A Companion to
Coalgebraic Weak Bisimulation
for Action-Type Systems[*]

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Abstract

We propose a coalgebraic definition of weak bisimulation for classes of coalgebras obtained from bifunctors in the category Set. Weak bisimilarity for a system is obtained as strong bisimilarity of a transformed system. The particular transformation consists of two steps: First, the behavior on actions is lifted to behavior on finite words. Second, the behavior on finite words is taken modulo the hiding of internal or invisible actions, yielding behavior on equivalence classes of words closed under silent steps. The coalgebraic definition is validated by two correspondence results: one for the classical notion of weak bisimulation of Milner, another for the notion of weak bisimulation for generative probabilistic transition systems as advocated by Baier and Hermanns.

1 Introduction

We present a definition of weak bisimulation for action type systems based on the general coalgebraic apparatus of bisimulation [1, 21, 36]. Action-type systems are systems that arise from bifunctors in the category Set. A typical and familiar example of an action-type system is a labelled transition system (LTS) (see, e.g., [22, 32]), but also many types of probabilistic systems (see, e.g., [24, 38, 17, 7, 37]) fall into this class. Informally, an action-type system in Set is a coalgebra that performs actions from a set A.

For the verification of system properties, behavior equivalences are often employed. One such behavior equivalence is strong bisimilarity. However strong bisimilarity is often too strong an equivalence. Weak bisimilarity, originally

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defined for LTSs in the work of Milner [27, 29], is a looser equivalence on systems that abstracts away from internal or invisible steps. In fact, weak bisimilarity for a labelled transition system $\mathcal{S}$ amounts to strong bisimilarity on the 'double-arrowed' system $\mathcal{S}'$ induced by $\mathcal{S}$. We generalize this idea for a coalgebraic definition of weak bisimulation. Our approach, given a system $\mathcal{S}$, consists of two stages.

1. First, we define a '$*$-extension' $\mathcal{S}'$ of $\mathcal{S}$ which is a system with the same carrier as $\mathcal{S}$, but with action set A^*, the set of all finite words over A. The system $\mathcal{S}'$ captures the behavior of $\mathcal{S}$ on finite traces.

2. Next, given a set of invisible actions $\tau \subseteq A$, we transform $\mathcal{S}'$ into a so-called 'weak τ-extension' $\mathcal{S}''$ which abstracts away from τ steps. Then we define weak bisimilarity on $\mathcal{S}$ as strong bisimilarity on the weak-τ-extension $\mathcal{S}''$.

Defining weak bisimulation for coalgebras has been studied before. There is early work by Rutten on weak bisimulation for while programs [35], succeeded by a syntactic approach to weak bisimulation by Rothe [33]. In the latter paper, weak bisimulation for a particular class of coalgebras was obtained by transforming a coalgebra into an LTS and making use of Milner's weak bisimulation there. This approach also supports a definition of weak homomorphisms and weak simulation relations. Later, in the work of Rothe and Mašulović [34], a complex, but interesting coalgebraic theory was developed leading to weak bisimulation for functors that weakly preserve pullbacks. They also consider a chosen 'observer' and hidden parts of a functor. However, in the case of probabilistic and similar systems, this does not lead to intuitive results and cannot be related to the concrete notions of weak bisimulation. The so-called skip relations used in [34] seem to be the major obstacle as it remains unclear how quantitative information can be incorporated. In the context of open maps, a category theoretical interpretation of weak bisimulation on presheaf models has been proposed in [15].

Indeed, the two-phase approach of defining weak bisimilarity for general systems is, amplifying Milner's original idea, rather natural. Our proposal for weak bisimilarity of action-type systems builds on the intuition in concrete cases. A drawback of our approach is that the definition of weak bisimulation is parameterized with a notion of a $*$-extension that does not come from a general categorical construction, but has to be tuned for the concrete type of systems at hand.

In this paper we focus on two particular examples of action-type systems: LTSs and the generative probabilistic systems [16, 17, 40]. The generative systems are closely related to LTSs, the difference is that all non-deterministic choices in an LTS are probabilistic choices in a generative system.

For LTSs, weak bisimulation is an established notion and the main motivation of the paper is to generalize this notion to coalgebras, as arbitrary as possible. Baier and Hermanns introduced, rather appealingly, the notion of weak bisimulation for generative probabilistic systems [7, 6, 8]. In this paper, we propose a notion of weak bisimulation at a high-level of abstraction that

justifies the definition of Baier and Hermanns for generative systems and illuminates the similarity between the notion of weak bisimulation for LTSs and of weak bisimulation for generative systems.

In the context of concrete probabilistic transition systems, there have been several other proposals for a notion of weak bisimulation, often relying on the particular model under consideration. For a detailed study of the different probabilistic models the reader is referred to [10, 11, 41, 40]. Segala [38, 37] proposes four notions of weak relations for his model of simple probabilistic automata. A detailed study of these relations can be found in [42]. It is a topic for further research to see how these notions fit into our general framework. Several groups of authors studied weak equivalences for the so-called alternating model of Hansson [20]. Philippou, Lee and Sokolsky [31] proposed the first notion of weak bisimulation in this setting. This work was extended to infinite systems by Desharnais, Gupta, Jagadeesan and Panangaden [14]. The same authors also provided a metric analogue of weak bisimulation [13]. Recently, Andova and Willemse studied branching bisimulation for the alternating model [4, 5], and together with Baeten [3] provided a complete axiomatization of this process equivalence in a process algebra setting. However, the alternating probabilistic automata are not coalgebras (see [40]) and therefore do not qualify for our definition.

Weak bisimulation was also considered for Markov chains in both discrete time [9, 39] and continuous time [9, 26]. Markov chains are not exactly action type coalgebras, since they are fully probabilistic non-labelled systems. However, the notion of weak bisimulation from [39] is based on the notion of weak bisimulation for generative probabilistic systems that is central to our paper. It is interesting to note that the notion of weak bisimulation by Baier and Hermanns has attracted attention in the security community and has been applied to security issues such as non-interference and secure information flow [2, 39, 23]. For the latter paper [23], as we will see for the present paper too, the coincidence of weak bisimulation and branching bisimulation in the setting of generative systems is crucial. Transition systems with both actions and generally distributed time delay occurring as labels are studied in [25] as well as a notion of weak bisimulation taking non-deterministic and sequential composition into account.

Below, we prove, not only for the case of labelled transition systems, but also for generative probabilistic systems that our coalgebraic definition corresponds to the concrete one of [29] and [7]. Despite the appeal of the coalgebraic definition of weak bisimulation, the proofs of correspondence results vary from straightforward to technically involved. For example, the relevant theorem for labelled transition systems takes less than a page, whereas proving the correspondence result for generative probabilistic systems takes in its present form more than twenty pages (additional machinery included).

The paper is organized as follows: Section 2 gathers the preliminary definitions and results. Section 3 is the kernel of the paper presenting the definition of coalgebraic weak bisimulation. We show that our definition of weak bisimilarity leads to Milner's weak bisimilarity for LTSs in Section 4. Section 5 is devoted to the correspondence result for the class of generative systems of the notion of

weak bisimilarity of Baier and Hermanns and our coalgebraic definition. This section is a technically involved part of the paper and is divided in several parts, discussing in detail generative probabilistic systems and their concrete and coalgebraic weak bisimulation. In Section 5.1 we study some basic notions, such as paths and cones of generative systems, and their properties. Section 5.2 establishes that the probability distributions defining a generative probabilistic system extend to measures on a certain σ-algebra of paths. In Section 5.3 we present the concrete definitions of weak bisimulation for generative systems by Baier and Hermanns, as well as branching bisimulation, and we gather and prove some properties of these relations (in concrete terms) that we need for our correspondence result. Section 5.4 presents the coalgebraic weak bisimulation for generative probabilistic systems which in Section 5.5 is compared to the concrete notion of weak bisimulation. At the end, Section 6 draws some conclusions. Last, but not least, one will find several appendices. The theme that connects them is the notion of weak pullback preservation—a technical condition that is helpful in relating concrete and coalgebraic bisimulations. We recall the definitions of pullbacks and their preservation in Appendix A. We prove weak pullback preservation of the distribution functor (without restricting to finite support) in Appendix B. This is an interesting side-contribution of the paper. Its place is in an appendix in order not to distract the main line of the story. In Appendix C we investigate the weak pullback preservation of the functor appearing in Section 5. Interestingly, this functor does not preserve weak pullbacks, but it preserves total weak pullbacks, a notion that turns out to be important in our investigations.

2 Systems and bisimilarity

We are treating systems from a coalgebraic point of view. Usually, in this context, a system is considered a coalgebra of a given Set endofunctor. For an introduction to the theory of coalgebra the reader is referred to the introductory articles by Rutten, Jacobs, and Gumm [36, 21, 19]. However, in our investigation of weak bisimilarity it is essential to explicitly specify the set of executable actions. Therefore we shall rather start from a so-called bifunctor instead of a Set endofunctor, cf [12].

A *bifunctor* is any functor $\mathcal{F} \colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$. If $\mathcal{F}$ is a bifunctor and A is a fixed set, then a Set endofunctor $\mathcal{F}_A$ is defined by

$$\mathcal{F}_A S = \mathcal{F}(A, S), \qquad \mathcal{F}_A f = \mathcal{F}\langle id_A, f \rangle \text{ for } f \colon S \to T. \tag{1}$$

We formulate the next simple proposition for further reference.

Proposition 2.1 *Let $\mathcal{F}$ be a bifunctor, and let A_1, A_2 be two fixed sets and $f \colon A_1 \to A_2$ a mapping. Then f induces a natural transformation $\eta^f \colon \mathcal{F}_{A_1} \Rightarrow \mathcal{F}_{A_2}$ defined by $\eta_S^f = \mathcal{F}\langle f, id_S \rangle$.* $\qquad\qquad\square$

We next define action-type coalgebras i.e. action-type systems based on bifunctors.

Definition 2.2 *Let $\mathcal{F}$ be a bifunctor. If S and A are sets and α is a function, $\alpha : S \to \mathcal{F}_A(S)$, then the triple $\langle S, A, \alpha \rangle$ is called an action type $\mathcal{F}_A$ coalgebra. A homomorphism between two $\mathcal{F}_A$-coalgebras $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ is a function $h : S \to T$ satisfying $\mathcal{F}_A h \circ \alpha = \beta \circ h$. The $\mathcal{F}_A$-coalgebras together with their homomorphisms form a category, which we denote by $\mathsf{Coalg}_{\mathcal{F}}^A$.*

Next we present two basic types of systems, labelled transition systems and generative systems, which will be treated in more detail in Section 4 and Section 5. We give their concrete definitions first.

