On compositions and paths for coalgebras

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Abstract

This report discusses the possibility of defining composed steps and paths in coalgebras. Both of these notions are relevant for defining some semantic relations for coalgebras like weak bisimulation and trace semantics. We present some observations, ideas, and small results on the mentioned topics.

1 Introduction

In this report we collect some notes on compositions and paths for coalgebras. We assume familiarity with the theory of coalgebras [Rut96, JR96, Rut00] and with basic category theory (see e.g. [Mac71, Bor94]). As leading examples we consider probabilistic systems. Most of the probabilistic systems are coalgebras of a functor [VR99, Mos99, BSV04, SVW05].

The notions of compositions and paths are remotely related to trace semantics. For some probabilistic systems there is a notion of trace semantics i.e. trace distribution [Seg95a, Seg95b, Bai98]. For coalgebras in general it is difficult to define what traces are. Interesting solutions for some classes of coalgebras have been recently proposed [Jac04, HJ05b, Jac05, HJ05a]. We believe that an important building block for defining trace semantics as well as weak bisimulation for coalgebras is the notion of a composite step. In a coalgebra, for each state, the transition function represents the one-step behavior of the state. In order to define traces or weak steps, one needs to consider sequences of steps i.e. composite steps. In Section 2 we discuss how composition of coalgebras, i.e. composition of steps in a coalgebra can be defined for some types of coalgebras.

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We next emphasize the importance of the notion of paths. Consider labelled transition systems. A (finite) path π is an alternating sequence of states and labels of the form $s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_n} s_n$. We notice that many of the semantic relations can be defined via a transformation on paths in the following way. Assume T is a function defined on paths. Then a T-semantics can be defined as: two states s and t are T-equivalent if and only if the image under T of the set of paths that start in s equals the image under T of the set of paths that start in t. Indeed, if the transformation T is the consistent coloring, then we get bisimilarity. If T is weak consistent coloring, then we get branching bisimilarity. If T maps a path π as above to its trace $a_1 \ldots a_n$, then the semantic relation obtained is the trace equivalence. For probabilistic systems the notion of a path is also used in many occasions, for example for weak bisimulation [Seg95a, BH97, SVW05]. For these reasons, we investigate what a proper notion of a path in coalgebra might be. We focus on this in Section 3.

To this end we would like to stress that the mentioned investigations are only preliminary. We collect some observations, ideas and small results that might turn out useful in the future.

2 Composition of coalgebras

Consider a transition system $\langle S, \alpha : S \to \mathcal{P}S \rangle$ with $s, t, u \in S$. The outgoing transitions from the state s are shown in the left diagram below.
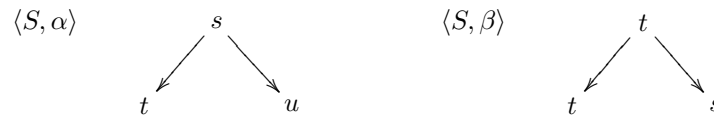
Moreover, assume that there is a transition system $\langle S, \beta : S \to \mathcal{P}S \rangle$ ($\beta = \alpha$ is possible) in which the state t allows the transitions shown in the right diagram above, and the state u is terminating. Transitions that correspond to the sequential composition of $\langle S, \alpha \rangle$ and $\langle S, \beta \rangle$ from the state s are as shown below.
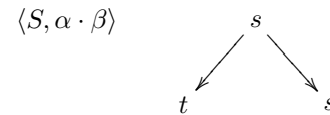
In this case, we compose by composing the transition relation $\longrightarrow$. Similarly, in the case of labelled transition systems, LTS, we consider $\langle S, \alpha \rangle$ and $\langle S, \beta \rangle$ with the transitions from s and t in the two systems as below, and u a terminating state.

The composition in state s is then described by the following transitions (now with labels from A^2).

$$\langle S, \alpha \cdot \beta \rangle$$

$$
\begin{array}{ccc}
 & s & \\
{}^{ab}\swarrow & & \searrow^{ac} \\
t & & s
\end{array}
$$
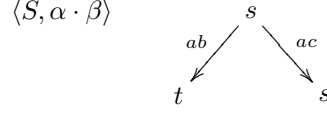
In general, if two systems $\langle S, \alpha : S \to \mathcal{F}S \rangle$ and $\langle S, \beta : S \to \mathcal{F}S \rangle$ are given, then we wonder which system (on S) behaves as if a step from $\langle S, \alpha \rangle$ is followed by a step from $\langle S, \beta \rangle$. We wish to define such a system $\langle S, \alpha \cdot \beta \rangle$. We could always define this composition to be of type $\mathcal{F}\mathcal{F}$, by $\alpha \cdot \beta = \mathcal{F}\beta \circ \alpha$, i.e.

$$S \xrightarrow{\alpha} \mathcal{F}S \xrightarrow{\mathcal{F}\beta} \mathcal{F}\mathcal{F}S. \tag{1}$$

However, in the case of transition systems we get a composed system of type $\mathcal{P}$ and not $\mathcal{P}\mathcal{P}$, and in the case of LTS we get a composed system of type $\mathcal{P}(A^2 \times \mathcal{I}d)$ and not of type $\mathcal{P}(A \times \mathcal{I}d)\mathcal{P}(A \times \mathcal{I}d)$. This is due to the richer structure of $\mathcal{P}$, namely it is a *monad*. Moreover there is a *distributive law* $\pi : (A \times \mathcal{I}d)\mathcal{P} \Longrightarrow \mathcal{P}(A \times \mathcal{I}d)$. Distributive laws have various applications in the theory of coalgebras (c.f. [Bar04]). In this section we shall see how composition of systems can be defined for systems of type $\mathcal{T}\mathcal{F}$ where $\mathcal{T}$ is a monad, with a distributive law $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$. We start by introducing the notions that we need for the sequel.

2.1 Monads and distributive laws

We start by introducing some basic notions and properties.

Definition 2.1. *A monad in a category $\mathbf{C}$ is a triple $\langle T, \eta, \mu \rangle$ where $\mathcal{T}$ is a $\mathbf{C}$ endofunctor, and $\eta : \mathcal{I}d \Longrightarrow \mathcal{T}$, $\mu : \mathcal{T} \circ \mathcal{T} \Longrightarrow \mathcal{T}$ are natural transformations, called the unit and the multiplication, respectively, such that the following diagrams commute.*

$$
\begin{array}{ccccc}
\mathcal{T} & \xRightarrow{\mathcal{T}\eta} & \mathcal{T}^2 & \xLeftarrow{\eta\mathcal{T}} & \mathcal{T} \\
\| & & {\scriptstyle \mu} \big\Downarrow {\scriptstyle (unit\,\mathcal{T})} & & \| \\
{\scriptstyle (unit\,\mathcal{T})} & & & & {\scriptstyle (unit\,\mathcal{T})} \\
& \searrow_{id} & \mathcal{T} & \swarrow_{id} &
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{T}^3 & \xRightarrow{\mathcal{T}\mu} & \mathcal{T}^2 \\
{\scriptstyle \mu\mathcal{T}}\big\Downarrow & {\scriptstyle (mult.\mathcal{T})} & \big\Downarrow{\scriptstyle \mu} \\
\mathcal{T}^2 & \xRightarrow{\mu} & \mathcal{T}
\end{array}
$$

The two parts of the left diagram are the unit laws, and the right diagram is the multiplication law, or the associativity, of the monad.

Example 2.2. A typical example of a monad in Set is the powerset monad $\langle \mathcal{P}, \{\}, \bigcup \rangle$ where $\{\} : \mathcal{I}d \Longrightarrow \mathcal{P}$ is the singleton natural transformation given by $\{\}_X(x) = \{x\}$, and $\bigcup : \mathcal{P}^2 \Longrightarrow \mathcal{P}$ is the union natural transformation given by $\bigcup_X(Y) = \bigcup_{Z \in Y} Z$ for any $Y \in \mathcal{P}\mathcal{P}X$.

The distribution functor can also be equipped with a monad structure, namely $\langle \mathcal{D}, \eta, \mu \rangle$ is a monad for $\eta : \mathcal{I}d \Longrightarrow \mathcal{D}$ being the Dirac natural transformation given by $\eta_X(x) = \mu_x^1$, and the multiplication $\mu : \mathcal{D}^2 \Longrightarrow \mathcal{D}$ is given by $\mu_X(\nu) = \bar{\nu}$ for $\nu \in \mathcal{D}\mathcal{D}X$ and $\bar{\nu} \in \mathcal{D}X$ defined by

$$\bar{\nu}(x) = \sum_{\xi \in \mathcal{D}X} \nu(\xi) \cdot \xi(x).$$

Simple derivations suffice to check that the unit and the multiplication laws hold in this case as well.

Let $\mathcal{F}$ and $\mathcal{G}$ be any endofunctors on a category $\mathbf{C}$. A distributive law of $\mathcal{F}$ over $\mathcal{G}$ is a natural transformation $\lambda : \mathcal{F}\mathcal{G} \Longrightarrow \mathcal{G}\mathcal{F}$. For the sequel we will use distributive laws of a functor over a monad, whose definition we give next.

