,) && \text{(Initiality of } 0 \text{ in } \mathcal{K}\ell(T)) \\
&= \mathcal{K}\ell(F)^n(\perp_{F0,F0}) && \text{(Composition is left-strict)} \\
&\leq \mathcal{K}\ell(F)^n(\mathrm{id}) = \mathrm{id} \ . && (\mathcal{K}\ell(F) \text{ is locally monotone})
\end{aligned}$$

$\square$

Lemma 3.5 * *We have* $\big(\mathcal{K}\ell(F)J\alpha_n\big)^P = \mathcal{K}\ell(F)\big((J\alpha_n)^P\big)$. *Hence the lower cone in Diagram (4) is the image of the upper cone under* $\mathcal{K}\ell(F)$.

Proof. It is easy to check that $\big(\mathcal{K}\ell(F)J\alpha_n, \mathcal{K}\ell(F)\big((J\alpha_n)^P\big)\big)$ indeed form an embedding-projection pair. Therein we use the monotonicity of $\mathcal{K}\ell(F)$'s action on arrows. □

Remark 3.6 The limit-colimit coincidence result of [22] is often applied to a (co)algebraic setting (see [20]). There it is common to assume the local continuity of a functor, such as $\mathcal{K}\ell(F)(\bigvee_i f_i) = \bigvee_i\big(\mathcal{K}\ell(F)f_i\big)$. For our main Theorem 3.1 we do not need that local continuity: the principal reason is that in Diagram (1) the lower cocone is already a colimit.

4 Finite traces for coalgebras

The previous section gives a combined initiality/finality result. The finality part is most interesting, and has already been exploited in [9] for the special case where the monad T is the powerset one $\mathcal{P}$. Here we shall investigate this situation more systematically. In particular we observe a concrete construction of the unique arrow (which we call the "*finite trace*") induced by the finality result in the previous section. This construction, together with the examples in the following section, shall clarify the computational meaning of the arrow and justify its name.

Corollary 4.1 * *Let* $\alpha : FA \xrightarrow{\cong} A$ *be an initial* F*-algebra in* **Sets**. *Given a coalgebra*

$$X \xrightarrow{c} \mathcal{K}\ell(F)X \quad \text{in } \mathcal{K}\ell(T), \qquad \text{that is,} \qquad X \xrightarrow{c} TFX \quad \text{in } \mathbf{Sets},$$

there exists a unique map tr_c *which makes the following diagram in* $\mathcal{K}\ell(T)$ *commute.*

In $\mathcal{K}\ell(T)$,

$$\begin{tikzcd} \mathcal{K}\ell(F)X \arrow[r, dashed, "\mathcal{K}\ell(F)(\mathsf{tr}_c)"] & \mathcal{K}\ell(F)A \\ X \arrow[u, "c"] \arrow[r, dashed, "\mathsf{tr}_c"'] & A \arrow[u, "\cong"', "J\alpha^{-1}"] \end{tikzcd},$$

equivalently in **Sets**,

$$\begin{tikzcd} TFX \arrow[r, dashed, "TF\mathsf{tr}_c"] & TFTA \arrow[d, "T\pi_A"] \\ & T^2FA \arrow[d, "\mu_{FA}"] \\ & TFA \\ X \arrow[uuu, "c"] \arrow[r, dashed, "\mathsf{tr}_c"'] & TA \arrow[u, "\cong"', "T\alpha^{-1}"] \end{tikzcd}.$$

The map tr_c *is called the finite trace of the coalgebra* c.