Definition 2.3 *A labelled transition system, or LTS for short, is a triple $\langle S, A, \to \rangle$ where S and A are sets and $\to \; \subseteq S \times A \times S$. We speak of S as the set of states, of A as the set of labels or actions the system can perform and of $\to$ as the transition relation. As usual we denote $s \xrightarrow{a} s'$ whenever $\langle s, a, s' \rangle \in \; \to$.*

When replacing the transition relation of an LTS by a "probabilistic transition relation", the so-called generative probabilistic systems are obtained.

Definition 2.4 *A generative probabilistic system is a triple $\langle S, A, P \rangle$ where S and A are sets and $P : S \times A \times S \to [0,1]$ with the property that for $s \in S$,*

$$\sum_{a \in A, \; s' \in S} P(s, a, s') \in \{0, 1\}. \tag{2}$$

We speak of S as the set of states, of A as the set of labels or actions the system can perform and of P as the probabilistic transition relation. Condition (2) states that for all $s \in S$, $P(s, \_, \_)$ is either a distribution over $A \times S$ or $P(s, \_, \_) = 0$, i.e. s is a terminating state. As usual we denote $s \xrightarrow{a[p]} s'$ whenever $P(s, a, s') = p$, and $s \xrightarrow{a} s'$ for $P(s, a, s') > 0$.

Remark 2.5 In order to clarify the condition (2) let us recall that the sum of an arbitrary family $\{x_i \mid i \in I\}$ of non-negative real numbers is defined as

$$\sum_{i \in I} x_i = \sup\{\sum_{i \in J} x_i \mid J \subseteq I, J \text{ finite}\}.$$

Note that, if $\sum_{i \in I} x_i < \infty$, then the set $\{x_i \mid i \in I, x_i \neq 0\}$ is at most countably infinite.

Let us turn to the coalgebraic side. LTSs can be viewed as coalgebras corresponding to the bifunctor

$$\mathcal{L} = \mathcal{P}(\mathcal{I}d \times \mathcal{I}d).$$

Namely, if $\langle S, A, \to \rangle$ is an LTS, then $\langle S, A, \alpha \rangle$, where $\alpha : S \to \mathcal{L}_A(S)$ is defined by

$$\langle a, s' \rangle \in \alpha(s) \iff s \xrightarrow{a} s'$$

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is an $\mathcal{L}_A$-coalgebra, and vice-versa. Furtheron, we will freely use $\xrightarrow{a}$ notation when talking about $\mathcal{L}_A$-coalgebras. Also the generative systems can be considered as coalgebras corresponding to the bifunctor

$$\mathcal{G} = \mathcal{D}(\mathcal{I}d \times \mathcal{I}d) + 1.$$

Here $\mathcal{D}$ denotes the distribution functor, that is, $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$

$$\mathcal{D}X = \{\mu : X \to [0,1] \mid \textstyle\sum_{x \in X} \mu(x) = 1\}$$

$$(\mathcal{D}f)(\mu)(y) = \textstyle\sum_{f(x)=y} \mu(x), \quad f : X \to Y, \mu \in \mathcal{D}X, y \in Y.$$

If $\langle S, A, P \rangle$ is a generative system, then $\langle S, A, \alpha \rangle$ is a $\mathcal{G}_A$-coalgebra where $\alpha : S \to \mathcal{G}_A(S)$ is given by

$$\alpha(s)(a, s') = P(s, a, s'),$$

and vice-versa. Thereby we interpret the singleton set 1 as the set containing the zero-function on $A \times S$. Note that $\alpha(s)$ is the zero-function if and only if s is a terminating state.

In the literature it is common to restrict to generative systems $\langle S, A, \alpha \rangle$ where for any state s the function $\alpha(s)$ has finite support. The restriction to finite support guarantees existence of a final coalgebra. However, in many respects, in particular when the existence of a final coalgebra is not needed, this restriction is not necessary.

An important notion in this paper is that of a bisimulation relation between two systems. We recall here the general definition of bisimulation in coalgebraic terms.

Definition 2.6 *Let $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ be two $\mathcal{F}_A$-coalgebras. A bisimulation between $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ is a relation $R \subseteq S \times T$, such that there exists a map $\gamma : R \to \mathcal{F}_A R$ making the projections π_1 and π_2 coalgebra homomorphisms between the respective coalgebras, i.e. making the following diagram commute:*

$$
\begin{array}{ccccc}
S & \xleftarrow{\ \pi_1\ } & R & \xrightarrow{\ \pi_2\ } & T \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\
\mathcal{F}_A S & \xleftarrow[\mathcal{F}_A \pi_1]{} & \mathcal{F}_A R & \xrightarrow[\mathcal{F}_A \pi_2]{} & \mathcal{F}_A T
\end{array}
$$

Two states $s \in S$ and $t \in T$ are bisimilar, notation $s \sim t$ if they are related by some bisimulation between $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$.

Often we will consider bisimulations that are equivalence relations on a single coalgebra $\langle S, A, \alpha \rangle$.

In general, hence also for functors $\mathcal{F}_A$ and $\mathcal{G}_A$ arising from bifunctors $\mathcal{F}$ and $\mathcal{G}$, it holds that a natural transformation $\eta : \mathcal{F}_A \Rightarrow \mathcal{G}_A$ determines a functor $\mathcal{T} : \mathsf{Coalg}_{\mathcal{F}}^A \to \mathsf{Coalg}_{\mathcal{G}}^A$ defined by

$$\mathcal{T}(\langle S, A, \alpha \rangle) = \langle S, A, \eta_S \circ \alpha \rangle, \quad \mathcal{T}f = f. \tag{3}$$

We will refer to the functor $\mathcal{T}$ as the *functor induced* by the natural transformation η. Functors induced by natural transformations preserve homomorphisms and thus preserve bisimulation relations, in particular bisimilarity (cf. [36]).

LTSs and generative systems come equipped with their concrete notions of bisimulation relations, cf. [28] and [24, 17], respectively, which we present next.

Definition 2.7 *Let $\langle S, A, \to \rangle$ be an LTS. An equivalence relation $R \subseteq S \times S$ is a (strong) bisimulation on $\langle S, A, \to \rangle$ if and only if whenever $\langle s, t \rangle \in R$ then for all $a \in A$ the following holds:*

$s \xrightarrow{a} s'$ implies that there exists $t' \in S$ with $t \xrightarrow{a} t'$ and $\langle s', t' \rangle \in R$.

Two states s and t of an LTS are called bisimilar if and only if they are related by some bisimulation relation. Notation $s \sim_\ell t$.

For generative systems we have the following definition of bisimulation.

Definition 2.8 *Let $\langle S, A, P \rangle$ be a generative system. An equivalence relation $R \subseteq S \times S$ is a (strong) bisimulation on $\langle S, A, P \rangle$ if and only if whenever $\langle s, t \rangle \in R$ then for all $a \in A$ and for all equivalence classes $C \in S/R$*

$$P(s, a, C) = P(t, a, C). \tag{4}$$

Here we have put $P(s, a, C) = \sum_{s' \in C} P(s, a, s')$. Two states s and t of a generative system are bisimilar if and only if they are related by some bisimulation relation. Notation $s \sim_g t$.

The concrete notion of bisimilarity for LTSs and generative systems and the respective notions of bisimilarity obtained from Definition 2.6 coincide. For the case of LTSs a direct proof was given, for example, by Rutten [36]. For generative systems this fact goes back to the work of De Vink and Rutten [43] where Markov systems were considered, and was treated in [10] for generative systems with finite support.

We will now describe a general procedure to obtain coincidence results of this kind. This method already appeared implicitly in [11]. It applies to LTSs as well as to generative systems in their full generality. We will also use the method to obtain a concrete characterization of bisimilarity for another, more complex, functor, in Section 5.

Definition 2.9 *Let $R \subseteq S \times T$ be a relation, and $\mathcal{F}$ a Set functor. The relation R can be lifted to a relation $\equiv_{\mathcal{F},R} \subseteq \mathcal{F}S \times \mathcal{F}T$ defined by*

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R \colon \mathcal{F}\pi_1(z) = x, \ \mathcal{F}\pi_2(z) = y.$$

The following lemma is obvious from Definition 2.6.

Lemma 2.10 *A relation $R \subseteq S \times T$ is a bisimulation between the $\mathcal{F}_A$ systems $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ if and only if*

$$\langle s, t \rangle \in R \implies \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t). \tag{5}$$

$\square$

Note that the condition (5) is an abstract formulation of what is commonly referred to as a transfer condition.

For the sequel, weak pullback preservation will be of some importance. We recall the definitions of (weak) pullbacks and some needed properties concerning their preservation in Appendix A. One particular kind of pullbacks, *total pullbacks*, are important for our investigations. A total pullback is a weak pullback with surjective legs.

A characterization of bisimilarity will follow from the next lemma.

Lemma 2.11 *If the functor $\mathcal{F}$ weakly preserves total pullbacks and R is an equivalence on S, then $\equiv_{\mathcal{F},R}$ is the pullback in Set of the cospan*

$$\mathcal{F}S \xrightarrow{\ \mathcal{F}c\ } \mathcal{F}(S/R) \xleftarrow{\ \mathcal{F}c\ } \mathcal{F}S \tag{6}$$

where $c \colon S \to S/R$ is the canonical morphism mapping each element to its equivalence class.

Proof Since R is an equivalence relation and therefore reflexive, the left diagram below is a pullback diagram with epi legs, i.e., a total pullback.