Definition 2.3. *Let $\mathcal{F}$ be a functor and $\mathcal{T}$ a monad. A plain distributive law of $\mathcal{F}$ over $\mathcal{T}$ is a distributive law of $\mathcal{F}$ over the functor $\mathcal{T}$. A distributive law of $\mathcal{F}$ over the monad $\mathcal{T}$ is a natural transformation $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$ that preserves the monad structure, i.e., the following diagrams commute.*

$$
\begin{array}{ccc}
\mathcal{F}X \xrightarrow{\ \mathcal{F}\eta_X\ } \mathcal{F}\mathcal{T}X & \qquad & \mathcal{F}\mathcal{T}\mathcal{T}X \xrightarrow{\ \lambda_{\mathcal{T}X}\ } \mathcal{T}\mathcal{F}\mathcal{T}X \xrightarrow{\ \mathcal{T}\lambda_X\ } \mathcal{T}\mathcal{T}\mathcal{F}X \\
\quad \searrow_{(a)} \quad \downarrow_{\lambda_X} & \qquad & {\scriptstyle \mathcal{F}\mu_X}\downarrow \qquad\qquad (b) \qquad\qquad \downarrow{\scriptstyle \mu_{\mathcal{F}X}} \\
\eta_{\mathcal{F}X}\quad \mathcal{T}\mathcal{F}X & \qquad & \mathcal{F}\mathcal{T}X \xrightarrow{\qquad\quad \lambda_X \qquad\quad} \mathcal{T}\mathcal{F}X
\end{array}
$$

Example 2.4. Let $\mathcal{F} = \underline{A} \times \mathcal{I}d$ and $\mathcal{T} = \mathcal{D}$ be the distribution monad. Then there exists a distributive law $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$ i.e. $\lambda : \underline{A} \times \mathcal{D} \Longrightarrow \mathcal{D}(\underline{A} \times \mathcal{I}d)$, given by

$$\lambda_X(\langle a, \mu \rangle)(\langle b, s \rangle) = \begin{cases} \mu(s) & a = b \\ 0 & \text{otherwise.} \end{cases}$$

This distributive law preserves the monad structure. Note that $\lambda_X(\langle a, \mu \rangle) = \mu_a^1 \times \mu$, for $\times$ denoting the product of distributions, and μ_a^1 the Dirac distribution for $a \in A$.

When dealing with the powerset monad, a distributive law comes for free, as in the next lemma.

Lemma 2.5. *([Jac04, HJ05b]) Let $\mathcal{F}$ be any weak pullback preserving functor. Then there exists a distributive law $\pi : \mathcal{F}\mathcal{P} \Longrightarrow \mathcal{P}\mathcal{F}$ of $\mathcal{F}$ over the powerset monad $\mathcal{P}$, called a power law. The power law π is given by*

$$\pi_X(v) = \{ u \in \mathcal{F}X \mid \langle u, v \rangle \in \mathrm{Rel}(\mathcal{F})(\in_X) \} \tag{2}$$

for any set X and $v \in \mathcal{F}\mathcal{P}X$. $\qquad\qquad\square$

Example 2.6. In particular, for $\mathcal{F} = \underline{A} \times \mathcal{I}d$, Lemma 2.5 provides us with a power law $\pi : \underline{A} \times \mathcal{P} \Longrightarrow \mathcal{P}(\underline{A} \times \mathcal{I}d)$ which according to the definition of relation lifting for $\underline{A} \times \mathcal{I}d$ is given by

$$\pi_X(\langle a, X' \rangle) = \{ \langle a, x \rangle \mid x \in X' \} \tag{3}$$

for any $a \in A$ and any $X' \subseteq X$.

Recently, Hasuo and Jacobs [HJ05a] have shown existence of a distributive law $\delta : \mathcal{F}\mathcal{D} \Longrightarrow \mathcal{D}\mathcal{F}$ for polynomial functors $\mathcal{F}$, providing a generalization of Example 2.4 in the sense of Lemma 2.5.

Assume $\mathcal{T}$ is a monad, $\mathcal{F}$ a functor, and assume there exists a (plain) distributive law $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$. Following [Jac04] we define families of maps $\lambda_X^n : \mathcal{F}^n\mathcal{T}X \to \mathcal{T}\mathcal{F}^nX$, indexed by sets, for all $n \in \mathbb{N}$ by

$$\lambda_X^0 = id_{\mathcal{T}X}, \quad \lambda_X^{n+1} = \lambda_{\mathcal{F}^nX} \circ \mathcal{F}\lambda_X^n \tag{4}$$

i.e.

$$
\begin{array}{ccc}
\mathcal{F}^{n+1}\mathcal{T}X & \xrightarrow{\ \lambda_X^{n+1}\ } & \mathcal{T}\mathcal{F}^{n+1}X \\
& & \\
\mathcal{F}\lambda_X^n \searrow & (4) & \nearrow \lambda_{\mathcal{F}^nX} \\
& \mathcal{F}\mathcal{T}\mathcal{F}^nX &
\end{array}
$$

Lemma 2.7. *Let* $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$ *be a (plain) distributive law. For all* $n \in \mathbb{N}$, *from (4) we get a (plain) distributive law*

$$\lambda^n : \mathcal{F}^n\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}^n.$$

Proof We first show the naturality of λ^n for $n \in \mathbb{N}$, by induction. For $n = 0$ the statement is trivial saying that $id : \mathcal{T} \Longrightarrow \mathcal{T}$. For $n = 1$ the statement is the naturality of λ. Assume λ^n is natural, and let $f : X \to Y$. Then we have that the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{F}^{n+1}\mathcal{T}X & \xrightarrow{\mathcal{F}^{n+1}\mathcal{T}f} & \mathcal{F}^{n+1}\mathcal{T}Y \\
\mathcal{F}\lambda_X^n \downarrow & \mathcal{F}(\text{nat.}\lambda^n) & \downarrow \mathcal{F}\lambda_Y^n \\
\mathcal{F}\mathcal{T}\mathcal{F}^nX & \xrightarrow{\mathcal{F}\mathcal{T}\mathcal{F}^nf} & \mathcal{F}\mathcal{T}\mathcal{F}^nY \\
\lambda_{\mathcal{F}^nX} \downarrow & (\text{nat.}\lambda) & \downarrow \lambda_{\mathcal{F}^nY} \\
\mathcal{T}\mathcal{F}^{n+1}X & \xrightarrow{\mathcal{T}\mathcal{F}^{n+1}f} & \mathcal{T}\mathcal{F}^{n+1}Y
\end{array}
$$

giving the naturality of λ^{n+1}. It remains to show that if λ preserves the monad structure of $\mathcal{T}$, then λ^n also does. We show this also by induction on n. Again, the case $n = 0$ is trivial, and the case $n = 1$ is the statement for λ. We need to show that both (a) and (b) from Definition 2.3 are satisfied for λ^{n+1} assuming that they are for λ^i if $i \le n$. We obtain (a) from the following diagram

$$
\begin{array}{ccc}
\mathcal{F}^{n+1}X & \xrightarrow{\ \mathcal{F}^{n+1}\eta_X\ } & \mathcal{F}^{n+1}\mathcal{T}X \\
& \mathcal{F}((a)\lambda^n) & \\
\mathcal{F}\eta_{\mathcal{F}^nX} \searrow & \searrow \mathcal{F}\lambda_X^n & \downarrow \lambda_X^{n+1} \\
& \mathcal{F}\mathcal{T}\mathcal{F}^nX \ (\text{def.}\lambda^{n+1}) & \\
& ((a)\lambda) \quad \lambda_{\mathcal{F}^nX} \searrow & \\
\eta_{\mathcal{F}^{n+1}X} & & \mathcal{T}\mathcal{F}^{n+1}X
\end{array}
$$

and (b) from the diagram below

$$
\begin{array}{ccccc}
\mathcal{F}^{n+1}\mathcal{TT}X & \xrightarrow{\ \lambda^{n+1}_{\mathcal{T}X}\ } & \mathcal{TF}^{n+1}\mathcal{T}X & \xrightarrow{\ \mathcal{T}\lambda^{n+1}_{X}\ } & \mathcal{TTF}^{n+1}X
\end{array}
$$

(def.λ^{n+1}) $\quad$ $\mathcal{FTF}^{n}\mathcal{T}X$ $\quad$ (nat.λ) $\quad$ $\mathcal{TFTF}^{n}X$ $\quad$ $\mathcal{T}(\text{def.}\lambda^{n+1})$

$\mathcal{F}\lambda^{n}_{\mathcal{T}X}$, $\lambda_{\mathcal{F}^{n}\mathcal{T}X}$, $\mathcal{TF}\lambda^{n}_{X}$, $\mathcal{T}\lambda_{\mathcal{F}^{n}X}$

$\mathcal{FTT}\mathcal{F}^{n}X$, $\mathcal{FT}\lambda^{n}_{X}$, $\lambda_{\mathcal{TF}^{n}X}$

$\mathcal{F}^{n+1}\mu_{X}$ $\quad$ $\mathcal{F}((\text{b})\lambda^{n})$ $\quad$ $\mathcal{F}\mu_{\mathcal{F}^{n}X}$ $\quad$ $((\text{b})\lambda)$ $\quad$ $\mu_{\mathcal{F}^{n+1}X}$

$\mathcal{FTF}^{n}X$

$\mathcal{F}\lambda^{n}_{X}$ $\quad$ (def.λ^{n+1}) $\quad$ $\lambda_{\mathcal{F}^{n}X}$

$$
\begin{array}{ccc}
\mathcal{F}^{n+1}\mathcal{T}X & \xrightarrow{\ \lambda^{n+1}_{X}\ } & \mathcal{TF}^{n+1}X
\end{array}
$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 2.8. *For λ^{k} defined by (4) and for all natural numbers $n, m \in \mathbb{N}$, it holds that*

$$
\lambda^{n+m}_{X} = \lambda^{n}_{\mathcal{F}^{m}X} \circ \mathcal{F}^{n}\lambda^{m}_{X}. \tag{5}
$$

Proof We prove the property by induction on n. We have

$$
\lambda^{0+m}_{X} = \lambda^{m}_{X} = id \circ \lambda^{m}_{X} = \lambda^{0}_{\mathcal{F}^{m}X} \circ \mathcal{F}^{0}\lambda^{m}_{X}.
$$

We show that if it holds for the pair $\langle n, m \rangle$, then it does for $\langle n+1, m \rangle$, by the commutativity of the following diagram.