Proof. The statement is the finality Theorem 3.1 itself. Translation of the diagram in $\mathcal{K}\ell(T)$ to that in **Sets**, and vice versa, is straightforward. □

More concretely, we shall construct the finite trace $\mathsf{tr}_c : X \to TA$ as the supremum of "n-th trace" tr_c^n. Let us explain the intuition for the case $T = \mathcal{P}$. The set $\mathsf{tr}_c^n(x)$ consists of "possible behaviors from state $x \in X$ which *terminate* within n steps". Therefore its supremum $\mathsf{tr}_c(x)$ is the set of "possible behaviors from state x which eventually terminate within a finite number of steps", hence its name "*finite* trace". For other monads we suitably substitute the word "set" above: for $T = \mathcal{D}$ that will be "probability distribution".

Definition 4.2 (n-fold iteration of coalgebras) Let $c : X \to TFX$ in **Sets**, i.e. $c : X \to \mathcal{K}\ell(F)X$ in $\mathcal{K}\ell(T)$, be a coalgebra. Its *n-fold iteration*

$$X \xrightarrow{\quad c^n \quad} \mathcal{K}\ell(F)^n X \quad \text{in } \mathcal{K}\ell(T), \qquad \text{that is,} \qquad X \xrightarrow{\quad c^n \quad} TF^n X \quad \text{in } \mathbf{Sets},$$

is defined inductively as $c^0 \stackrel{\text{def}}{=} \mathrm{id}$ and $c^{n+1} \stackrel{\text{def}}{=} \mathcal{K}\ell(F)c^n \circ c$ in $\mathcal{K}\ell(T)$.

The idea is that one transition of c^n corresponds to n successive transitions of the original coalgebra c. Note that the use of the distributive law π—implicit in $\mathcal{K}\ell(F)$—is crucial here.

Definition 4.3 ***(n-th trace of coalgebras)** For a coalgebra $c : X \to TFX$ in **Sets**, we define its *n-th trace*

$$X \xrightarrow{\quad \mathsf{tr}_c^n \quad} TA \quad \text{in } \mathbf{Sets}$$

as follows:

$$\begin{tikzcd}
X \arrow[r, "c^n"] \arrow[ddr, "\mathsf{tr}_c^n"'] & \mathcal{K}\ell(F)^n X \arrow[d, "\mathcal{K}\ell(F)^n(\,!\,)"] \\
& \mathcal{K}\ell(F)^n 0 \arrow[d, "J\alpha_n"] \\
\boxed{\text{In } \mathcal{K}\ell(T)} & A
\end{tikzcd}
\qquad \text{that is,} \qquad
\begin{tikzcd}
X \arrow[r, "c^n"] \arrow[ddddrr, bend right, "\mathsf{tr}_c^n"'] & TF^nX \arrow[r, "TF^n(\,!\,)"] \arrow[dddr, "\overline{T}F^n(\,\text{¡}\,)"'] & TF^nT0 \arrow[d, "T\pi^n"] \\
& & T^2F^n0 \arrow[d, "\mu"] \\
& & TF^n0 \arrow[d, "T\alpha_n"] \\
\boxed{\text{In } \mathbf{Sets}} & & TA
\end{tikzcd}\ ,$$

where the first $!$ is to the final object 0 in $\mathcal{K}\ell(T)$; the second $!$ is to the final object $T0 \cong 1$ in **Sets**; and ¡ is the unique arrow $0 \to X$ in **Sets**. The map $\overline{T}F^n(\,\text{¡}\,)$ here is just the name we give to the composite $\mu \circ T\pi^n \circ TF^n(\,!\,)$.

Proposition 4.4 * *The finite trace map $\mathsf{tr}_c : X \to TA$ is the supremum of n-th traces $\mathsf{tr}_c = \bigvee_{n<\omega} \mathsf{tr}_c^n$ taken in the dcpo $\mathcal{K}\ell(T)(X, A)$.*