Since $\mathcal{F}$ weakly preserves total pullbacks, the right diagram is a weak pullback diagram. By Definition 2.9 the map $\omega \colon \mathcal{F}R \to \equiv_{\mathcal{F},R}$, $\omega(z) = \langle \mathcal{F}\pi_1(z), \mathcal{F}\pi_2(z) \rangle$ is well-defined, surjective, and it makes the two upper triangles of the next

diagram commute:

$$\xymatrix{ & \equiv_{\mathcal{F},R} & \\ & \mathcal{F}R \ar[u]^{\omega} & }$$

As the lower square commutes and ω is surjective, the outer square of the above diagram also commutes, and by the existence of ω from the weak pullback $\mathcal{F}R$ to $\equiv_{\mathcal{F},R}$, $\equiv_{\mathcal{F},R}$ is a weak pullback as well. However, since it has projections as legs it is a pullback. □

Suppose that a functor $\mathcal{F}$ weakly preserves total pullbacks and assume that R is an equivalence bisimulation on S, i.e., R is both an equivalence relation and a bisimulation on S, such that $\langle s,t\rangle \in R$. The pullback in Set of the cospan (6) is the set $\{\,\langle x,y\rangle \mid \mathcal{F}c(x) = \mathcal{F}c(y)\,\}$. By Lemma 2.11 this set coincides with the lifted relation $\equiv_{\mathcal{F},R}$. Thus $x \equiv_{\mathcal{F},R} y \iff \mathcal{F}c(x) = \mathcal{F}c(y)$. Therefore, we obtain the transfer condition for the particular notion of bisimulation if we succeed in expressing concretely $(\mathcal{F}c \circ \alpha)(s) = (\mathcal{F}c \circ \alpha)(t)$ in terms of the representation of $\alpha(s)$ and $\alpha(t)$.

To illustrate the method, we will use it in showing the well-known correspondence of coalgebraic and concrete bisimulation for LTSs.

Lemma 2.12 *An equivalence relation R on a set S is a coalgebraic bisimulation on the LTS $\langle S, A, \alpha\rangle$ according to Definition 2.6 for the functor $\mathcal{L}_A$ if and only if it is a concrete bisimulation according to Definition 2.7.*

Proof It is easy to show that the LTS functor $\mathcal{L}_A$ preserves weak pullbacks (see e.g. [40]). For $X \in \mathcal{L}_A(S)$, i.e. $X \subseteq A \times S$, we have $\mathcal{L}_A(c)(X) = \mathcal{P}\langle id_A, c\rangle(X) = \langle id_A, c\rangle(X) = \{\langle a, c(x)\rangle \mid \langle a, x\rangle \in X\}$. Using Lemma 2.10 we get that an equivalence $R \subseteq S \times S$ is a coalgebraic bisimulation for an LTS $\langle S, A, \alpha\rangle$ if and only if

$$\langle s,t\rangle \in R \implies \{\,\langle a, c(s')\rangle \mid \langle a, s'\rangle \in \alpha(s)\,\} = \{\,\langle a, c(t')\rangle \mid \langle a, t'\rangle \in \alpha(t)\,\}$$

or, equivalently

$$\langle s,t\rangle \in R \implies (\, s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{a} t' \wedge \langle s', t'\rangle \in R \,).$$

which is the transfer condition from Definition 2.7. □

The most difficult part in establishing the correspondence result for generative systems is proving the weak pullback preservation for the distribution functor.

Proposition 2.13 *The functor $\mathcal{D}$ preserves weak pullbacks.* □

Appendix B is dedicated to the proof of this proposition. As a consequence we get that the functor for generative systems $\mathcal{G}_A$ preserves weak pullbacks. An application of Lemma 2.10 and some simple derivations now suffice to show the correspondence result.

Lemma 2.14 *An equivalence relation R on a set S is a coalgebraic bisimulation on the generative system $\langle S, A, \alpha \rangle$ according to Definition 2.6 for the functor $\mathcal{G}_A$ if and only if it is a concrete bisimulation according to Definition 2.8.* □

We end this section with a small discussion on the assumption of Lemma 2.10. Often we require a functor to weakly preserve pullbacks, so that it will be "well-behaved". For example, for bisimilarity being an equivalence. It can easily be seen that the milder condition of weakly preserving total pullbacks suffices for bisimilarity to be an equivalence. Moreover, we have relaxed the weak pullback preservation condition since in Section 5 we will need a bisimilarity characterization of a functor that transforms total pullbacks to weak pullbacks, but does not preserve weak pullbacks.

3 Weak bisimulation for action-type coalgebras

In this section we present a general definition of weak bisimulation for action-type systems. Our idea arises as a generalization of the notions of weak bisimulation for concrete types of systems. In our opinion, a weak bisimulation on a given system is a strong bisimulation on a suitably transformed system obtained from the original one.

Weak bisimulation in concrete cases deals with hiding actions. Therefore we focus on weak bisimulation for action-type coalgebras. Recall that we have defined action-type coalgebras in Definition 2.2 as triples $\langle S, A, \alpha \rangle$ such that $\langle S, \alpha : S \to \mathcal{F}_A S \rangle$ is a coalgebra for the functor $\mathcal{F}_A$ induced by a bifunctor $\mathcal{F}$, as in Equation (1).

We proceed with the definition of weak bisimulation for action-type coalgebras. The definition consists of two phases. First we define the notion of a *$*$-extended system*, that captures the behavior of the original system when extending from the given set of actions A to A^*, the set of finite words over A. The $*$-extension should emerge from the original system in a faithful way (which will be made precise below). The second phase considers invisibility. Given a subset $\tau \subseteq A$ of invisible actions, we restrict the $*$-extension to visible behavior only, by defining its *weak-τ-extended system*. Then a weak bisimulation relation on the original system is obtained as a bisimulation relation on the weak-τ-extension.

Definition 3.1 *Let $\mathcal{F}$ and $\mathcal{G}$ be two bifunctors. Let Φ be a map assigning to every $\mathcal{F}_A$-coalgebra $\langle S, A, \alpha \rangle$, a $\mathcal{G}_{A^*}$ system $\langle S, A^*, \alpha' \rangle$, on the same set of states S, such that the following conditions are met*

(1) Φ is injective, i.e. $\Phi(\langle S, A, \alpha \rangle) = \Phi(\langle S, A, \beta \rangle) \Rightarrow \alpha = \beta$;

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(2) Φ *preserves and reflects bisimilarity, i.e.* $s \sim t$ *in the system* $\langle S, A, \alpha \rangle$ *if and only if* $s \sim t$ *in the transformed system* $\Phi(\langle S, A, \alpha \rangle)$.

Then Φ *is called a* $*$*-translation, notation* $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$. *The* $\mathcal{G}_{A^*}$*-coalgebra* $\Phi(\langle S, A, \alpha \rangle)$ *is said to be a* $*$*-extension of the* $\mathcal{F}_A$*-coalgebra* $\langle S, A, \alpha \rangle$.

From the conditions (1) and (2) in Definition 3.1 it follows that the original system is "embedded" in its $*$-extension, cf. [10, 11, 41]. The fact that a $*$-translation may lead to systems of a new type, viz. of the bifunctor $\mathcal{G}$, might seem counterintuitive at first sight. However, this extra freedom is exploited in Section 5 when the starting functor itself is not expressive enough to allow for a $*$-extension.

A way to obtain $*$-translations follows from a previous result. Namely, if $\lambda \colon \mathcal{F}_A \Rightarrow \mathcal{G}_{A^*}$ is a natural transformation with injective components and the functor $\mathcal{F}_A$ preserves weak pullbacks, then the induced functor (see Equation (3)) is a $*$-translation [10, 11]. However, we shall see later that $*$-translations emerging from natural transformations do not suffice.

Having described how to extend an $\mathcal{F}_A$ system to its $*$-extension we show how to hide invisible actions. Fix a set of invisible actions $\tau \subseteq A$. Consider the function $h_\tau \colon A^* \to (A \setminus \tau)^*$ induced by $h_\tau(a) = a$ if $a \notin \tau$ and $h_\tau(a) = \varepsilon$ for $a \in \tau$ (where ε denotes the empty word). The function h_τ is deleting all the occurrences of elements of τ in a word of A^*. We put $A_\tau = (A \setminus \tau)^*$. By Proposition 2.1, we get the following.

Corollary 3.2 *The transformation* $\eta^\tau \colon \mathcal{G}_{A^*} \Rightarrow \mathcal{G}_{A_\tau}$ *given by* $\eta_S^\tau = \mathcal{G}\langle h_\tau, id_S \rangle$ *is natural.* $\square$

Let Ψ_τ be the functor from $\mathsf{Coalg}_{\mathcal{G}}^{A^*}$ to $\mathsf{Coalg}_{\mathcal{G}}^{A_\tau}$ induced by the natural transformation η^τ, i.e. $\Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \alpha'' \rangle$ for $\alpha'' = \eta_S^\tau \circ \alpha'$ and $\Psi_\tau f = f$ for any morphism $f : S \to T$. As mentioned above, the induced functor preserves bisimilarity. The composition of a $*$-translation Φ and the hiding functor Ψ_τ is denoted by $\Omega_\tau = \Psi_\tau \circ \Phi$ and is called a weak-τ-translation. The resulting system $\langle S, A_\tau, \eta_S^\tau \circ \alpha' \rangle$ is called a weak-τ-extension of $\langle S, A, \alpha \rangle$.

The transformation to a weak-τ-extension is presented in the following scheme.

$$
\begin{array}{ccccc}
S & & S & & S \\
\alpha \downarrow & \xrightarrow{\ \Phi\ } & \downarrow \alpha' & \xrightarrow{\ \Psi_\tau\ } & \downarrow \\
\mathcal{F}_A S & & \mathcal{G}_{A^*} S & & \alpha'' = \eta_S^\tau \circ \alpha' \\
& & & & \downarrow \\
\mathcal{F}_A \text{ - coalgebra} & \mathcal{G}_{A^*} \text{ - coalgebra} & & \mathcal{G}_{A_\tau} S
\end{array}
$$

$$\mathcal{G}_{A_\tau} \text{ - coalgebra}$$

A weak-τ-translation, or equivalently, the pair $\langle \Phi, \tau \rangle$, yields a notion of weak bisimulation with respect to Φ and τ.

Definition 3.3 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ a $*$-translation and $\tau \subseteq A$. Let $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ be two $\mathcal{F}_A$ systems. A relation $R \subseteq S \times T$ is a weak bisimulation with respect to $\langle \Phi, \tau \rangle$ if and only if it is a bisimulation between $\Omega_\tau(\langle S, A, \alpha \rangle)$ and $\Omega_\tau(\langle T, A, \beta \rangle)$. Two states $s \in S$ and $t \in T$ are weakly bisimilar with respect to $\langle \Phi, \tau \rangle$, notation $s \approx_\tau t$, if they are related by some weak bisimulation with respect to $\langle \Phi, \tau \rangle$.*

Concrete examples of weak bisimulation will be discussed in Section 4 and Section 5. We continue with verifying that weak bisimulations $\approx_\tau$ posses the intuitively expected properties.