$$
\begin{array}{ccc}
\mathcal{F}^{n+m+1}\mathcal{T}X & \xrightarrow{\ \lambda^{n+m+1}_{X}\ } & \mathcal{TF}^{n+m+1}
\end{array}
$$

$\mathcal{F}\lambda^{n+m}_{X}$ $\quad$ (def.λ^{n+m+1}) $\quad$ $\lambda_{\mathcal{F}^{n+m}X}$

$(\mathcal{F}(IH))$ $\quad$ $\mathcal{FTF}^{n+m}X$ $\quad$ (def.λ^{n+1})

$\mathcal{F}\lambda^{n}_{\mathcal{F}^{m}X}$

$\mathcal{F}^{n+1}\lambda^{m}_{X}$ $\quad$ $\mathcal{F}^{n+1}\mathcal{TF}^{m}X$ $\quad$ $\lambda^{n+1}_{\mathcal{F}^{m}X}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Example 2.9. Consider again the setting of Example 2.6, and the given power law $\pi : \mathcal{FP} \Longrightarrow \mathcal{PF}$ for $\mathcal{F} = \underline{A} \times \mathcal{I}d$. Since

$$
(\underline{A} \times \mathcal{I}d)^{n} \cong \underline{A^{n}} \times \mathcal{I}d,
$$

from Lemma 2.8, we get n-fold power law $\pi^{n} : \underline{A^{n}} \times \mathcal{P} \Longrightarrow \mathcal{P}(\underline{A^{n}} \times \mathcal{I}d)$, for each $n \in \mathbb{N}$. According to Equation (4), we can derive that it is given by

$$
\pi^{n}_{X}(\langle w, X' \rangle) = \{ \langle w, x \rangle \mid x \in X' \}
$$

for $w \in A^{n}$ and $X' \subseteq X$.

2.2 Composition

We can now define composition of coalgebras of type $\mathcal{TF}$ for a monad $\mathcal{T}$ with a distributive law $\lambda : \mathcal{FT} \Longrightarrow \mathcal{TF}$.

Let S be a given set. We consider the set of all systems with carrier set S of type $\mathcal{TF}^n$, for some $n \in \mathbb{N}$. Let $\langle S, \alpha \rangle$ and $\langle S, \beta \rangle$ be two such systems, $\alpha : S \to \mathcal{TF}^k S$, $\beta : S \to \mathcal{TF}^m S$. We define

$$\langle S, \gamma \rangle = \langle S, \alpha \rangle \cdot \langle S, \beta \rangle$$

for $\gamma : S \to \mathcal{TF}^{k+m} S$ as given by the following diagram

$$
S \xrightarrow{\alpha} \mathcal{TF}^k S \xrightarrow{\mathcal{TF}^k \beta} \mathcal{TF}^k \mathcal{TF}^m S \xrightarrow{\mathcal{T}\lambda^k_{\mathcal{F}^m S}} \mathcal{T}^2 \mathcal{F}^{k+m} S \qquad (6)
$$

with γ and $\mu_{\mathcal{F}^{k+m} S}$ down to $\mathcal{TF}^{k+m} S$.

The system $\langle S, \gamma \rangle$ is called the composition of $\langle S, \alpha \rangle$ and $\langle S, \beta \rangle$. When the carrier set is clear from the context, we shall often just write $\gamma = \alpha \cdot \beta$. Note that if $\mathcal{T}$ is the identity monad, then one obtains the obvious definition of composition as in (1).

The next lemma shows that the composition is a monoid operation.

Lemma 2.10. *Let $\mathcal{T}$ be a monad, $\mathcal{F}$ a functor, and assume that there exists a distributive law $\lambda : \mathcal{FT} \Longrightarrow \mathcal{TF}$. The following hold.*

  (i) *The composition of systems is associative.*

  (ii) *The system $\langle S, \eta_S : S \to \mathcal{T}S \rangle$ is a unit for the composition of systems.*

Proof

  (i) Let $\langle S, \alpha \rangle, \langle S, \beta \rangle$ and $\langle S, \gamma \rangle$ be such that $\alpha : S \to \mathcal{TF}^k S$, $\beta : S \to \mathcal{TF}^m S$ and $\gamma : S \to \mathcal{TF}^l S$. By the definition of the composition we have

$$
\begin{aligned}
(\alpha \cdot \beta) \cdot \gamma &= (\mu_{\mathcal{F}^{k+m} S} \circ \mathcal{T}\lambda^k_{\mathcal{F}^m S} \circ \mathcal{TF}^k \beta \circ \alpha) \cdot \gamma \\
&= \mu_{\mathcal{F}^{k+m+l} S} \circ \mathcal{T}\lambda^{k+m}_{\mathcal{F}^l S} \circ \mathcal{TF}^{k+m} \gamma \circ \mu_{\mathcal{F}^{k+m} S} \circ \mathcal{T}\lambda^k_{\mathcal{F}^m S} \circ \mathcal{TF}^k \beta \circ \alpha
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha \cdot (\beta \cdot \gamma) &= \alpha \cdot (\mu_{\mathcal{F}^{m+l} S} \circ \mathcal{T}\lambda^m_{\mathcal{F}^l S} \circ \mathcal{TF}^m \gamma \circ \beta) \\
&= \mu_{\mathcal{F}^{k+m+l} S} \circ \mathcal{T}\lambda^k_{\mathcal{F}^{m+l} S} \circ \mathcal{TF}^k \mu_{\mathcal{F}^{m+l} S} \circ \mathcal{TF}^k \mathcal{T}\lambda^m_{\mathcal{F}^l S} \\
&\quad \circ \mathcal{TF}^k \mathcal{TF}^m \gamma \circ \mathcal{TF}^k \beta \circ \alpha
\end{aligned}
$$

Hence, the associativity is a consequence of the commutativity of the following diagram.

$$\mathcal{T}\mathcal{F}^k\mathcal{T}\mathcal{F}^m S \xrightarrow{\ \mathcal{T}\lambda^k_{\mathcal{F}^m S}\ } \mathcal{T}^2\mathcal{F}^{k+m}S \xrightarrow{\ \mu_{\mathcal{F}^{k+m}S}\ } \mathcal{T}\mathcal{F}^{k+m}S$$

(with vertical maps)
$\mathcal{T}\mathcal{F}^k\mathcal{T}\mathcal{F}^m\gamma$, $\mathcal{T}(\mathrm{nat}.\lambda^k)$, $\mathcal{T}^2\mathcal{F}^{k+m}\gamma$, $(\mathrm{nat}.\mu)$, $\mathcal{T}\mathcal{F}^{k+m}\gamma$

$$\mathcal{T}\mathcal{F}^k\mathcal{T}\mathcal{F}^m\mathcal{T}\mathcal{F}^l S \xrightarrow{\ \mathcal{T}\lambda^k_{\mathcal{F}^m\mathcal{T}\mathcal{F}^l S}\ } \mathcal{T}^2\mathcal{F}^{k+m}\mathcal{T}\mathcal{F}^l S \xrightarrow{\ \mu_{\mathcal{F}^{k+m}\mathcal{T}\mathcal{F}^l S}\ } \mathcal{T}\mathcal{F}^{k+m}\mathcal{T}\mathcal{F}^l S$$

$\mathcal{T}\mathcal{F}^k\mathcal{T}\lambda^m_{\mathcal{F}^l S}$, $\mathcal{T}(\mathrm{nat}.\lambda^k)$, $\mathcal{T}^2\mathcal{F}^k\lambda^m_{\mathcal{F}^l S}$, $\mathcal{T}(4)$, $\mathcal{T}^2\lambda^{k+m}_{\mathcal{F}^l S}$, $(\mathrm{nat}.\mu)$, $\mathcal{T}\lambda^{k+m}_{\mathcal{F}^l S}$

$$\mathcal{T}\mathcal{F}^k\mathcal{T}^2\mathcal{F}^{m+l}S \xrightarrow{\ \mathcal{T}\lambda^k_{\mathcal{T}\mathcal{F}^{m+l}S}\ } \mathcal{T}^2\mathcal{F}^k\mathcal{T}\mathcal{F}^{m+l}S \xrightarrow{\ \mathcal{T}^2\lambda^k_{\mathcal{F}^{m+l}S}\ } \mathcal{T}^3\mathcal{F}^{k+m+l}S \xrightarrow{\ \mu_{\mathcal{T}\mathcal{F}^{k+m+l}S}\ } \mathcal{T}^2\mathcal{F}^{k+m+l}S$$

$\mathcal{T}\mathcal{F}^k\mu_{\mathcal{F}^{m+l}S}$, $\mathcal{T}((\mathrm{b})\lambda^k)$, $\mathcal{T}\mu_{\mathcal{F}^{k+m+l}S}$, $(\mathrm{mult}.\mathcal{T})$, $\mu_{\mathcal{F}^{k+m+l}S}$

$$\mathcal{T}\mathcal{F}^k\mathcal{T}\mathcal{F}^{m+l}S \xrightarrow{\ \mathcal{T}\lambda^k_{\mathcal{F}^{m+l}S}\ } \mathcal{T}^2\mathcal{F}^{k+m+l}S \xrightarrow{\ \mu_{\mathcal{F}^{k+m+l}S}\ } \mathcal{T}\mathcal{F}^{k+m+l}S$$

(ii) Let $\langle S,\alpha\rangle$ be such that $\alpha : S \to \mathcal{T}\mathcal{F}^k S$. The system $\langle S,\eta_S\rangle$ is a right unit since

$$S \xrightarrow{\ \alpha\ } \mathcal{T}\mathcal{F}^k S \xrightarrow{\ \mathcal{T}\mathcal{F}^k\eta_S\ } \mathcal{T}\mathcal{F}^k\mathcal{T}S$$

$\mathcal{T}((\mathrm{a})\lambda^k)$, $\mathcal{T}\lambda^k_S$, $\mathcal{T}\eta_{\mathcal{F}^k S}$, $\mathcal{T}^2\mathcal{F}^k S$, $(\mathrm{unit}\ \mathcal{T})$, $\mu_{\mathcal{F}^k S}$, id, $\alpha\cdot\eta_S$, $\mathcal{T}\mathcal{F}^k S$

and it is a left unit since

$$\eta_S\cdot\alpha$$
$$S \xrightarrow{\ \eta_S\ } \mathcal{T}S$$

α, $(\mathrm{nat}.\eta)$, $\mathcal{T}\alpha$, $(\mathrm{def.com.})$

$$\mathcal{T}\mathcal{F}^k S \xrightarrow{\ \eta_{\mathcal{T}\mathcal{F}^k S}\ } \mathcal{T}^2\mathcal{F}^k S \xrightarrow{\ \mu_{\mathcal{F}^k S}\ } \mathcal{T}\mathcal{F}^k S$$

$(\mathrm{unit}\ \mathcal{T})$, id

$\square$

Exponentiation of systems can also be defined, in the usual way: Let $\langle S,\alpha\rangle$ be a system of type $\mathcal{T}\mathcal{F}^n$. Then $\alpha^0 = \eta_S$ and $\alpha^{n+1} = \alpha^n \cdot \alpha$. By the associativity of the composition we have $\alpha^{n+m} = \alpha^n \cdot \alpha^m$, for any $n,m \in \mathbb{N}$.