Proof. By the proof of Proposition 2.2 we know that tr_c is the mediating arrow from the cone $(\beta_n : X \to \mathcal{K}\ell(F)^n 0)_{n<\omega}$, induced by $c : X \to \mathcal{K}\ell(F)X$, to the limit $\big((J\alpha_n)^P : A \to \mathcal{K}\ell(F)^n 0\big)_{n<\omega}$, where everything is in $\mathcal{K}\ell(T)$. By the proof of Theorem 3.1 the limit $\big((J\alpha_n)^P\big)_{n<\omega}$ is an **O**-limit: hence by the proof of Proposition 2.12 the mediating arrow tr_c is described as

$$\mathsf{tr}_c = \bigvee_{n<\omega} J\alpha_n \circ \beta_n \ .$$

We show $J\alpha_n \circ \beta_n = \mathsf{tr}_c^n$ by proving $\beta_n = \mathcal{K}\ell(F)^n(\,!\,) \circ c^n$ in $\mathcal{K}\ell(T)$. By induction: for $n = 0$ it is obvious due to the finality of 0. For the step case,

$$\begin{aligned} \beta_{n+1} &= \mathcal{K}\ell(F)\beta_n \circ c && \text{(Definition of } \beta_n\text{)} \\ &= \mathcal{K}\ell(F)\big(\mathcal{K}\ell(F)^n(\,!\,) \circ c^n\big) \circ c && \text{(Induction hypothesis)} \\ &= \mathcal{K}\ell(F)^{n+1}(\,!\,) \circ c^{n+1} \ . && \text{(Definition of } c^n\text{)} \end{aligned}$$ □

5 Examples

5.1 Satisfaction of order-theoretic assumptions

In this section we check that the monads $T = \mathcal{L}, \mathcal{P}, \mathcal{D}$ and shapely functors F indeed satisfy the global assumptions * in Section 3, so that we can apply our main technical result.

Proposition 5.1 *For $T \in \{\mathcal{L}, \mathcal{P}, \mathcal{D}\}$, the Kleisli category $\mathcal{K}\ell(T)$ is* $\mathbf{DCpo}_\perp$*-enriched with composition being left-strict.*

Proof. The dcpo structure of the homsets $\mathcal{K}\ell(T)(X, Y)$ comes from those of TY in a pointwise manner. It remains to show that composition in $\mathcal{K}\ell(T)$ is continuous and left-strict: this is laborious but straightforward. Notice that for $T = \mathcal{D}$, composition in $\mathcal{K}\ell(\mathcal{D})$ is described concretely as follows. For $X \xrightarrow{f} Y \xrightarrow{g} Z$,

$$(g \circ f)(x)(z) = \textstyle\sum_{y \in Y} f(x)(y) \cdot g(y)(z) \ .$$ □

For our main technical result in Section 3 it is enough to assume that $\mathcal{K}\ell(F)$ is locally monotone. However we can prove the following stronger statement, which says that the endofunctor $\mathcal{K}\ell(F)$ on the $\mathbf{DCpo}_\perp$-enriched category $\mathcal{K}\ell(T)$ is indeed an $\mathbf{DCpo}_\perp$*-enriched functor*.

Proposition 5.2 *The lifting $\mathcal{K}\ell(F)$ of a shapely functor F to $\mathcal{K}\ell(T)$ for $T \in \{\mathcal{L}, \mathcal{P}, \mathcal{D}\}$ is locally continuous. That is, the action of $\mathcal{K}\ell(F)$ on a homset is continuous. Moreover it is strict, i.e., preserves bottom elements.*

Proof. The proof is by induction on the construction of the shapely functor.

- $F = \mathrm{id}$, the identity functor. Then $\mathcal{K}\ell(F) = \mathrm{id}$ which satisfies the condition.
- $F = \Sigma$, a constant functor. Then $\mathcal{K}\ell(F)$ maps every arrow to the identity map on Σ in $\mathcal{K}\ell(T)$. This is obviously continuous and strict.
- $F = F_1 \times F_2$. First notice that, for $f : X \to TY$ in **Sets**, we obtain $\mathcal{K}\ell(F)f$

as the following composite in **Sets**.