Proposition 3.4 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ a $*$-translation, $\langle S, A, \alpha \rangle$ an $\mathcal{F}_A$-coalgebra, $\tau \subseteq A$ and let $\approx_\tau$ denote the weak bisimilarity on $\langle S, A, \alpha \rangle$ w.r.t. $\langle \Phi, \tau \rangle$. Then the following hold:*

(i) $\sim \; \subseteq \; \approx_\tau$ *for any $\tau \subseteq A$*
  i.e. strong bisimilarity implies weak bisimilarity.

(ii) $\sim \; = \; \approx_\emptyset$
  i.e. strong bisimilarity is weak bisimilarity in absence of invisible actions.

(iii) $\tau_1 \; \subseteq \; \tau_2 \Rightarrow \; \approx_{\tau_1} \; \subseteq \; \approx_{\tau_2}$ *for any $\tau_1, \tau_2 \subseteq A$,*
  i.e. the more actions are invisible, the coarser the weak bisimilarity gets.

Proof Let $\mathcal{F}, \mathcal{G}, \Phi, \langle S, A, \alpha \rangle$ and τ be as in the assumptions of the Lemma.

(i) Assume $s \sim t$ in $\langle S, A, \alpha \rangle$. Since Φ preserves bisimilarity (Definition 3.1) we have that $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle)$. Next, since Ψ_τ preserves bisimilarity we get $s \sim t$ in $\Psi_\tau \circ \Phi(\langle S, A, \alpha \rangle)$, which by Definition 3.3 means $s \approx_\tau t$ in $\langle S, A, \alpha \rangle$.

(ii) From (i) we get $\sim \; \subseteq \; \approx_\emptyset$. For the opposite inclusion, note that $h_\emptyset : A^* \to A^*$ is the identity map, hence the natural transformation $\eta^\emptyset$ from Corollary 3.2 is the identity natural transformation. Therefore the induced functor $\Psi_\emptyset$ is the identity functor on $\mathsf{Coalg}_\mathcal{G}^{A^*}$. Now assume $s \approx_\emptyset t$ in $\langle S, A, \alpha \rangle$. This means $s \sim t$ in $\Omega_\emptyset(\langle S, A, \alpha \rangle)$, i.e. $s \sim t$ in $\Psi_\emptyset \circ \Phi(\langle S, A, \alpha \rangle)$, i.e. $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle)$. Since, by Definition 3.1, every $*$-translation reflects bisimilarity we get $s \sim t$ in $\langle S, A, \alpha \rangle$.

(iii) Let $\tau_1 \subseteq \tau_2$. Consider the diagram

$$
\begin{array}{ccc}
A^* & \xrightarrow{\;h_{\tau_2}\;} & A_{\tau_2} \\
{\scriptstyle h_{\tau_1}}\Big\downarrow & \nearrow{\scriptstyle h_{\tau_1,\tau_2}} & \\
A_{\tau_1} & &
\end{array}
$$

where h_{τ_1,τ_2} is the map deleting all occurrences of elements of τ_2 in a word of A_{τ_1}. The diagram commutes since first deleting all occurrences of

elements of τ_1 followed by deleting all occurrences of elements of τ_2, in a word of A^* is the same as just deleting all occurrences of elements of τ_2. Let η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2} be the natural transformations induced by h_{τ_1}, h_{τ_2}, h_{τ_1,τ_2}, respectively (see Proposition 2.1 and Corollary 3.2). Then the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{G}_{A^*} & \xLongrightarrow{\eta^{\tau_2}} & \mathcal{G}_{A_{\tau_2}} \\
\eta^{\tau_1} \Big\Downarrow & \nearrow \eta^{\tau_1,\tau_2} & \\
\mathcal{G}_{A_{\tau_1}} & &
\end{array}
$$

Let Ψ_{τ_1}, Ψ_{τ_2}, Ψ_{τ_1,τ_2} be the functors induced by the natural transformations η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2}, respectively. By Equation (3) it holds that

$$\Psi_{\tau_2} = \Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \tag{7}$$

and they all preserve bisimilarity. Now assume $s \approx_{\tau_1} t$ in $\langle S, A, \alpha \rangle$. This means that $s \sim t$ in the system $\Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha \rangle)$. Then, since Ψ_{τ_1,τ_2} preserves bisimilarity we have $s \sim t$ in the system $\Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha \rangle)$ which by equation (7) is the system $\Psi_{\tau_2} \circ \Phi(\langle S, A, \alpha \rangle)$ and we find $s \approx_{\tau_2} t$ in $\langle S, A, \alpha \rangle$. $\qquad \square$

For further use, we introduce some more notation. For any $w \in A_\tau$, we put $B_w = h_\tau^{-1}(\{w\}) \subseteq A^*$. We refer to the sets B_w as blocks. Note that $B_w = \tau^* a_1 \tau^* \cdots \tau^* a_k \tau^*$ for $w = a_1 \ldots a_k \in A_\tau = (A \setminus \tau)^*$.

4   Weak bisimulation for LTSs

In this section we show that in the case of LTSs there exists a $*$-translation according to the Definition 3.1, such that weak bisimulation in the concrete case [28] coincides with weak bisimulation induced by this $*$-translation. First we recall the standard definition of concrete weak bisimulation for LTSs.

Definition 4.1 *Let $\langle S, A, \rightarrow \rangle$ be an LTS. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, \alpha \rangle$ if and only if $\langle s, t \rangle \in R$ implies that*

if $s \xrightarrow{a} s'$, then there exists $t' \in S$ with
$$t \xrightarrow{\tau}{}^* \circ \xrightarrow{a} \circ \xrightarrow{\tau}{}^* t' \text{ and } \langle s', t' \rangle \in R$$

for all $a \in A \setminus \{\tau\}$, and

if $s \xrightarrow{\tau} s'$, then there exists $t' \in S$ with $t \xrightarrow{\tau}{}^* t'$ and $\langle s', t' \rangle \in R$.

Two states s and t are called weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_\ell t$.

We now present a definition of a $*$-translation that will give rise to a notion of weak bisimulation that coincides with the standard one of Definition 4.1. Recall that $\mathcal{L}$, $\mathcal{L}_A$ are the functors for LTSs, as introduced in Section 2.

Definition 4.2 *Let Φ assign to every LTS, i.e. any $\mathcal{L}_A$-coalgebra $\langle S, A, \alpha \rangle$, the $\mathcal{L}_{A^*}$ coalgebra $\langle S, A^*, \alpha' \rangle$ where for $w = a_1 \ldots a_k \in A^*$, $k > 0$,*

$$\langle a_1 \ldots a_k, s' \rangle \in \alpha'(s) \quad \Longleftrightarrow \quad s \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \cdots \circ \xrightarrow{a_k} s'$$

and $\langle \varepsilon, s' \rangle \in \alpha'(s) \iff s = s'$. We use the notation $s \overset{w}{\Rightarrow} s'$ for $\langle w, s' \rangle \in \alpha'(s)$.

Hence, for $w = a_1 \ldots a_k$, we have $s \overset{w}{\Rightarrow} s'$ if and only if there exist states $s_1, \ldots, s_{k-1}$ such that

$$s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_{k-1}} s_{k-1} \xrightarrow{a_k} s'.$$

Furthermore, note that for $a \in A$, since no hiding applies, it holds that

$$s \xrightarrow{a} s' \text{ in } \langle S, A, \alpha \rangle \quad \text{if and only if} \quad s \overset{a}{\Rightarrow} s' \text{ in } \langle S, A, \alpha' \rangle = \Phi(\langle S, A, \alpha \rangle)$$

i.e.,

$$\langle a, s' \rangle \in \alpha(s) \quad \Longleftrightarrow \quad \langle a, s' \rangle \in \alpha'(s).$$

Proposition 4.3 *The assignment Φ from Definition 4.2 is a $*$-translation.*

Proof We need to prove that Φ is injective and reflects and preserves bisimilarity. Let $\Phi(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$, $\Phi(\langle S, A, \beta \rangle) = \langle S, A^*, \beta' \rangle$. Assume that $\alpha' = \beta'$. Then, for any state s,

$$\langle a, s' \rangle \in \alpha(s) \iff \langle a, s' \rangle \in \alpha'(s) \iff \langle a, s' \rangle \in \beta'(s) \iff \langle a, s' \rangle \in \beta(s).$$

Hence $\alpha(s) = \beta(s)$, i.e., $\alpha = \beta$.

For the reflection of bisimilarity, let $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$. Hence there exists an equivalence bisimulation relation R such that $\langle s, t \rangle \in R$ and (according to Definition 2.7) for all $w \in A^*$,

if $s \overset{w}{\Rightarrow} s'$ then there exists $t' \in S$ such that $t \overset{w}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$.

Assume $s \xrightarrow{a} s'$ in $\langle S, A, \alpha \rangle$. Then $s \overset{a}{\Rightarrow} s'$ in $\langle S, A, \alpha' \rangle$ and therefore there exists $t' \in S$ with $\langle s', t' \rangle \in R$ and $t \overset{a}{\Rightarrow} t'$, i.e., $t \xrightarrow{a} t'$. Hence, R is a bisimulation on $\langle S, A, \alpha \rangle$ i.e. $s \sim t$ in the original system.

For the preservation of bisimulation, let $s \sim t$ in $\langle S, A, \alpha \rangle$ and let R be an equivalence bisimulation relation such that $\langle s, t \rangle \in R$. Assume $s \overset{w}{\Rightarrow} s'$, for some word $w \in A^*$. We show by induction on the length of w that there exists t' with $t \overset{w}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$. If w has length 0, then $w = \varepsilon$, $s' = s$ and we take $t' = t$. Assume w has length $k + 1$, i.e. $w = a \cdot w'$ for $a \in A, w' \in A^*$. Pick s'' such that $s \xrightarrow{a} s'' \overset{w'}{\Rightarrow} s'$. Since $\langle s, t \rangle \in R$ we can pick t'' such that $t \xrightarrow{a} t''$ and $\langle s'', t'' \rangle \in R$. By the inductive hypothesis, for w' we can choose t' such

that $t'' \overset{w'}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$. Note that $t \overset{a}{\longrightarrow} t'' \overset{w'}{\Rightarrow} t'$, i.e., $t \overset{w}{\Rightarrow} t'$. Hence R is a bisimulation on $\langle S, A^*, \alpha' \rangle$ and $s \sim t$ holds in the $*$-extension. $\square$

Note that if $\mathcal{T}$ is a functor induced by a natural transformation η, in the context of Equation (3), and if $\langle S, A, \alpha \rangle, \langle S, A, \beta \rangle$ are two systems such that, for some $s \in S$, $\alpha(s) = \beta(s)$, then, clearly,

$$\alpha'(s) = \eta_S(\alpha(s)) = \eta_S(\beta(s)) = \beta'(s) \tag{8}$$

for $\langle S, A, \alpha' \rangle = \mathcal{T}(\langle S, A, \alpha \rangle)$, $\langle S, A, \beta' \rangle = \mathcal{T}(\langle S, A, \beta \rangle)$.