Remark 2.11. We note that the definition of composition as well as Lemma 2.10 are related to Kleisli categories. For a monad $\mathcal{T} = \langle \mathcal{T}, \eta, \mu \rangle$, by $\mathsf{Set}_\mathcal{T}$ we denote the Kleisli category associated to $\mathcal{T}$. Its objects are sets and morphisms $f : X \to_\mathcal{T} Y$ are functions $f : X \to \mathcal{T}Y$. The identity morphism is then η_X for any set X and two morphisms $f : X \to_\mathcal{T} Y$ and $g : Y \to_\mathcal{T} Z$ compose to a morphism $g \circ_\mathcal{T} f : X \to_\mathcal{T} Z$ given by

$$g \circ_\mathcal{T} f = \mu_Z \circ \mathcal{T}g \circ f.$$

Given a Set endofunctor $\mathcal{F}$ with a distributive law $\lambda : \mathcal{F}\mathcal{T} \Longrightarrow \mathcal{T}\mathcal{F}$, we can lift $\mathcal{F}$ to an endofunctor $\mathcal{F}_\mathcal{T}$ on the Kleisli category which acts as follows

$$\mathcal{F}_\mathcal{T}(X) = \mathcal{F}X \qquad \mathcal{F}_\mathcal{T}(f) = \lambda_Y \circ \mathcal{F}f$$

for $f : X \to_\mathcal{T} Y$ a morphism in $\mathsf{Set}_\mathcal{T}$. By Lemma 2.7 and Lemma 2.8 also $\mathcal{F}^k$ lifts to a functor $\mathcal{F}_\mathcal{T}^k$ on the Kleisli category. Then the composition of coalgebras $\alpha \cdot \beta$ corresponds to composing some morphisms in $\mathsf{Set}_\mathcal{T}$ in particular $\alpha \cdot \beta = \mathcal{F}_\mathcal{T}^k \beta \circ_\mathcal{T} \alpha$. Hence, the proof of Lemma 2.10 could also be given via the Kleisli category. There is only an obligation to prove that $\mathcal{F}^k$ lifts in $\mathsf{Set}_\mathcal{T}$ to the exponent of the lifting of $\mathcal{F}$, i.e.

$$\mathcal{F}_\mathcal{T}^k = (\mathcal{F}_\mathcal{T})^k$$

which can be done by induction, by the definition of λ^k and by Lemma 2.8.

We next provide examples that show how the composition is defined for LTSs and for generative probabilistic systems.

Example 2.12. The functor defining the LTSs is of a form $\mathcal{T}\mathcal{F}$ for $\mathcal{T} = \mathcal{P}$, the powerset monad, and $\mathcal{F} = \underline{A} \times \mathcal{I}d$. By Lemma 2.5 the composition (and exponentiation) is defined for LTSs. Moreover, since

$$(\underline{A^k} \times \mathcal{I}d) \circ (\underline{A^m} \times \mathcal{I}d) \cong (\underline{A^{k+m}} \times \mathcal{I}d) \tag{7}$$

if $\langle S, \alpha : S \to \mathcal{P}(\underline{A^k} \times \mathcal{I}d) \rangle$ and $\langle S, \beta : S \to \mathcal{P}(\underline{A^m} \times \mathcal{I}d) \rangle$, then $\langle S, \alpha \cdot \beta : S \to \mathcal{P}(\underline{A^{k+m}} \times \mathcal{I}d) \rangle$. Some derivations suffice to see that

$$(\alpha \cdot \beta)(s) = \{\langle uv, t \rangle \mid \exists r \in S : \langle u, r \rangle \in \alpha(s), \langle v, t \rangle \in \beta(r)\}$$

i.e., $s \xrightarrow{uv} t$ in $\langle S, \alpha \cdot \beta \rangle$ if and only if there exists $r \in S$ such that $s \xrightarrow{u} r$ in $\langle S, \alpha \rangle$ and $r \xrightarrow{v} t$ in $\langle S, \beta \rangle$, for arbitrary state $s \in S$ and arbitrary words $u \in A^k$, $v \in A^m$.

Example 2.13. The generative probabilistic systems are defined by the functor $\mathcal{D}(\underline{A} \times \mathcal{I}d)$ i.e., by a functor $\mathcal{T}\mathcal{F}$ for $\mathcal{F}$ as in the previous example, and $\mathcal{T} = \mathcal{D}$ being the distribution monad. Example 2.4 provides also a distributive law of $\mathcal{F}$ over the monad $\mathcal{T}$. Hence, composition and exponentiation of generative probabilistic systems is also defined. Using the isomorphism (7), and the definition

of composition, after some derivations we get that if $\langle S, \alpha : S \to \mathcal{D}(\underline{A^k} \times \mathcal{I}d)\rangle$
and $\langle S, \beta : S \to \mathcal{D}(\underline{A^m} \times \mathcal{I}d)\rangle$, then $\langle S, \alpha \cdot \beta : S \to \mathcal{D}(\underline{A^{k+m}} \times \mathcal{I}d)\rangle$ is given by

$$(\alpha \cdot \beta)(s)(uv, t) = \sum_{r \in S} \alpha(s)(u, r) \cdot \beta(r)(v, t)$$

for $u \in A^k, v \in A^m$ and $s, t \in S$.

3   Paths

In this section we investigate possible definitions of paths for coalgebras. The
case of LTS makes us believe that having a good definition of paths brings pos-
sibilities of defining various semantic relations.

Assume we have a system $\langle S, \alpha \rangle$ of type $\mathcal{F}$. A (finite) path could be a sequence
of transitions

$$s_0 \ \alpha(s_0) \ s_1 \ \alpha(s_1) \ s_2 \cdots s_{n-1} \ \alpha(s_{n-1}) \ s_n \tag{8}$$

where, each s_i is "reachable" from $\alpha(s_{i-1})$ for $i \geq 1$. In case of transition
systems, i.e. the powerset functor, it is intuitively clear that reachable means
"belongs to" i.e. we require $s_i \in \alpha(s_{i-1})$.

Still, the usual notions of paths for TS and LTS are *linear*, being sequences of
states and actions, unlike those from (8). In an LTS a path is an alternating
sequence of states and actions, usually represented as

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{n-1} \xrightarrow{a_n} s_n.$$

Similarly, in a transition system a path is only a sequence of states

$$s_0 \to s_1 \to s_2 \cdots \to s_n.$$

Moreover, in a simple Segala system [Seg95b], a path is a sequence

$$s_0 \xrightarrow{a_1} \mu_1 \ s_1 \xrightarrow{a_2} \mu_2 \ s_2 \cdots s_{n-1} \xrightarrow{a_n} \mu_n \ s_n$$

where $s_i \in \text{supp}(\mu_i)$. This definition of a path is semi-linear. On the one hand,
it exploits the similarity with LTS and therefore shows linearity and, on the
other hand, it involves whole distributions over states.

For generative probabilistic systems the usual notion of a path (see e.g. [BH97,
SVW05]) is also linear. A path in a generative system is an alternating sequence

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{n-1} \xrightarrow{a_n} s_n$$

such that the probability of performing each transition $s_{i-1} \xrightarrow{a_i} s_i$ is greater
than 0, for $i \geq 1$. This is very much different than what comes out of (8). The

advantage of the used definition is that it is indeed linear, the disadvantage is that it looses probabilistic information. The behavior of the state is no longer determined by the set of paths, nor by the set of paths of length one, in contrast to the LTS case.

In this section we will present some general observations that show, for example, the general principles that lead to this linear but incomplete definition of paths for generative systems.

Jacobs [Jac04] defines paths in a semi-linear fashion for systems of type $\mathcal{PF}$, where $\mathcal{F}$ preserves weak pullbacks, as sequences

$$\langle u_0, u_1, \ldots, u_n \rangle \in \prod_{i=0}^{n} \mathcal{F}^i S$$

such that for $i \geq 0$ we have

$$\langle u_{i+1}, u_i \rangle \in \operatorname{Rel}(\mathcal{F})^i ((id \times \alpha)^{-1} \in_{\mathcal{F}S}). \tag{9}$$

Jacobs' definition of a path implicitly uses the existing power law and the fact that $\in_X$ is the reachability relation.