$$\begin{tikzcd} F_1X \times F_2X \arrow[r, "\mathcal{K}\ell(F_1)f \times \mathcal{K}\ell(F_2)f"] \arrow[dr, "\mathcal{K}\ell(F)f"'] & TF_1Y \times TF_2Y \arrow[d, "\mathsf{dst}_{F_1Y,F_2Y}"] \\ & T(F_1Y \times F_2Y) \end{tikzcd}$$

Because the order in $\mathcal{K}\ell(T)(FX, FY)$ is a pointwise one, it suffices to show the following: $\mathsf{dst} : TX \times TY \to T(X \times Y)$, as a map of dcpo's, is continuous and strict. It is easy to check that this is indeed the case: see Example 2.8.

- $F = \coprod_{j \in J} F_j$. For $f : X \to TY$ in **Sets**, we obtain the map $\mathcal{K}\ell(F)f$ as the composite $[T\kappa_j]_{j \in J} \circ \coprod_{j \in J} \mathcal{K}\ell(F_j)(f)$ in **Sets**. Since the order on the homsets is pointwise, it suffices to show that each $T\kappa_j : TF_jY \to T(\coprod_{j \in J} F_jY)$ is continuous and strict. This is easy. □

5.2 *Concrete examples*

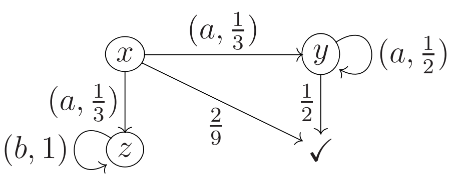
Many known concrete dynamic systems are in fact TF-coalgebras for F shapely and $T \in \{\mathcal{L}, \mathcal{P}, \mathcal{D}\}$, to which we can apply our finality result. For example,

- LTS's with explicit termination (see e.g. [4,3]) are TF-coalgebras for $T = \mathcal{P}$ and $F = 1 + \Sigma \times \_$;
- generative probabilistic transition systems [25,23] are TF-coalgebras for $T = \mathcal{D}$ and $F = 1 + \Sigma \times \_$.

In this section we take a step further ahead from the previous section to instantiate a shapely functor F, principally with $1 + \Sigma \times \_$. Then we observe that the finite trace map induced by our finality result coincides with the usual or natural notion of (finite) traces defined for those familiar types of systems.

Example 5.3 (Generative probabilistic systems) Let $T = \mathcal{D}$ and $F = 1 + \Sigma \times \_$, where $1 = \{\checkmark\}$. The initial F-algebra $[\mathtt{nil}, \mathtt{cons}] : 1 + \Sigma \times \Sigma^* \xrightarrow{\cong} \Sigma^*$ in **Sets** consists of the *lists* over Σ.

The following is an example of a coalgebra $c : X \to \mathcal{D}FX$.



$$c(x) = \begin{bmatrix} \checkmark & \mapsto & 2/9 \\ (a, y) & \mapsto & 1/3 \\ (a, z) & \mapsto & 1/3 \end{bmatrix}$$

The behavior of the state x is: it transits to y outputting a with the probability of $1/3$, the same to z, and it terminates with the probability of $2/9$. The remaining $1/9$ is best understood as the probability that x gets into *deadlock*.

Now the commutation of the diagram in Corollary 4.1—which defines the finite trace map $\mathsf{tr}_c : X \to \mathcal{D}(\Sigma^*)$—is equivalent to the following equation.

For $x \in X, a \in \Sigma$ and $\sigma \in \Sigma^*$,

$$\mathsf{tr}_c(x)(\langle\rangle) = c(x)(\checkmark) \ , \qquad \mathsf{tr}_c(x)(a \cdot \sigma) = \textstyle\sum_{y \in X} c(x)(a, y) \cdot \mathsf{tr}_c(y)(\sigma) \ .$$

In fact, for the above concrete example the distribution $\mathsf{tr}_c(x)$ is such that: $\langle\rangle \mapsto 2/9$ and $a^n \mapsto 1/(3 \cdot 2^n)$. Out of the remaining $4/9$, $1/9$ is the probability that x gets into deadlock at the first transition, and $1/3$ is the probability that x goes to z and keeps outputting b without termination (*livelock*).