Having $*$-translations induced by natural transformations is desirable, since such *-translations are functorial and also obtained by a categorical construct. However, the following simple example shows that the $*$-translation Φ from Definition 4.2 violates (8). Therefore it can not be induced by a natural transformation.

Example 4.4 Let $S = \{s_1, s_2, s_3\}$ and $A = \{a, b, c\}$. Consider the LTSs:

$$\langle S, A, \alpha \rangle : s_1 \overset{a}{\longrightarrow} s_2 \overset{b}{\longrightarrow} s_3 \quad \text{and} \quad \langle S, A, \beta \rangle : s_1 \overset{a}{\longrightarrow} s_2 \overset{c}{\longrightarrow} s_3.$$

Obviously $\alpha(s_1) = \beta(s_1)$. However, $\alpha'(s_1) = \{\langle \varepsilon, s_1 \rangle, \langle a, s_2 \rangle, \langle ab, s_3 \rangle\}$ while $\beta'(s_1) = \{\langle \varepsilon, s_1 \rangle, \langle a, s_2 \rangle, \langle ac, s_3 \rangle\}$.

We next show that the coalgebraic and the concrete definitions coincide in the case of LTS.

Theorem 4.5 *Let $\langle S, A, \alpha \rangle$ be an LTS. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ with respect to the pair $\langle \Phi, \{\tau\} \rangle$ if and only if $s \approx_\ell t$.*

Proof Assume $s \approx_{\{\tau\}} t$ for $s, t \in S$ of an LTS $\langle S, A, \alpha \rangle$. This means that $s \sim t$ in the LTS $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha' \rangle$, i.e., there exists an equivalence bisimulation R on this system with $\langle s, t \rangle \in R$.

As usual, α' is such that $\langle S, A^*, \alpha' \rangle = \Phi(\langle S, A, \alpha \rangle)$. Here we have $\eta_S^{\{\tau\}} = \mathcal{L}(h_{\{\tau\}}, id_S) = \mathcal{P}(h_{\{\tau\}}, id_S)$ and

$$
\begin{aligned}
(\eta_S^{\{\tau\}} \circ \alpha')(s) &= \eta_S^{\{\tau\}}(\alpha'(s)) \\
&= \mathcal{P}(h_{\{\tau\}}, id_S)(\alpha'(s)) \\
&= \{\langle h_{\{\tau\}}(w), s' \rangle \mid \langle w, s' \rangle \in \alpha'(s)\} \\
&= \{\langle u, s' \rangle \mid \exists w \in B_u : s \overset{w}{\Rightarrow} s'\}
\end{aligned}
$$

We denote the transition relation of the weak-τ-system $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha' \rangle$ by $\Rightarrow_\tau$, i.e., for $w \in A_\tau$,

$$s \overset{w}{\Rightarrow}_\tau s' \iff \langle w, s' \rangle \in (\eta_S^{\{\tau\}} \circ \alpha')(s).$$

The above shows that for a word $w = a_1 \ldots a_k \in A_\tau$

$$s \stackrel{w}{\Rightarrow}_\tau s' \iff \exists v \in B_w = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* \colon s \stackrel{v}{\Rightarrow} s'.$$

We will show that the relation R is a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 4.1. Let $s \stackrel{a}{\longrightarrow} s'$ $(a \neq \tau)$. Then $s \stackrel{a}{\Rightarrow} s'$, implying $s \stackrel{a}{\Rightarrow}_\tau s'$. Since R is a bisimulation on the weak-τ-system, there exists t' such that $t \stackrel{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. We only need to note here that $\stackrel{a}{\Rightarrow}_\tau = \stackrel{\tau}{\longrightarrow}^* \circ \stackrel{a}{\longrightarrow} \circ \stackrel{\tau}{\longrightarrow}^*$. In case $s \stackrel{\tau}{\longrightarrow} s'$ we have $s \stackrel{\varepsilon}{\Rightarrow} s'$ implying now $s \stackrel{\varepsilon}{\Rightarrow}_\tau s'$. Hence, there exists t' such that $t \stackrel{\varepsilon}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. Since $\stackrel{\varepsilon}{\Rightarrow}_\tau = \stackrel{\tau}{\longrightarrow}^*$, we have proved that R is a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 4.1.

For the opposite, let R be a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 4.1 such that $\langle s, t \rangle \in R$. It is easy to show that for any $a \in A$, if $s \stackrel{\tau}{\longrightarrow}^* \circ \stackrel{a}{\longrightarrow} \circ \stackrel{\tau}{\longrightarrow}^* s'$ then there exists t' such that $t \stackrel{\tau}{\longrightarrow}^* \circ \stackrel{a}{\longrightarrow} \circ \stackrel{\tau}{\longrightarrow}^* t'$ and $\langle s', t' \rangle \in R$. Hence, if $s \stackrel{a}{\Rightarrow}_\tau s'$ then there exists t' with $t \stackrel{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. Based on this, a simple inductive argument on k leads to the conclusion that for any word $w = a_1 \ldots a_k \in A_\tau$, if $s \stackrel{w}{\Rightarrow}_\tau s'$ then there exists a t' such that $t \stackrel{w}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$, i.e. R is a bisimulation on the weak-τ-system and hence $s \approx_{\{\tau\}} t$. □

5 Weak bisimulation for generative systems

In this section we deal with generative systems and their weak bisimilarity. We first focus on the concrete definition of weak bisimulation by Baier and Hermanns [7, 6, 8]. Inspired by it, we provide a functor that suits for a definition of a ∗-translation for generative systems. This way we obtain a coalgebraic definition of weak bisimulation for this type of systems. We show that our definition, although at first sight much stronger, coincides with the definition of Baier and Hermanns for finite systems. Unlike in the case of LTSs, for generative systems the ∗-translation needs to leave its original class of systems, which justifies the generality of the definition.

This section is divided into several parts that lead to the correspondence result: First we introduce paths in a generative system and establish some notions and properties of paths. Next we define a measure on the set of paths, where we basically follow the lines of Baier and Hermanns [8, 6]. Furthermore, we present the definition of weak bisimulation by Baier and Hermanns, and we prove some properties of weak bisimulation relations that will be used later on (without restricting to finite state systems as in [8, 6]). Then we define a translation and prove that it is a ∗-translation providing us with a notion of weak-τ-bisimulation. The final part of this section is devoted to the question of correspondence of the notion of weak-τ-bisimulation defined by means of the given ∗-translation and the concrete notion proposed by Baier and Hermanns.

5.1 Paths and cones in a generative system

Let $\langle S, A, P \rangle$ be a generative system. A finite path π of $\langle S, A, P \rangle$ is an alternating sequence $\langle s_0, a_1, s_1, a_2, \ldots, a_k, s_k \rangle$, where $k \in \mathbb{N}_0$, $s_i \in S$, $a_i \in A$, and $P(s_{i-1}, a_i, s_i) > 0$, $i = 1, \ldots, k$. We will denote a finite path $\pi = \langle s_0, a_1, s_1, a_2, \ldots, a_k, s_k \rangle$ more suggestively by

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s_k \, .$$

Moreover, in the situation above, we put

$$\text{length}(\pi) = k, \ \text{first}(\pi) = s_0, \ \text{last}(\pi) = s_k, \ \text{trace}(\pi) = a_1 a_2 \cdots a_k \, .$$

The path $\varepsilon_{s_0} = (s_0)$ will be understood as the empty path starting at s_0. We will often write just ε for an arbitrary empty path. Similar to the finite case, an infinite path π of $\langle S, A, P \rangle$ is an infinite sequence $\langle s_0, a_1, s_1, a_2, \ldots \rangle$, where $s_i \in S$, $a_i \in A$ and $P(s_{i-1}, a_i, s_i) > 0$, $i \in \mathbb{N}$, and will be written as

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots$$

Again we set $\text{first}(\pi) = s_0$. A path π is called complete if it is either infinite or it is finite with $\text{last}(\pi)$ a terminating state, i.e. $P(\text{last}(\pi), \_, \_) = 0$.

The sets of all (finite or infinite) paths, of all finite paths and of all complete paths will be denoted by Paths, FPaths and CPaths, respectively. Moreover, if $s \in S$, we write

$$
\begin{aligned}
\text{Paths}(s) &= \big\{ \pi \in \text{Paths} \mid \text{first}(\pi) = s \big\}, \\
\text{FPaths}(s) &= \big\{ \pi \in \text{FPaths} \mid \text{first}(\pi) = s \big\}, \\
\text{CPaths}(s) &= \big\{ \pi \in \text{CPaths} \mid \text{first}(\pi) = s \big\}.
\end{aligned}
$$

We next define sets of concatenated paths. If $\Pi_1, \Pi_2 \subseteq \text{FPaths}$, we define

$$\Pi_1 \cdot \Pi_2 = \big\{ \pi_1 \cdot \pi_2 \mid \pi_1 \in \Pi_1, \pi_2 \in \Pi_2, \text{last}(\pi_1) = \text{first}(\pi_2) \big\} \, ,$$

where $\pi_1 \cdot \pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} \cdots \xrightarrow{a_n} s_n$ for $\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k$ and $\pi_2 \equiv s_k \xrightarrow{a_{k+1}} \cdots \xrightarrow{a_n} s_n$.

The set $\text{Paths}(s)$ is partially ordered by the prefix relation. For $\pi, \pi' \in \text{Paths}(s)$ we write $\pi \preceq \pi'$ if and only if the path π is a prefix of the path π'.