In general, let $\mathcal{T}$ be a monad and $\mathcal{F}$ a functor, with a distributive law $\lambda : \mathcal{FT} \Longrightarrow \mathcal{TF}$. Moreover, let a family of reachability relations be given $\mathcal{R} = \{R_X\}$ where $R_X \subseteq X \times \mathcal{T}X$ for any set X. Intuitively, this is how we form paths of $\langle S, \alpha : S \to \mathcal{TFS} \rangle$ w.r.t. $\mathcal{R}$: We pick up a first element $u_0 \in S$. We apply the transition function to it and get $\alpha(u_0) \in \mathcal{TFS}$. We pick up a next element $u_1 \in \mathcal{FS}$ such that $\langle u_1, \alpha(u_0) \rangle \in R_{\mathcal{FS}}$. Then we continue from u_1, we apply now $\mathcal{F}\alpha$ to it in order to get its transitions and we land in $\mathcal{F}\alpha(u_1) \in \mathcal{FTFS}$. An application of the distributive law gets us back on the right track, and we have $\lambda_{\mathcal{FS}}(\mathcal{F}\alpha(u_1)) \in \mathcal{TF}^2 S$. At this point we are in position to again pick up a next element for our path. We choose $u_2 \in \mathcal{F}^2 S$ such that $\langle u_2, \lambda_{\mathcal{FS}}(\mathcal{F}\alpha(u_1)) \rangle \in R_{\mathcal{F}^2 S}$. Proceeding this way, we build possible paths.

Hence, paths for systems of type $\mathcal{TF}$ with $\mathcal{T}$ being a monad and a corresponding distributive law, w.r.t a family $\mathcal{R}$ are sequences

$$\langle u_0, u_1, \ldots, u_n \rangle \in \prod_{i=0}^{n} \mathcal{F}^i S$$

such that for all $i = 0, \ldots, n-1$ we have

$$\langle u_{i+1}, u_i \rangle \in (id \times \lambda_{\mathcal{FS}}^i \mathcal{F}^i \alpha)^{-1} R_{\mathcal{F}^{i+1}S} = R^{\alpha,i}. \tag{10}$$

Remark 3.1. We note that Jacobs' paths are indeed paths according to (10) in case of the powerset monad and the membership relations. Let $\mathcal{T} = \mathcal{P}$, π^n

the n-fold power law, and $R_X = \in_X$. Moreover, let $\langle S, \alpha \rangle$ be a $\mathcal{P}\mathcal{F}$ coalgebra, for a weak pullback preserving functor $\mathcal{F}$. Then

$$\mathrm{Rel}(\mathcal{F})^i(\in_{\mathcal{F}S}) = (id \times \pi^i_{\mathcal{F}S})^{-1}(\in_{\mathcal{F}^{i+1}S}) \qquad (11)$$

as can be derived from [Jac04, Lemma 4.2]. This implies that

$$
\begin{aligned}
\mathrm{Rel}(\mathcal{F})^i((id \times \alpha)^{-1}(\in_{\mathcal{F}S})) \quad &\overset{(*)}{=} \quad (id \times \mathcal{F}^i \alpha)^{-1}\, \mathrm{Rel}(\mathcal{F})^i(\in_{\mathcal{F}S}) \\
&\overset{(11)}{=} \quad (id \times \mathcal{F}^i \alpha)^{-1}(id \times \pi^i_{\mathcal{F}S})^{-1}(\in_{\mathcal{F}^{i+1}S}) \\
&= \quad (id \times \pi^i_{\mathcal{F}S}\mathcal{F}^i \alpha)^{-1}(\in_{\mathcal{F}^{i+1}S})
\end{aligned}
$$

where the equality $(*)$ holds by the properties of relation liftings (see [Jac02, JH03]).

Hence, both notions of paths coincide for LTS, and they also correspond to the usual notion of paths for LTS, as shown in the next example.

Example 3.2. Let $\mathcal{T} = \mathcal{P}$, $\mathcal{F} = \underline{A} \times \mathcal{I}d$ and $R_X = \in_X$. Let π^n be the distributive laws from Example 2.9. A sequence $\langle u_0, \ldots, u_n \rangle$ is a path, $u_i \in A^i \times S$ if for all $i \geq 0$

$$\langle u_{i+1}, u_i \rangle \in (id \times \pi^i_{\mathcal{F}S}\mathcal{F}^i \alpha)^{-1} R_{\mathcal{F}^{i+1}S}.$$

i.e.

$$u_{i+1} \in \pi^i_{\mathcal{F}S}((\underline{A}^i \times \alpha)(u_i)).$$

Now, since $u_i = \langle w_i, s_i \rangle \in A^i \times S$, and $(\underline{A}^i \times \alpha)(u_i) = \langle w_i, \alpha(s_i) \rangle$, from Example 2.9, we get

$$
\begin{aligned}
\pi^i_{\mathcal{F}S}(\langle w_i, \alpha(s_i) \rangle) \quad &= \quad \{\langle w_i, u \rangle \mid u \in \alpha(s_i)\} \\
&= \quad \{\langle w_i, \langle a, s' \rangle \rangle \mid \langle a, s' \rangle \in \alpha(s_i)\} \\
&\cong \quad \{\langle w_i \cdot a, s' \rangle \mid s \overset{a}{\longrightarrow} s'\}.
\end{aligned}
$$

Hence, $\langle u_0, \ldots, u_n \rangle$ is a path, $u_i = \langle w_i, s_i \rangle \in A^i \times S$ if and only if for all $i \geq 0$ it holds that $w_{i+1} = w_i \cdot a$ for some $a \in A$ and $s_i \overset{a}{\longrightarrow} s_{i+1}$.

The definition of paths for systems of type $\mathcal{T}\mathcal{F}$ depends on a family of relations $\mathcal{R}$. We do not know what characterizes a good family $\mathcal{R} = \{R_X \subseteq X \times \mathcal{T}X\}$ of relations for reachability. In any case, every such family of relations should satisfy the following condition. For any natural number n, any system $\langle S, \alpha \rangle$ of type $\mathcal{T}\mathcal{F}$ and any state $s \in S$

$$(id \times \alpha^n)^{-1}(R_{\mathcal{F}^n S}) \cap \mathcal{F}^n S \times \{s\} = R^{\alpha,n} \circ \cdots \circ R^{\alpha,1} \cap \mathcal{F}^n S \times \{s\} \qquad (12)$$

The condition provides a link between the notion of composition of systems and the notion of a path. It can be rewritten to

$$\{u \mid \langle u, \alpha^n(s) \rangle \in R_{\mathcal{F}^n S}\} =$$
$$\{u \mid \exists u_1, \ldots, u_{n-1} \colon \langle s, u_1, \ldots, u_{n-1}, u \rangle \text{ is a path in } \langle S, \alpha \rangle \text{ w.r.t } R_X\}$$

expressing that reachable elements from $\alpha^n(s)$ are exactly those that can be reached by a path of length n.

We next show that for $\mathcal{T}$ being a submonad of $\mathcal{P}$ in the sense that there exists a natural transformation $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$, families of reachability relations come naturally.

3.1 Submonads of $\mathcal{P}$

Since the powerset monad, together with the family of membership relations, seems to play a special role in defining paths of systems, it makes sense to study in more detail submonads of $\mathcal{P}$ i.e. monads that can be naturally mapped to the powerset monad.

Lemma 3.3. *The following are equivalent:*

(i) *The monad $\mathcal{T}$ is submonad of $\mathcal{P}$, i.e. there exists a natural transformation $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$.*

(ii) *There exists a family $\mathcal{R} = \{R_X \subseteq X \times \mathcal{T}X\}$ of relations indexed by sets such that for all sets X, Y and all $f : X \to Y$*

$$(f \times id)R_X = (id \times \mathcal{T}f)^{-1}R_Y. \tag{13}$$

Proof

$(i) \implies (ii)$ If $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$, then we can define a family of relations

$$R_X = (id \times \sigma_X)^{-1}(\in_X) \subseteq X \times \mathcal{T}X \tag{14}$$

which satisfies (13) since for any $y \in Y$ and $u \in \mathcal{T}X$

$$
\begin{aligned}
\langle y, u \rangle \in (id \times \mathcal{T}f)^{-1}((id \times \sigma_Y)^{-1}(\in_Y)) & \\
\iff \quad & \langle y, (\mathcal{T}f)u \rangle \in (id \times \sigma_Y)^{-1}(\in_Y) \\
\iff \quad & \langle y, \sigma_Y((\mathcal{T}f)u) \rangle \in \in_Y \\
\iff \quad & y \in \sigma_Y((\mathcal{T}f)u) \\
\overset{(\mathrm{nat.}\sigma)}{\iff} \quad & y \in (\mathcal{P}f)(\sigma_X(u)) = f(\sigma_X(u)) \\
\iff \quad & \langle y, u \rangle = \langle f(x), u \rangle \wedge x \in \sigma_X(u) \\
\iff \quad & \langle y, u \rangle \in (f \times id)(id \times \sigma_X)^{-1}(\in_X).
\end{aligned}
$$

$(ii) \implies (i)$ If $R_X \subseteq X \times \mathcal{T}X$ is a family of relations with the property (13), then we define $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$ by

$$\sigma_X : \mathcal{T}X \to \mathcal{P}X, \quad \sigma_X(u) = \{x \in X \mid \langle x, u \rangle \in R_X\}. \tag{15}$$

We have, for $u \in \mathcal{T}X$,

$$
\begin{aligned}
\sigma_Y(\mathcal{T}f(u)) &= \{y \in Y \mid \langle y, \mathcal{T}f(u) \rangle \in R_Y\} \\
&\overset{(13)}{=} \{y \in Y \mid \langle y, u \rangle \in (f \times id)(R_X)\} \\
&= \{y \in f(X) \mid y = f(x) \wedge \langle x, u \rangle \in R_X\} \\
&= \{f(x) \in f(X) \mid \langle x, u \rangle \in R_X\} \\
&= f(\sigma_X(u)) \\
&= (\mathcal{P}f)(\sigma_X(u)).
\end{aligned}
$$