Example 5.4 (Deterministic systems with termination and deadlock) Let us take $T = \mathcal{L} = 1 + \_$ and $F = 1 + \Sigma \times \_$, where $\perp$ (deadlock) resides in the former 1 while $\checkmark$ (successful termination) resides in the latter 1. For a TF-coalgebra

$$X \xrightarrow{\quad c \quad} \{\perp\} + \{\checkmark\} + \Sigma \times X \ \ ,$$

the diagramatic definition of the finite trace $\mathsf{tr}_c : X \to \{\perp\} + \Sigma^*$ is spelled out as the following equation:

$$\mathsf{tr}_c(x) = \begin{cases} \perp & \text{if } c(x) = \perp, \\ \langle\rangle & \text{if } c(x) = \checkmark, \\ a \star \mathsf{tr}_c(y) & \text{if } c(x) = (a, y), \end{cases}$$

where $a \star u$ is the concatenation if $u \in \Sigma^*$, and $a \star \perp = \perp$.

The following two examples are investigated in the previous paper [9], to which we refer for more details.

Example 5.5 (LTS's with explicit termination) Let us take $T = \mathcal{P}$ and $F = 1 + \Sigma \times \_$. Then a TF-coalgebra is an LTS with explicit termination: it is also called a non-deterministic automaton. The finite trace map of this type of coalgebra gives its accepted languages.

Example 5.6 (Context-free grammar/languages) When $T = \mathcal{P}$ and $F = (\Sigma + \_)^*$, a TF-coalgebra is a context-free grammar (without finiteness assumptions). Its finite trace map gives the set of generated parse trees.

Remark 5.7 (LTS's without explicit termination) An LTS (without explicit termination) is a TF-coalgebra for $T = \mathcal{P}$ and $F = \Sigma \times \_$. Its finite trace map is not interesting because the initial F-algebra is 0; the finite trace is always trivial.

The result in [11]—a final coalgebra in **Sets** yields a weakly final coalgebra in $\mathcal{K}\ell(\mathcal{P})$—assigns a *(possibly infinite) trace* $X \to \mathcal{P}\Sigma^\omega$ to an LTS $X \to \mathcal{P}FX$. However a (possibly infinite) trace is not uniquely determined categorically.

We will now show another possible application of our main result, as an instantiation of Example 5.5. Namely, the finality result allows defining operations on $\mathcal{P}(\Sigma^*)$ by coinduction.

Example 5.8 (Parallel composition of languages) Let Σ be an alphabet. Given two languages $u, v \in \mathcal{P}(\Sigma^*)$ we want to define a language $u \parallel v$ called the *(shuffle) parallel composition of all possible interleavings*, such that:

$$\begin{array}{rcl} \langle\rangle \in u \parallel v & \stackrel{\text{def}}{\Longleftrightarrow} & \langle\rangle \in u \quad \text{and} \quad \langle\rangle \in v \ , \\ a \cdot w \in u \parallel v & \stackrel{\text{def}}{\Longleftrightarrow} & w \in \partial_a u \parallel v \quad \text{or} \quad w \in u \parallel \partial_a v \ . \end{array}$$

Here $\partial_a u = \{w \in \Sigma^* \mid a \cdot w \in u\}$ is the so-called Brzozowski derivative [6]. For example, $\{a, ab\} \parallel \{\langle\rangle, c\} = \{a, ab, ac, ca, cab, acb, abc\}$. Then the operation $\parallel$ is a map

$$\mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*) \longrightarrow \mathcal{P}(\Sigma^*) \text{ in } \mathbf{Sets},\ i.e.\quad \mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*) \longrightarrow \Sigma^* \text{ in } \mathcal{K}\ell(\mathcal{P}).$$