Note that if $\pi \prec \pi'$ then π is a finite path, and if $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$, then either $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. The complete paths are exactly the maximal elements in this partial order. For every $\pi \in \text{Paths}(s)$, there exists a $\pi' \in \text{CPaths}(s)$ such that $\pi \preceq \pi'$.

The following statement will be used at several occasions throughout this section.

Lemma 5.1 *For any state $s \in S$, the set $\text{FPaths}(s)$ is at most countable.*

Proof Let $\text{FPaths}_n(s)$ denote the set of finite paths starting from s with length n. Clearly, $\text{FPaths}(s) = \cup_{n \in \mathbb{N}} \text{FPaths}_n(s)$. The statement follows from the observation that for any state s and any $n \in \mathbb{N}$ the set $\text{FPaths}_n(s)$ is at most countable. This observation can be proven by induction on n as follows. We have $\text{FPaths}_0(s) = \{\epsilon\}$ and

$$\text{FPaths}_{n+1}(s) = \bigcup_{\langle a, s' \rangle : P(s,a,s') > 0} s \xrightarrow{a} s' \cdot \text{FPaths}_n(s')$$

which is at most countable by the inductive hypothesis and by the fact that $P(s, a, s') > 0$ for at most countably many a and s' (see Lemma B.1 in Appendix B). $\qquad\square$

Definition 5.2 *For a finite path $\pi \in \text{FPaths}(s)$, let $\pi{\uparrow}$ denote the set*

$$\pi{\uparrow} = \{\xi \in \text{CPaths}(s) \mid \pi \preceq \xi\}$$

also called the cone of complete paths generated by the finite path π.

Note that always $\pi{\uparrow} \neq \emptyset$. Let

$$\text{Cones}(s) = \{\pi{\uparrow} \mid \pi \in \text{FPaths}(s)\} \subseteq \mathcal{P}(\text{CPaths}(s))$$

denote the set of all cones. By Lemma 5.1 this set is at most countable. For the study of weak bisimulation for generative systems a thorough understanding of the geometry of cones is crucial. To begin with, we have the following elementary property:

Lemma 5.3 *Let $\pi_1, \pi_2 \in \text{FPaths}(s)$. Then the cones $\pi_1{\uparrow}$ and $\pi_2{\uparrow}$ are either disjoint or one is a subset of the other. In fact,*

$$\pi_1{\uparrow} \cap \pi_2{\uparrow} = \begin{cases} \pi_2{\uparrow} & \text{if } \pi_1 \preceq \pi_2 \\ \pi_1{\uparrow} & \text{if } \pi_2 \preceq \pi_1 \\ \emptyset & \text{if } \pi_1 \not\preceq \pi_2 \text{ and } \pi_2 \not\preceq \pi_1 \end{cases}$$

Moreover, we have $\pi_1{\uparrow} = \pi_2{\uparrow}$ if and only if either

$$\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k, \ \pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n \qquad (9)$$

for $n \geq k \geq 0$, and

$$P(s_{i-1}, a_i, s_i) = 1, \ i = k+1, \ldots, n \qquad (10)$$

or vice-versa.

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Proof Let $\hat{\pi} \in \pi_1\!\!\uparrow \cap \ \pi_2\!\!\uparrow$, $\hat{\pi} \in \mathrm{CPaths}(s)$. Then $\pi_1 \preceq \hat{\pi}$ and $\pi_2 \preceq \hat{\pi}$. This implies that $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. Assume $\pi_1 \preceq \pi_2$. Then

$$\pi \in \pi_2\!\!\uparrow \quad \Longleftrightarrow \quad \pi_2 \preceq \pi \implies \pi_1 \preceq \pi \quad \Longleftrightarrow \quad \pi \in \pi_1\!\!\uparrow$$

i.e., $\pi_2\!\!\uparrow \ \subseteq \pi_1\!\!\uparrow$ and therefore $\pi_1\!\!\uparrow \cap \ \pi_2\!\!\uparrow \ = \pi_2\!\!\uparrow$.

It is clear that (9) and (10) imply $\pi_1\!\!\uparrow \ = \pi_2\!\!\uparrow$. Assume $\pi_1\!\!\uparrow \ = \pi_2\!\!\uparrow$. Then $\pi_1\!\!\uparrow \cap \ \pi_2\!\!\uparrow \ \neq \emptyset$ and therefore $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. Assume $\pi_1 \preceq \pi_2$, $\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k$, $\pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n$. If for some $i \in \{k+1, \dots, n\}$ it happens that $P(s_{i-1}, a_i, s_i) < 1$, then there exists an action $a_i' \in A$ and a state $s_i' \in S$ such that $\langle a_i', s_i' \rangle \neq \langle a_i, s_i \rangle$ and

$$\pi_2' \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_{i-1}} s_{i-1} \xrightarrow{a_i'} s_i'$$

is a path in $\langle S, A, P \rangle$. Since $i \geq k+1$ we have $\pi_1 \preceq \pi_2'$. However, this path is not prefix related to π_2, i.e., we have $\pi_2' \not\preceq \pi_2$ and $\pi_2 \not\preceq \pi_2'$. Therefore $\pi_2'\!\!\uparrow \cap \ \pi_1\!\!\uparrow \ = \pi_2'\!\!\uparrow$ and $\pi_2'\!\!\uparrow \cap \ \pi_2\!\!\uparrow \ = \emptyset$ contradicting $\pi_1\!\!\uparrow \ = \pi_2\!\!\uparrow$. $\qquad \square$

Let $\Pi \subseteq \mathrm{FPaths}(s)$. We say that Π is *minimal* if for any two $\pi_1, \pi_2 \in \Pi$, $\pi_1 \neq \pi_2$, we have $\pi_1\!\!\uparrow \cap \ \pi_2\!\!\uparrow \ = \emptyset$. Hence in a minimal set of paths Π no path of Π is a proper prefix of another path of Π. We will express that Π is minimal by writing $\min(\Pi)$. As example note that every singleton set $\{\pi\}$, $\pi \in \mathrm{FPaths}(s)$, is minimal. Also every subset of $\mathrm{CPaths}(s)$ is minimal, too.

For $\Pi \subseteq \mathrm{FPaths}(s)$ we denote by $\Pi\!\!\uparrow$ the set

$$\Pi\!\!\uparrow \ = \bigcup_{\pi \in \Pi} \pi\!\!\uparrow .$$

Then the fact $\min(\Pi)$ just means that $\Pi\!\!\uparrow$ is actually the disjoint union of all $\pi\!\!\uparrow$, $\pi \in \Pi$, i.e.

$$\min(\Pi) \quad \Longleftrightarrow \quad \Pi\!\!\uparrow \ = \bigsqcup_{\pi \in \Pi} \pi\!\!\uparrow ,$$

where, here and in the sequel, the symbol $\sqcup$ denotes disjoint unions. It is an immediate consequence of the definition that,

$$\min(\Pi), \ \Pi' \subseteq \Pi \quad \implies \quad \min(\Pi').$$

However, if Π_1 and Π_2 are minimal, their union need not necessarily be minimal, even if $\Pi_1 \cap \Pi_2 = \emptyset$. We will use the notation

$$\Pi = \biguplus_{i \in I} \Pi_i$$

to express that

$$\Pi_i \subseteq \mathrm{FPaths}(s), \ i \in I, \ \Pi = \bigsqcup_{i \in I} \Pi_i \ \text{and} \ \min(\Pi) .$$

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Note that if $\Pi = \biguplus_{i \in I} \Pi_i$, also $\min(\Pi_i)$ for all $i \in I$. In particular this notation applies to minimal subsets Π written as the union of their one-element subsets:

$$\min(\Pi) \quad \implies \quad \Pi = \biguplus_{\pi \in \Pi} \{\pi\}.$$

Observe that the following two properties hold, as can be readily checked.

- If $\Pi = \biguplus_{i \in I} \Pi_i$, then $\Pi{\uparrow} = \bigsqcup_{i \in I} \Pi_i{\uparrow} = \bigsqcup_{i \in I, \pi \in \Pi_i} \pi{\uparrow}$.

- We have $\Pi = \biguplus_{i \in I} \Pi_i$ if and only if

  - $\forall i \in I : \min(\Pi_i)$, and
  - $\forall i, j \in I : i \neq j \implies \Pi_i \cap \Pi_j = \emptyset$, and
  - $\forall i, j \in I : i \neq j \implies \forall \pi_i \in \Pi_i, \forall \pi_j \in \Pi_j : \pi_i \not\preceq \pi_j$ and $\pi_j \not\preceq \pi_i$.

Let $\Pi \subseteq \mathrm{FPaths}(s)$. Put $\Pi{\downarrow} = \{\pi \in \Pi \mid \forall \pi' \in \Pi : \pi' \not\prec \pi\}$.

Lemma 5.4 *For any subset $\Pi \subseteq \mathrm{FPaths}(s)$, it holds that $\Pi{\downarrow} \subseteq \Pi$, $\min(\Pi{\downarrow})$ and $\Pi{\uparrow} = (\Pi{\downarrow}){\uparrow}$.*

Proof It is clear that $\Pi{\downarrow}$ is minimal, and that $\Pi{\downarrow} \subseteq \Pi$. Therefore also $(\Pi{\downarrow}){\uparrow} \subseteq \Pi{\uparrow}$. Take $\pi \in \Pi$. Since the prefix ordering does not allow for infinite descending chains, there exists $\pi' \in \Pi{\downarrow}$ with $\pi' \preceq \pi$. So we have $\pi{\uparrow} \subseteq \pi'{\uparrow}$ and this way we get $\Pi{\uparrow} \subseteq (\Pi{\downarrow}){\uparrow}$. $\qquad\square$

5.2 The measure Prob

We proceed with the construction of a probability measure Prob out of the distribution P of a generative system $\langle S, A, P \rangle$ on a certain σ-algebra on $\mathrm{CPaths}(s)$. This method was used in many papers, also in [8, 6], and before that in [37], where the setting is slightly different and/or only a part of the story is given. Here we give complete proofs for our setting. As a standard reference for measure theoretic notions and results we use the monograph [44]. An important measure theoretic result is the extension theorem which states that any pre-measure (σ-additive, monotone function with value zero for the empty set) on a semi-ring extends in a unique way to a measure on the σ-field generated by the semi-ring. Slightly different versions of this theorem apply to different definitions of the notion "semi-ring". For our purposes, the definition of a semi-ring from [44] fits best. Namely, a family of subsets of a given set S is a semi-ring if it contains the empty set, is closed under finite intersection and the set difference of any two of its elements is a disjoint union of at most countably many elements of the semi-ring.