$\square$

In the proof above, if σ satisfying (13) exists, then we call the family of relations R_X defined by (14), the family associated to σ. Conversely, if R_X exists satisfying (13), then we say that the natural transformation σ from (15) is associated to the family. These assignments are inverses to each other. Assume σ exists, R_X is defined by (14) and a natural transformation σ' is defined by (15) associated to R_X. Then

$$
\begin{aligned}
\sigma'_X(u) &= \{x \in X \mid \langle x, u \rangle \in R_X\} \\
&= \{x \in X \mid \langle x, \sigma_X(u) \rangle \in \in_X\} \\
&= \{x \in X \mid x \in \sigma_X(u)\} \\
&= \sigma_X(u).
\end{aligned}
$$

The opposite also holds, if R_X exists, σ is associated to it by (15), and a family of relations R'_X is associated to σ by (14), then

$$
\begin{aligned}
R'_X &= \{\langle x, u \rangle \mid \langle x, \sigma_X(u) \rangle \in \in_X\} \\
&= \{\langle x, u \rangle \mid x \in \sigma_X(u)\} \\
&= \{\langle x, u \rangle \mid \langle x, u \rangle \in R_X\} \\
&= R_X.
\end{aligned}
$$

Moreover, the condition (13) implies that

$$
(f \times \mathcal{T}f)R_X \subseteq R_Y \tag{16}
$$

which is equivalent to the condition that the family $\mathcal{R} = \{R_X\}$ is functorial, i.e. $\mathcal{R}$ determines a functor making the following diagram commute.

The next lemma shows that a family of relations associated to a submonad of $\mathcal{P}$ can be used instead of the membership family, in case the submonad natural transformation is a monad map. We first define the notion of a monad map.

Definition 3.4. *Let $\langle \mathcal{T}, \eta^{\mathcal{T}}, \mu^{\mathcal{T}} \rangle$ and $\langle \mathcal{M}, \eta^{\mathcal{M}}, \mu^{\mathcal{M}} \rangle$ be two monads. A monad map from $\mathcal{T}$ to $\mathcal{M}$ is a natural transformation $\lambda : \mathcal{T} \implies \mathcal{M}$ such that it preserves the monad structures, i.e., the following two diagrams commute.*

$$
\begin{array}{ccc}
X \xrightarrow{\ \eta^{\mathcal{T}}_X\ } \mathcal{T}X & \qquad & \mathcal{T}\mathcal{T}X \xrightarrow{\ \lambda_{\mathcal{T}X}\ } \mathcal{M}\mathcal{T}X \xrightarrow{\ \mathcal{M}\lambda_X\ } \mathcal{M}\mathcal{M}X \\
\searrow_{(c)} \quad \downarrow \lambda_X & & \mu^{\mathcal{T}}_X \downarrow \qquad (d) \qquad \downarrow \mu^{\mathcal{M}}_X \\
\ \ \eta^{\mathcal{M}}_X \searrow \mathcal{M}X & & \mathcal{T}X \xrightarrow{\hspace{3cm} \lambda_X \hspace{3cm}} \mathcal{M}X
\end{array}
$$

Lemma 3.5. *Let $\sigma : \mathcal{T} \implies \mathcal{P}$ be a monad map and let $\mathcal{R}$ be the family associated to σ, i.e. $R_X = (\mathrm{id} \times \sigma_X)^{-1}(\in_X)$. Let $\lambda : \mathcal{FT} \implies \mathcal{TF}$ be a distributive law. Then $\mathcal{R}$ satisfies condition (12).*

Proof Assume $\sigma : \mathcal{T} \implies \mathcal{P}$ is a monad map, i.e., according to Definition 3.4, the next diagrams commute

$$
\begin{array}{ccc}
X \xrightarrow{\ \eta_X\ } \mathcal{T}X & \qquad & \mathcal{T}\mathcal{T}X \xrightarrow{\ \sigma_{\mathcal{T}X}\ } \mathcal{P}\mathcal{T}X \xrightarrow{\ \mathcal{P}\sigma_X\ } \mathcal{P}\mathcal{P}X \\
\searrow_{(c)} \quad \downarrow \sigma_X & & \mu_X \downarrow \qquad (d) \qquad \downarrow \bigcup_X \\
\ \ \{\}_X \searrow \mathcal{P}X & & \mathcal{T}X \xrightarrow{\hspace{3cm} \sigma_X \hspace{3cm}} \mathcal{P}X
\end{array}
$$

where η and μ are the unit and the multiplication of the monad $\mathcal{T}$. We prove condition (12) by induction on n. Let $\langle S, \alpha \rangle$ be a $\mathcal{TF}$ system and $s \in S$. For $n = 0$ we have

$$
\begin{aligned}
\{u \mid \langle u, \alpha^0(s) \rangle \in R_S\} &= \{u \mid \langle u, \eta_S(s) \rangle \in R_S\} \\
&= \{u \mid \langle u, \sigma_S(\eta_S(s)) \rangle \in \in_S\} \\
&\overset{(c)}{=} \{u \mid u \in \{s\}\} \\
&= \{s\}
\end{aligned}
$$

and $\{s\}$ is exactly the set of all elements of S that can be reached by a path of length 0 w.r.t R_X, starting in s.

Assume (12) holds for n. Note that

$$
\{u \mid \exists u_1, \ldots, u_n \colon \langle s, u_1, \ldots, u_{n-1}, u_n, u \rangle \text{ is a path in } \langle S, \alpha \rangle \text{ w.r.t } R_X\}
$$

$$
\begin{aligned}
&= \{u \mid \exists u_1, \ldots, u_{n-1}, u_n \colon \langle s, u_1, \ldots, u_{n-1}, u_n \rangle \text{ is a path in } \langle S, \alpha \rangle \text{ w.r.t } R_X \\
&\quad \text{and } \langle u, u_n \rangle \in R^{\alpha, n}\} \tag{17} \\
&\overset{(IH)}{=} \{u \mid \exists u_n : \langle u_n, \alpha^n(s) \rangle \in R_{\mathcal{F}^n X} \wedge \langle u, u_n \rangle \in R^{\alpha, n}\}
\end{aligned}
$$

From the commutativity of the following diagram

$$
\begin{array}{c}
X \xrightarrow{\quad\alpha^{n+1}\quad} \\
\end{array}
$$

we get that, for a monad map σ

$$
\sigma_{\mathcal{F}^{n+1}X} \circ \alpha^{n+1} = \bigcup\nolimits_{\mathcal{F}^{n+1}X} \circ \mathcal{P}\sigma_{\mathcal{F}^{n+1}X} \circ \mathcal{P}\lambda^n_{\mathcal{F}X} \circ \mathcal{P}\mathcal{F}^n\alpha \circ \sigma_{\mathcal{F}^nX} \circ \alpha^n \qquad (18)
$$

and furthermore

$$\langle u, \alpha^{n+1}(s)\rangle \in R_{\mathcal{F}^{n+1}S}$$

$$\iff \quad \langle u, \sigma_{\mathcal{F}^{n+1}S} \circ \alpha^{n+1}(s)\rangle \in \in_{\mathcal{F}^{n+1}S}$$

$$\iff \quad u \in \sigma_{\mathcal{F}^{n+1}S} \circ \alpha^{n+1}(s)$$

$$\overset{(18)}{\iff} \quad u \in \bigcup\nolimits_{\mathcal{F}^{n+1}S} \circ \mathcal{P}\sigma_{\mathcal{F}^{n+1}S} \circ \mathcal{P}\lambda^n_{\mathcal{F}S} \circ \mathcal{P}\mathcal{F}^n\alpha \circ \sigma_{\mathcal{F}^nS} \circ \alpha^n(s)$$

$$\overset{(*)}{\iff} \quad \exists u_n \in \mathcal{F}^nS : u_n \in \sigma_{\mathcal{F}^nS} \circ \alpha^n(s) \wedge u \in \sigma_{\mathcal{F}^{n+1}S} \circ \lambda^n_{\mathcal{F}S} \circ \mathcal{F}^n\alpha(u_n)$$

$$\iff \quad \exists u_n \in \mathcal{F}^nS : \langle u_n, \alpha^n(s)\rangle \in R_{\mathcal{F}^nX} \wedge$$
$$\quad\quad \langle u, \lambda^n_{\mathcal{F}S} \circ \mathcal{F}^n\alpha(u_n)\rangle \in (id \times \sigma_{\mathcal{F}^{n+1}S})^{-1}(\in_{\mathcal{F}^{n+1}S})$$

$$\iff \quad \exists u_n \in \mathcal{F}^nS : \langle u_n, \alpha^n(s)\rangle \in R_{\mathcal{F}^nX} \wedge$$
$$\quad\quad \langle u, \lambda^n_{\mathcal{F}S} \circ \mathcal{F}^n\alpha(u_n)\rangle \in R_{\mathcal{F}^{n+1}S}$$

$$\iff \quad \exists u_n \in \mathcal{F}^nS : \langle u_n, \alpha^n(s)\rangle \in R_{\mathcal{F}^nX} \wedge \langle u, u_n\rangle \in R^{\alpha,n}$$

$$\overset{(17)}{\iff} \quad \exists u_1, \ldots, u_n : \langle s, u_1, \ldots, u_{n-1}, u_n, u\rangle \text{ is a path in}$$
$$\quad\quad \langle S, \alpha\rangle \text{ w.r.t } R_X$$

where the equivalence marked with $(*)$ holds by the definition of the powerset functor. This completes the proof. $\square$