We obtain the map $\parallel$ via coinduction (Theorem 3.1), by defining a suitable $\mathcal{P}(1 + \Sigma \times \_)$-coalgebra structure on $\mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*)$.

$$\begin{array}{rcl} \mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*) & \longrightarrow & \mathcal{P}\big(1 + \Sigma \times \big(\mathcal{P}(\Sigma^*) \times \mathcal{P}(\Sigma^*)\big)\big) \\ (u, v) & \longmapsto & \left[\begin{array}{l} \{\checkmark \mid \langle\rangle \in u \cap v\} \\ \quad \cup \quad \big\{\, \big(a,\ (\partial_a u,\ v)\big) \mid a \in \Sigma \,\big\} \\ \quad \cup \quad \big\{\, \big(a,\ (u,\ \partial_a v)\big) \mid a \in \Sigma \,\big\} \end{array}\right] \end{array}$$

The following equations can be proved by coinduction, for languages $u, v \in \mathcal{P}(\Sigma^*)$, the empty language $0 = \emptyset$ and the unit language $1 = \{\langle\rangle\}$.

$$\begin{array}{lcl} u \parallel 0 = 0 & & u \parallel (v \cup w) = (u \parallel v) \cup (u \parallel w) \\ & u \parallel v = v \parallel u & \\ u \parallel 1 = u & & u \parallel (v \parallel w) = (u \parallel v) \parallel w \end{array}$$

For example, in order to prove associativity of parallel composition, consider the relation on $\mathcal{P}\Sigma^*$,

$$R = \{\, (\, u \parallel (v \parallel w)\, ,\ (u \parallel v) \parallel w\,) \ \mid u, v, w \in \mathcal{P}(\Sigma^*)\}$$

together with the coalgebra structure $R \to \mathcal{P}(1 + \Sigma \times R)$ given by

$$(x, y) \quad \mapsto \quad \left[\begin{array}{l} \{\checkmark \mid \langle\rangle \in x \ , \text{ which is equivalent to } \langle\rangle \in y\} \\ \quad \cup \quad \big\{\, \big(a,\ (\,\partial_a(x)\, ,\ \partial_a(y)\,)\big) \mid a \in \Sigma\big\} \end{array}\right] \ .$$

One can then show that both projections $r_1, r_2 : R \to \mathcal{P}(\Sigma^*)$ are homomorpshisms in $\mathcal{K}\ell(\mathcal{P})$ from R to the final $\mathcal{K}\ell(1 + \Sigma \times \_)$-coalgebra Σ^*.[4] By finality we have $r_1 = r_2$: this proves the associativity of the parallel composition.

The case of "probabilistic languages" is more complex: defining parallel composition of probabilistic languages $u, v \in \mathcal{D}(\Sigma^*)$ and investigating their properties is a topic of our current research.

[4] A nice characterisation of $\mathcal{K}\ell(1 + \Sigma \times \_)$-colagebra homomorphisms is presented in [18].

6 Conclusion and future work

In this paper we re-examine the finite trace semantics of [9] and put the subject in a wider perspective. The paper:

- extends the approach used for the powerset monad $\mathcal{P}$ to other monads with suitable order structure,
- identifies the Smyth-Plotkin style limit-colimit coincidence in Kleisli categories as the relevant underlying structure.

A next challange to this approach is to apply it to combined monads, producing trace semantics for suitably combined computational behaviours. An interesting example is combining classical and probabilistic non-determinism [27,21]. It has been shown (see [26]) that the simple composition $\mathcal{PD}$ has no monad structure: to make it a monad the authors propose to take the so-called indexed-valuation monad instead of the subdistribution monad. Another example are the F-automata [16] where the combination of type $\mathcal{PP}$ is used. Describing finite traces of such combined monads is a non-trivial matter which we postpone to a follow-up paper.

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