Lemma 5.5 *The set $\mathrm{Cones}(s) \cup \{\emptyset\}$ is a semi-ring.*

Proof Clearly, $\text{Cones}(s) \cup \{\emptyset\}$ contains the empty set and it is closed under intersection, by Lemma 5.3. We need to check that the set-difference of any two of its elements is a disjoint union of at most countably many elements of $\text{Cones}(s) \cup \{\emptyset\}$. Let $\pi_1\!\uparrow, \pi_2\!\uparrow \in \text{Cones}(s)$. We consider $\pi_1\!\uparrow \setminus \pi_2\!\uparrow$. Since $\pi_1\!\uparrow \setminus \pi_2\!\uparrow = \pi_1\!\uparrow \setminus (\pi_1\!\uparrow \cap \pi_2\!\uparrow)$, by Lemma 5.3, the only interesting case is $\pi_1\!\uparrow \cap \pi_2\!\uparrow = \pi_2\!\uparrow \neq \pi_1\!\uparrow$ which implies $\pi_1 \prec \pi_2$. Let

$$\Pi = \{\pi \mid \pi = \pi' \cdot \text{last}(\pi') \xrightarrow{a} s', \pi_1 \preceq \pi' \prec \pi_2, \pi \not\preceq \pi_2\}.$$

Then $\pi_1\!\uparrow \setminus \pi_2\!\uparrow = \Pi\!\uparrow = \sqcup_{\pi \in \Pi} \pi\!\uparrow$. This union is at most countable since the set Π is at most countable by Lemma 5.1. $\square$

Now we are ready to introduce the desired extension of P to a measure. By Lemma 5.3, a function $\text{Prob} : \text{Cones}(s) \cup \{\emptyset\} \to [0,1]$ is well-defined by $\text{Prob}(\emptyset) = 0$, $\text{Prob}(\varepsilon\!\uparrow) = \text{Prob}(\text{CPaths}(s)) = 1$ and

$$\text{Prob}(C) = P(s, a, s') \cdot \text{Prob}(C'), \text{ for } C = \pi\!\uparrow, \pi = s \xrightarrow{a} s' \cdot \pi', C' = \pi'\!\uparrow$$

Lemma 5.6 *The function* Prob *is a pre-measure[1] on the semi-ring* $\text{Cones}(s) \cup \{\emptyset\}$.

Proof By definition $\text{Prob}(\emptyset) = 0$. Further we need to check σ-additivity and monotonicity.

For the σ-additivity, assume

$$\pi\!\uparrow = \bigsqcup_{i \in I} \pi_i\!\uparrow \tag{11}$$

for some at most countable index set I. We need to show that $\text{Prob}(\pi\!\uparrow) = \sum_{i \in I} \text{Prob}(\pi_i\!\uparrow)$.

If $|I| = 1$, then the property is trivially satisfied. Therefore we assume that $|I| > 1$. In particular this means that π is not terminating.

There exists (via a Lemma of Zorn argument) a partial function depth[2] that assigns to some finite paths an ordinal number, satisfying:

1. If $\xi \in \text{FPaths}(s)$ is such that $\pi_i \preceq \xi$ for some $i \in I$, or if ξ terminates, then $\text{depth}(\xi) = 0$.

2. Otherwise, if ξ is a finite path such that all its one step successors $\{\xi' \mid \xi \preceq \xi', \text{length}(\xi') = \text{length}(\xi) + 1\}$ have assigned depth then also ξ belongs to the domain of depth and

$$\text{depth}(\xi) = \sup\{\text{depth}(\xi') \mid \xi \preceq \xi', \text{length}(\xi') = \text{length}(\xi) + 1\} + 1. \tag{12}$$

---

[1] In [44] pre-measures are also called measures.
[2] The function depth has also been defined and used in a proof of a similar property by Segala [37].

Actually the function depth applied to a finite path ξ captures how deep in the cone generated by ξ one must go in order to be sure that all extensions of the path under consideration belong to some $\pi_i\uparrow$ for $i \in I$ or terminate. In other words, if depth(ξ) is defined, and if Ξ is the set of paths that extend ξ in at least depth(ξ) steps, then any path that extends any path in Ξ belongs to some of the cones $\pi_i\uparrow$ for $i \in I$ or terminates.

We first show, by reducing to contradiction, that our starting finite path π has been assigned a value for depth. Assume that π has not been assigned a value for depth. Let $\pi^0 = \pi$. For each $i > 0$ let π^i be a path such that length(π^i) = length(π^{i-1}) + 1, $\pi^{i-1} \preceq \pi^i$ and π^i has not been assigned a value for depth. Such a chain under the prefix ordering exists since if for some i all paths that extend π^i in one step would had been assigned depth, then π^i would also have been assigned a depth. Consider the infinite complete path π^∞ such that for all $i > 0$, $\pi^i \preceq \pi^\infty$. By definition $\pi^\infty \in \pi\uparrow$. By (11), there exists $i \in I$ such that $\pi^\infty \in \pi_i\uparrow$, implying that $\pi_i \preceq \pi^\infty$ and hence $\pi_i = \pi^n$ for some $n \geq 0$. However, then depth(π^n) = depth(π_i) = 0 contradicting that π^n has no value for depth assigned.

Let $\hat{\pi}$ be any non-terminating path and let $\{\pi_o \mid o \in O\}$ be the set of paths that extend $\hat{\pi}$ in one step, which means that

$$\forall o \in O\colon\ \hat{\pi} \prec \pi_o, \text{length}(\pi_o) = \text{length}(\hat{\pi}) + 1. \tag{13}$$

Then

$$\hat{\pi}\uparrow = \bigsqcup_{o \in O} \pi_o\uparrow \tag{14}$$

and

$$
\begin{aligned}
\sum_{o \in O} \text{Prob}(\pi_o\uparrow) &= \sum_{a \in A, s' \in S} \text{Prob}(\hat{\pi}\uparrow) \cdot P(\text{last}(\hat{\pi}), a, s') \\
&= \text{Prob}(\hat{\pi}\uparrow) \cdot \sum_{a \in A, s' \in S} P(\text{last}(\hat{\pi}), a, s') \\
&= \text{Prob}(\hat{\pi}\uparrow) \tag{15}
\end{aligned}
$$

since $\hat{\pi}$ does not end in a terminating state, i.e. $\sum_{a \in A, s \in S} P(\text{last}(\hat{\pi}), a, s) = 1$.

We will now show, by induction on depth, that if $\hat{\pi}$ is a finite path which has been assigned a value for depth and if

$$\hat{\pi}\uparrow = \bigsqcup_{i \in I' \subseteq I} \pi_i\uparrow, \tag{16}$$

for some $I' \subseteq I$, then $\text{Prob}(\hat{\pi}\uparrow) = \sum_{i \in I' \subseteq I} \text{Prob}(\pi_i\uparrow)$. Assume $\hat{\pi}$ is a path with depth($\hat{\pi}$) = 0 satisfying the assumption above. Then either $\hat{\pi}$ terminates or $\hat{\pi}\uparrow = \pi_i\uparrow$ for some $i \in I'$ and therefore $|I'| = 1$ and the additivity holds trivially. Now assume depth($\hat{\pi}$) = α and α is a successor ordinal (by definition α can not be a limit ordinal). This implies that $\hat{\pi}$ is not terminating. Moreover assume that the property holds for any path of the discussed form with depth smaller than α and let $\{\pi_o \mid o \in O\}$ be the set of paths that extend $\hat{\pi}$ in one step.

By (16) we have that
$$\forall i \in I' : \ \hat{\pi} \preceq \pi_i. \tag{17}$$

Moreover, from (16) and (14), using Lemma 5.3 we easily conclude that
$$\forall i \in I', \exists! o \in O : \ \pi_o \preceq \pi_i \tag{18}$$

and
$$\forall o \in O, \exists i \in I' : \ \pi_o \preceq \pi_i. \tag{19}$$

Let
$$I'_o = \{i \in I' \mid \pi_o \preceq \pi_i\}.$$

From (16), (18) and (19), we get that $I'_o \neq \emptyset$,
$$I' = \bigsqcup_{o \in O} I'_o \qquad \text{and} \qquad \pi_o{\uparrow} = \bigsqcup_{i \in I'_o} \pi_i{\uparrow} \ \text{for } o \in O. \tag{20}$$

Then we get

$$
\begin{aligned}
\mathrm{Prob}(\hat{\pi}{\uparrow}) \ &\overset{(15)}{=} \ \sum_{o \in O} \mathrm{Prob}(\pi_o{\uparrow}) \\
&\overset{(I.H.)}{=} \ \sum_{o \in O} \sum_{i \in I'_o} \mathrm{Prob}(\pi_i{\uparrow}) \\
&\overset{(20)}{=} \ \sum_{i \in I'} \mathrm{Prob}(\pi_i{\uparrow}).
\end{aligned}
$$

where the inductive hypothesis is applicable since by (12) and (13), $\mathrm{depth}(\pi_o) < \alpha$ for all $o \in O$ and $I'_o \subseteq I' \subseteq I$. This completes the proof of σ-additivity.

To see that Prob is monotonic assume $\pi_1{\uparrow} \subseteq \pi_2{\uparrow}$. Then, by Lemma 5.3, we have two possibilities. The first one is $\pi_2 \prec \pi_1$ and since $P(s, a, t) \leq 1$ for all $s, t \in S, a \in A$, from the definition of Prob we get $\mathrm{Prob}(\pi_1{\uparrow}) \leq \mathrm{Prob}(\pi_2{\uparrow})$. The second possibility is $\pi_1{\uparrow} = \pi_2{\uparrow}$, in which case $\mathrm{Prob}(\pi_1{\uparrow}) = \mathrm{Prob}(\pi_2{\uparrow})$. $\qquad\square$

Corollary 5.7 *The function* Prob *extends uniquely to a probability measure on the σ-algebra on* CPaths(s) *generated by* Cones(s) $\cup \{\emptyset\}$. *We will denote this measure again by* Prob. $\qquad\square$

Remark 5.8 Note that, although paths are more or less just alternating sequences of elements of S and A, whether an alternating sequence of states and actions is a path depends on the distribution P. Therefore the function Prob itself, but also the σ-algebra where it is defined and in fact already the base set CPaths(s) depends heavily on P.