Example 3.6. We have a natural transformation supp $: \mathcal{D} \Longrightarrow \mathcal{P}$ mapping any distribution to its support set, i.e., $\mathrm{supp}_X(\mu) = \mathrm{supp}(\mu) = \{x \in X \mid \mu(x) > 0\}$ for any $\mu \in \mathcal{D}X$. The associated family of relations with the property (13), by (14) is

$$
\begin{aligned}
R_X &= (id \times \mathrm{supp}_X)^{-1}(\in_X) \\
&= \{\langle x, \mu\rangle \mid \langle x, \mathrm{supp}(\mu)\rangle \in \in_X\} \\
&= \{\langle x, \mu\rangle \mid x \in \mathrm{supp}(\mu)\}.
\end{aligned}
$$

We now show that, under a reasonable assumption, if $\mathcal{R}$ is the family of relations associated to a natural transformation $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$, then one can define paths via (10) or via (9) with $\mathcal{R}$ instead of $\in$ obtaining the same notion. For this we introduce first the notion of a *map of distributive laws*. We say that $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$ is a map of the distributive laws λ and π, notation $\sigma : \lambda \Longrightarrow \pi$, if the next diagram commutes.

$$
\begin{array}{ccc}
\mathcal{FTX} & \xrightarrow{\mathcal{F}\sigma_X} & \mathcal{FPX} \\
{\scriptstyle \lambda_X}\big\downarrow & & \big\downarrow{\scriptstyle \pi_X} \\
\mathcal{TFX} & \xrightarrow{\sigma_{\mathcal{F}X}} & \mathcal{PFX}
\end{array}
\tag{19}
$$

In a sense, a map of distributive laws shows that two distributive laws are compatible, or imitate each other along the natural transformation σ.

Lemma 3.7. *Let $\mathcal{T}$ be a monad, $\mathcal{F}$ a functor, $\lambda : \mathcal{FT} \Longrightarrow \mathcal{TF}$ a (plain) distributive law, and $\pi : \mathcal{FP} \Longrightarrow \mathcal{PF}$ the power law. Moreover, let $\sigma : \lambda \Longrightarrow \pi$ and let $\mathcal{R} = \{R_X\}$ be the associated family of relations. Then*

$$
(id \times \lambda_X^i)^{-1}(R_{\mathcal{F}^i X}) = \mathrm{Rel}(\mathcal{F})^i(R_X).
\tag{20}
$$

Proof Let λ, σ, π be as in the assumption of the lemma, such that (19) holds. We first show, by induction on n, that the following diagram commutes for all $n \in \mathbb{N}$.

$$
\begin{array}{ccc}
\mathcal{F}^n \mathcal{T} X & \xrightarrow{\mathcal{F}^n \sigma_X} & \mathcal{F}^n \mathcal{P} X \\
{\scriptstyle \lambda_X^n}\big\downarrow & & \big\downarrow{\scriptstyle \pi_X^n} \\
\mathcal{T}\mathcal{F}^n X & \xrightarrow{\sigma_{\mathcal{F}^n X}} & \mathcal{P}\mathcal{F}^n X
\end{array}
\tag{21}
$$

where λ^n and π^n are obtained from λ and π, respectively, by (4). For $n = 0$ it holds trivially, for $n = 1$ it holds by assumption. Assume it holds for n. Then we have

$$
\begin{aligned}
\pi_X^{n+1} \circ \mathcal{F}^{n+1} \sigma_X \quad &\overset{(4)}{=} \quad \pi_{\mathcal{F}^n X} \circ \mathcal{F}\pi_X^n \circ \mathcal{F}^{n+1} \sigma_X \\
&= \quad \pi_{\mathcal{F}^n X} \circ \mathcal{F}(\pi_X^n \circ \mathcal{F}^n \sigma_X) \\
&\overset{(IH)}{=} \quad \pi_{\mathcal{F}^n X} \circ \mathcal{F}(\sigma_{\mathcal{F}^n X} \circ \lambda_X^n) \\
&= \quad \pi_{\mathcal{F}^n X} \circ \mathcal{F}\sigma_{\mathcal{F}^n X} \circ \mathcal{F}\lambda_X^n \\
&\overset{(19)}{=} \quad \sigma_{\mathcal{F}^{n+1} X} \circ \lambda_{\mathcal{F}^n X} \circ \mathcal{F}\lambda_X^n \\
&\overset{(4)}{=} \quad \sigma_{\mathcal{F}^{n+1} X} \circ \lambda_X^{n+1}.
\end{aligned}
$$

Next we show (20). Let $i \in \mathbb{N}$. We have

$$
\mathrm{Rel}(\mathcal{F})^i(R_X) = (id \times \mathcal{F}^i \sigma_X)^{-1}(\mathrm{Rel}(\mathcal{F})^i(\in_X))
$$

and

$$\langle u, v \rangle \in (id \times \lambda_X^i)^{-1} R_{\mathcal{F}^i X} = (id \times \lambda_X^i)^{-1}(id \times \sigma_{\mathcal{F}^i X})^{-1}(\in_{\mathcal{F}^i X})$$

$$\iff \quad \langle u, (\sigma_{\mathcal{F}^i X} \circ \lambda_X^i)(v) \rangle \in \in_{\mathcal{F}^i X}$$

$$\overset{(21)}{\iff} \quad \langle u, (\pi_X^i \circ \mathcal{F}^i \sigma_X)(v) \rangle \in \in_{\mathcal{F}^i X}$$

$$\overset{(*)}{\iff} \quad \langle u, \mathcal{F}^i \sigma_X(v) \rangle \in \operatorname{Rel}(\mathcal{F})^i(\in_X)$$

$$\iff \quad \langle u, v \rangle \in \operatorname{Rel}(\mathcal{F})^i(R_X)$$

where the equivalence marked with $(*)$ holds since from [Jac04, Lemma 4.2] we have

$$(id \times \pi_X^i)^{-1}(\in_{\mathcal{F}^i X}) = \operatorname{Rel}(\mathcal{F})^i(\in_X).$$

$\square$

It can easily be seen, as in Remark 3.1, that Equation (20) implies equivalence of the Conditions (10) and (9).

Maps between distributive laws exist in the nature, as the following example demonstrates.

Example 3.8. Let $\lambda : \underline{A} \times \mathcal{D} \Longrightarrow \mathcal{D}(\underline{A} \times \mathcal{I}d)$ and $\pi : \underline{A} \times \mathcal{P} \Longrightarrow \mathcal{P}(\underline{A} \times \mathcal{I}d)$ be defined as in Example 2.4 and Example 2.6, respectively. Consider the support natural transformation $\operatorname{supp} : \mathcal{D} \Longrightarrow \mathcal{P}$. Then supp is a map of distributive laws, $\operatorname{supp} : \lambda \Longrightarrow \pi$, since one can directly verify that

$$\pi_X \circ \mathcal{F} \operatorname{supp}_X = \operatorname{supp}_{\mathcal{F}X} \circ \lambda_X.$$

Hence, for generative probabilistic systems there is one notion of a path with respect to the support relations. Moreover, it can be seen that this one notion corresponds to the usual linear notion of a path for generative systems.

The following result suggests that in the case of the powerset monad, the family of membership relations deserves to be called the family of reachability relations.

Lemma 3.9. *There exist exactly two families of relations R_X that satisfy (13), associated to the powerset monad $\mathcal{P}$. These are $R_X = \emptyset$ for all sets X, and $R_X = \in_X$.*

Proof Consider $\varepsilon_X : \mathcal{P}X \to \mathcal{P}X$ for any set X, defined by $\varepsilon_X(X') = \emptyset$. Then $\varepsilon : \mathcal{P} \Longrightarrow \mathcal{P}$ is a natural transformation, and $R_X = \emptyset = (id \times \varepsilon_X)^{-1}(\in_X)$. Hence, by the proof of Lemma 3.3, $R_X = \emptyset$ satisfies (13). Furthermore, $R_X = \in_X = (id \times \hat{id}_X)^{-1}(\in_X)$ for $\hat{id}$ denoting the identity natural transformation $\hat{id} : \mathcal{P} \Longrightarrow \mathcal{P}$, and therefore $R_X = \in_X$ also satisfies (13). In the remainder of the proof we show that no other family of relations satisfies (13).

Assume $R_X \subseteq X \times \mathcal{T}X$, for any set X is a family of relations that satisfy (13), and assume that for some set Y, $R_Y \neq \emptyset$. We will first show that

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(a) $R_X \neq \emptyset$ for all sets $X \neq \emptyset$.

Let $X \neq \emptyset$, and choose a function $f : Y \to X$. Then $(f \times id)R_Y \neq \emptyset$ and so by (13) $(id \times \mathcal{P}f)^{-1}R_X \neq \emptyset$ implying that $R_X \neq \emptyset$.

Next we will show that

(b) $R_X \cap (X \times \{\emptyset\}) = \emptyset$ for all sets $X \neq \emptyset$.

Assume that for some set $X \neq \emptyset$, $R_X \cap (X \times \{\emptyset\}) \neq \emptyset$. Then there exists $x \in X$ with $\langle x, \emptyset \rangle \in R_X$, implying by (13), that for any set Z and any map $f : X \to Z$, the pair $\langle f(x), \emptyset \rangle \in R_Z$. This further implies that for any set Z, $Z \times \{\emptyset\} \subseteq R_Z$, and therefore

$$Z \times \{\emptyset\} \subseteq (id \times \mathcal{P}f)^{-1}R_Z = (f \times id)R_X. \tag{22}$$

Now choose a set Z and a map $f : X \to Z$ which is not surjective. Since $R_X \subseteq X \times \mathcal{P}X$, we have $(f \times id)R_X \subseteq f(X) \times \mathcal{P}X$ but $Z \times \{\emptyset\} \not\subseteq f(X) \times \mathcal{P}X$, contradicting (22). Hence, we have shown (b).

The third step in the proof is to show:

(c) For any set X, and any $x \in X$ we have $\langle x, \{x\} \rangle \in R_X$.

Let Y be a set such that $R_Y \neq \emptyset$ and let $\langle y', Y' \rangle \in R_Y$. Then, by (b) $Y' \neq \emptyset$. Consider now an arbitrary set X with $x \in X$. Define a function $f : Y \to X$ by $f(y) = x$ for all $y \in Y$. Note that $(\mathcal{P}f)(Y') = \{x\}$ since $Y' \neq \emptyset$ and we get

$$\langle x, \{x\} \rangle = \langle f(y'), (\mathcal{P}f)(Y') \rangle = (f \times \mathcal{P}f)(\langle y', Y' \rangle) \in (id \times \mathcal{P}f)(f \times id)R_Y \subseteq R_X$$

and (c) is proven.

Our next step is to prove that

(d) $\langle x, X \rangle \in R_X$ for all $x \in X$.

Take $x' \in X$ and consider the constant map $f : X \to X$, $f(x) = x'$ for all $x \in X$. Then, by (c) we have

$$(id \times \mathcal{P}f)(\langle x', X \rangle) = \langle x', \{x'\} \rangle \in R_X$$

which means that $\langle x', X \rangle \in (id \times \mathcal{P}f)^{-1}R_X = (f \times id)R_X$ and so there exists $x'' \in X$ such that $\langle x'', X \rangle \in R_X$. Since $(f \times \mathcal{P}f)R_X \subseteq R_Y$ for any map $f : X \to Y$ (see (16)), we have $(f \times \mathcal{P}f)(\langle x'', X \rangle) \in R_X$ for any $f : X \to X$. In particular, for arbitrary permutation f on X we have

$$(f \times \mathcal{P}f)(\langle x'', X \rangle) = \langle f(x''), f(X) \rangle = \langle f(x''), X \rangle \in R_X$$

which shows that (d) holds.

Next consider a set $X \neq \emptyset$ and let $x' \in X$, $X' \subseteq X$ such that $x' \in X'$. Define a map $f : X \to X$ by

$$f(x) = \begin{cases} x & x \in X' \\ x' & x \notin X' \end{cases}$$

Then $f(X) = X'$, $f(x') = x'$ and

$$\langle x', X' \rangle = \langle f(x'), f(X) \rangle \in (f \times \mathcal{P}f)R_X \subseteq R_X$$

i.e. we have shown that $\in_X \subseteq R_X$.

It remains to show the opposite inclusion. Assume that $\langle x', X' \rangle \in R_X$ and $x' \notin X'$. By (b) we have that $X' \neq \emptyset$. Choose $x'' \in X'$ and define $f : X \to X$ by

$$f(x) = \begin{cases} x & x \in X' \\ x'' & x \notin X' \end{cases}$$

Then we have $\langle x', X' \rangle \in R_X$ implying $\langle x', X \rangle \in (id \times \mathcal{P}f)^{-1}R_X = (f \times id)R_X$. This implies that $x' \in f(X)$, contradicting the assumption $x' \notin X'$. Hence, $R_X \subseteq \in_X$. $\qquad \square$

We next point that for any submonad of $\mathcal{P}$ there is a largest natural transformation that witnesses the submonad property. It corresponds to a largest family of relations. First we order the families of relations and the natural transformations. Let $\mathcal{R} = \{R_X \subseteq X \times \mathcal{T}X\}$ and $\mathcal{Q} = \{Q_X \subseteq X \times \mathcal{T}X\}$. We define

$$\mathcal{R} \leq \mathcal{Q} \quad \Longleftrightarrow \quad R_X \subseteq Q_X$$

for all sets X. Furthermore, let $\lambda : \mathcal{T} \Longrightarrow \mathcal{P}$ and $\tau : \mathcal{T} \Longrightarrow \mathcal{P}$. Define

$$\lambda \leq \tau \iff \lambda_X(u) \subseteq \tau_X(u)$$

for all sets X and all $u \in \mathcal{T}X$. One directly verifies that if $\mathcal{R}, \mathcal{Q}$ are the families of relations associated to the natural transformations λ, τ from $\mathcal{T}$ to $\mathcal{P}$, respectively, then

$$\mathcal{R} \leq \mathcal{Q} \quad \Longleftrightarrow \quad \lambda \leq \tau.$$

For any monad $\mathcal{T}$, there exists the empty natural transformation $\varepsilon : \mathcal{T} \Longrightarrow \mathcal{P}$ given by $\varepsilon_X(u) = \emptyset$ for all sets X and all $u \in \mathcal{T}X$. Furthermore, if $\{\mathcal{R}^i \mid i \in I\}$ is a collection of families of relations that satisfy (13), then $\mathcal{R}$ with components $R_X = \bigcup_{i \in I} R_X^i$ also satisfies (13), and $\mathcal{R}^i \leq \mathcal{R}$ for all $i \in I$. As a consequence we get the following property.

Lemma 3.10. *For any monad $\mathcal{T}$, there exists a largest family of relations $\mathcal{R} = \{R_X \subseteq X \times \mathcal{T}X\}$ with the property (13), and a corresponding largest natural transformation $\sigma : \mathcal{T} \Longrightarrow \mathcal{P}$.* $\qquad \square$

Example 3.11. The family $\mathcal{R}$ corresponding to the support natural transformation supp $: \mathcal{D} \Longrightarrow \mathcal{P}$ is the largest family of relations that satisfies (13).

Assume $\lambda : \mathcal{D} \Longrightarrow \mathcal{P}$. Let X be arbitrary set and $\mu \in \mathcal{D}X$ a distribution.

Choose a function $f : X \to X$ such that $f(x) = x$ for $x \in \mathrm{supp}(\mu)$, i.e., $f|_{\mathrm{supp}(\mu)} = id$ and $f(X) = \mathrm{supp}(\mu)$. Such a function exists since the support of a distribution is never empty. Then we have $(\mathcal{D}f)(\mu) = \mu$ since for $x \notin \mathrm{supp}(\mu)$ we have $f^{-1}(x) \cap \mathrm{supp}(\mu) = \emptyset$, and for $x \in \mathrm{supp}(\mu)$ it holds $f^{-1}(x) \cap \mathrm{supp}(\mu) = \{x\}$. Since λ is natural, $\mathcal{P}f \circ \lambda_X = \lambda_X \circ \mathcal{D}f$ and in particular

$$\lambda_X(\mu) = \lambda_X(\mathcal{D}f(\mu)) = (\mathcal{P}f)(\lambda_X(\mu)) = f(\lambda_X(\mu)) \subseteq f(X) = \mathrm{supp}(\mu)$$

proving that $\lambda \leq \mathrm{supp}$.

4 Concluding remarks

We have discussed composition of coalgebras and paths for coalgebras. The interest in compositions was already invoked when considering weak bisimulations [SVW05]. It was an important issue to know what does it mean to perform several consecutive steps from a state. Later, the work of Jacobs [Jac04] on trace semantics for coalgebras of type $\mathcal{P}\mathcal{F}$ for $\mathcal{F}$ a polynomial functor drew the author's attention to monads and distributive laws and made it easy to define composition of coalgebras of type $\mathcal{T}\mathcal{F}$ with $\mathcal{T}$ a monad and a corresponding distributive law. The author's ambition was to extend the results on traces for other coalgebras but of type $\mathcal{P}\mathcal{F}$. The definition of composition is a small step in this direction. In a recent work, Hasuo and Jacobs [HJ05b] proposed a different treatment of (finite) traces for $\mathcal{P}\mathcal{F}$ coalgebras. More recently, the same authors also obtained traces for coalgebras of type $\mathcal{D}_{\leq}\mathcal{F}$ [HJ05a]. The trace map in this new result is defined in terms of composition i.e. exponentiation of coalgebras. The same can be done for the trace map from [HJ05b]. The compositions are not an essential part of the results, but they do provide a nice presentation. Generalizing the traces result i.e. obtaining traces for more general coalgebras, for example of type $\mathcal{T}\mathcal{F}$ for any monad $\mathcal{T}$, a polynomial functor $\mathcal{F}$, with a corresponding distributive law, is an interesting direction for future work.

Coming back to weak bisimulations, we believe that compositions might also help in obtaining $*$-extensions (see [SVW05, SVW04]). For example, given an LTS coalgebra $\langle S, \alpha : S \to \mathcal{P}(A \times S) \rangle$, the $*$-extension $\langle S, \alpha^* : S \to \mathcal{P}(A^* \times S) \rangle$ can be expressed in terms of compositions by

$$\alpha^*(s) = \bigcup_{n \in \mathbb{N}} \alpha^n(s)$$

where α^n denotes the n-th exponent of the coalgebra $\langle S, \alpha \rangle$. There might be similar connections between exponents and $*$-extensions for general coalgebras as well.

Moreover, we studied ways to define paths in coalgebras. We are not yet convinced whether it is reasonable to define notions of linear behavior, such as

paths, for general coalgebras. The most general definition given by condition (8) does not seem to reflect intuition of what a linear path should be. Moreover, there is the question of how to pick next states, i.e. which states are "reachable" from a transition $\alpha(s)$.

Therefore, we discussed subclasses of coalgebras for which a definition of a path is possible. Such are the coalgebras of a monad, the coalgebras of type $\mathcal{P}\mathcal{F}$ [Jac04], and, as we have seen, also coalgebras of type $\mathcal{T}\mathcal{F}$ for $\mathcal{T}$ being a submonad of $\mathcal{P}$. An example of such coalgebras are the generative probabilistic systems, and for them we obtain the usual notion of a path. While studying the possibility of defining paths, we have come to some interesting observations for the submonads of $\mathcal{P}$. Still, we are not convinced that defining paths by "forgetting" parts of the behavior, as in the case of generative systems is a good idea. Application of these notions of paths for obtaining semantic relations remains an issue for future research. It could be a way to evaluate the notions of paths that we have considered.

Acknowledgements

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