The measure Prob induces a function on sets of finite paths, which we will also denote by Prob. We define $\mathrm{Prob} : \mathcal{P}(\mathrm{FPaths}(s)) \to [0, 1]$ by
$$\mathrm{Prob}(\Pi) = \mathrm{Prob}(\Pi{\uparrow}).$$

Note that $\Pi\uparrow$ is measurable since it is a countable union of cones. This notation is not in conflict with the already existing notation of the measure Prob. In fact, $\mathcal{P}(\text{FPaths}(s)) \cap \mathcal{P}(\text{CPaths}(s))$ consists entirely of Prob-measurable sets and on such sets both definitions coincide. To see this, note that if $\pi \in \text{FPaths}(s) \cap \text{CPaths}(s)$, then $\pi\uparrow = \{\pi\}$. Thus, if $\Pi \subseteq \text{FPaths}(s)$ and $\Pi \subseteq \text{CPaths}(s)$, we have

$$\Pi = \biguplus_{\pi \in \Pi} \{\pi\} = \biguplus_{\pi \in \Pi} \pi\uparrow = \Pi\uparrow \,,$$

and this union is at most countable.

It will always be clear from the context whether we mean the measure Prob or the just defined function Prob on sets of finite paths. Still, there is a word of caution in order: The function $\text{Prob} : \mathcal{P}(\text{FPaths}(s)) \to [0, 1]$ is, in general, not additive. However, looking at the properties of $\uplus$ introduced above (on page 19), we find that

$$\Pi = \biguplus_{i \in I} \Pi_i \quad \implies \quad \text{Prob}(\Pi) = \sum_{i \in I} \text{Prob}(\Pi_i) \,.$$

For this reason, we will overload the notation $\uplus$ and use it also for sets of cones generated by sets of finite paths, i.e. from now on we will freely write

$$\Pi\uparrow = \biguplus_{i \in I} \Pi_i\uparrow$$

if and only if it holds that $\Pi = \biguplus_{i \in I} \Pi_i$ for $\Pi, \Pi_i \subseteq \text{FPaths}(s)$.

We obtain that $\text{Prob}(\Pi) = \sum_{\pi \in \Pi} \text{Prob}(\pi\uparrow)$ for every minimal set Π. Moreover, by Lemma 5.4, we always have $\text{Prob}(\Pi) = \text{Prob}(\Pi\downarrow)$.

We next introduce some particular sets of paths. For $s \in S$, $S', S'' \subseteq S$ with $S' \subseteq S''$, and $W, W' \subseteq A^*$ with $W \subseteq W'$, by

$$s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S'$$

we denote the set of all finite paths that start in s, have a trace in W, end up in S', without passing a state in S'' having just performed a trace in the set W'. Formally,

$$s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' = \left\{ \pi \in \text{FPaths}(s) \mid \begin{array}{c} \text{last}(\pi) \in S', \text{trace}(\pi) \in W \\ \forall \xi \prec \pi : \text{trace}(\xi) \notin W' \vee \text{last}(\xi) \notin S'' \end{array} \right\}.$$

We write $\text{Prob}(s, W, \neg W, S', \neg S'') = \text{Prob}(s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S')$. Since $S' \subseteq S''$ and $W \subseteq W'$ we always have $\min(s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S')$. For notational convenience we will

drop redundant arguments whenever possible. Put

$$
\begin{aligned}
s \xrightarrow[\neg W']{W} S' &= s \xrightarrow[\substack{\neg W' \\ \neg S'}]{W} S', \\
s \xrightarrow[\neg S'']{W} S' &= s \xrightarrow[\substack{\neg W \\ \neg S''}]{W} S', \\
s \xrightarrow{W} S' &= s \xrightarrow[\substack{\neg W \\ \neg S'}]{W} S',
\end{aligned}
\tag{21}
$$

and, correspondingly,

$$
\begin{aligned}
\mathrm{Prob}(s, W, \neg W', S') &= \mathrm{Prob}(s, W, \neg W', S', \neg S'), \\
\mathrm{Prob}(s, W, S', \neg S'') &= \mathrm{Prob}(s, W, \neg W, S', \neg S''), \\
\mathrm{Prob}(s, W, S') &= \mathrm{Prob}(s, W, \neg W, S', \neg S') \,.
\end{aligned}
\tag{22}
$$

Note that

$$
s \xrightarrow{W} S' = \big\{ \pi \in \mathrm{FPaths}(s) \mid \mathrm{trace}(\pi) \in W, \mathrm{last}(\pi) \in S' \big\} \!\downarrow
$$

and hence

$$
\begin{aligned}
\mathrm{Prob}(s, W, S') &= \mathrm{Prob}(s \xrightarrow{W} S') \\
&= \mathrm{Prob}(\{ \pi \in \mathrm{FPaths}(s) \mid \mathrm{trace}(\pi) \in W, \mathrm{last}(\pi) \in S' \})
\end{aligned}
\tag{23}
$$

Also, for $a \in A, t \in S$, we have

$$
\mathrm{Prob}(s, \{a\}, \{t\}) = \begin{cases} \mathrm{Prob}(s \xrightarrow{a} t) = P(s, a, t), & \text{if } s \xrightarrow{a} t \\ \mathrm{Prob}(\emptyset) = 0, & \text{otherwise} \end{cases}
\tag{24}
$$

Let S', S'', W, W' be as above. Suppose $F \subseteq S$. Then we put

$$
F \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' = \bigsqcup_{s \in F} s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' \subseteq \mathrm{FPaths}
$$

In case that for every $s \in F$ the value of $\mathrm{Prob}(s, W, \neg W', S', \neg S'')$ is the same, we speak of this value as $\mathrm{Prob}(F, W, \neg W', S', \neg S'')$. Also, in this context, we shall freely apply shorthand as in (21) and (22).

The next technical property concerning sets of concatenated paths will be used at several occasions in the paper. Note that, whenever a concatenation $\pi_1 \cdot \pi_2$ is defined, we have $\mathrm{Prob}(\{\pi_1 \cdot \pi_2\}) = \mathrm{Prob}(\{\pi_1\}) \cdot \mathrm{Prob}(\{\pi_2\})$.

Proposition 5.9 *Let* $\Pi_1 \subseteq \mathrm{FPaths}(s)$, $\Pi_2 \subseteq \mathrm{FPaths}$ *and assume that the set of states* S *is represented as a disjoint union* $S = \bigsqcup_{i \in I} S_i$. *Denote* $\Pi_{1,S_i} = \{\pi_1 \in \Pi_1 \mid \mathrm{last}(\pi_1) \in S_i\}$, $\Pi_{2,t} = \{\pi_2 \in \Pi_2 \mid \mathrm{first}(\pi_2) = t\}$. *Assume that for every* $i \in I$

$$
\mathrm{Prob}(\Pi_{2,t'}) = \mathrm{Prob}(\Pi_{2,t''}), \ t', t'' \in S_i \,.
$$

Moreover, assume that Π_1, Π_2 *and* $\Pi_1 \cdot \Pi_2$ *are minimal. Then, for every choice of* $(t_i)_{i \in I} \in \prod_{i \in I} S_i$, *we have*

$$
\mathrm{Prob}(\Pi_1 \cdot \Pi_2) = \sum_{i \in I} \mathrm{Prob}(\Pi_{1,S_i}) \cdot \mathrm{Prob}(\Pi_{2,t_i}) \,.
$$

Proof Denote by $\Pi_{2,S_i} = \{\pi_2 \in \Pi_2 \mid \text{first}(\pi_2) \in S_i\}$ and by $\Pi_{1,t} = \{\pi_1 \in \Pi_1 \mid \text{last}(\pi_1) = t\}$. Under the assumptions of the proposition, we have

$$
\begin{aligned}
\text{Prob}(\Pi_1 \cdot \Pi_2) &= \text{Prob}(\biguplus_{\pi \in \Pi_1 \cdot \Pi_2} \pi{\uparrow}) \\
&= \text{Prob}(\biguplus_{i \in I}(\biguplus_{\pi \in \Pi_{1,S_i} \cdot \Pi_{2,S_i}} \pi{\uparrow})) \\
&= \text{Prob}(\biguplus_{i \in I}(\biguplus_{t \in S_i}(\biguplus_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \pi{\uparrow}))) \\
&= \sum_{i \in I} \sum_{t \in S_i} \sum_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \text{Prob}(\pi{\uparrow})
\end{aligned}
$$

Since, by minimality, $\Pi_{1,t} \times \Pi_{2,t} \cong \Pi_{1,t} \cdot \Pi_{2,t}$ via $(\pi_1, \pi_2) \mapsto \pi_1 \cdot \pi_2$, we have

$$
\begin{aligned}
\sum_{\pi \in \Pi_{1,t} \cdot \Pi_{2,t}} \text{Prob}(\pi{\uparrow}) &= \sum_{(\pi_1, \pi_2) \in \Pi_{1,t} \times \Pi_{2,t}} \text{Prob}(\pi_1 \cdot \pi_2{\uparrow}) \\
&= \sum_{\pi_1 \in \Pi_{1,t}} \sum_{\pi_2 \in \Pi_{2,t}} \text{Prob}(\pi_1{\uparrow}) \, \text{Prob}(\pi_2{\uparrow}) \\
&= \sum_{\pi_1 \in \Pi_{1,t}} \text{Prob}(\pi_1{\uparrow}) \cdot \sum_{\pi_2 \in \Pi_{2,t}} \text{Prob}(\pi_2{\uparrow}) \\
&= \text{Prob}(\Pi_{1,t}) \cdot \text{Prob}(\Pi_{2,t}).
\end{aligned}
$$

Since, by assumption, for every $i \in I$ the value of $\text{Prob}(\Pi_{2,t})$ does not depend on $t \in S_i$, it follows that

$$
\begin{aligned}
\text{Prob}(\Pi_1 \cdot \Pi_2) &= \sum_{i \in I} \sum_{t \in S_i} \text{Prob}(\Pi_{1,t}) \cdot \text{Prob}(\Pi_{2,t}) \\
&= \sum_{i \in I} \Big(\text{Prob}(\Pi_{2,t_i}) \cdot \sum_{t \in S_i} \text{Prob}(\Pi_{1,t}) \Big) \\
&= \sum_{i \in I} \text{Prob}(\Pi_{2,t_i}) \, \text{Prob}(\Pi_{1,S_i}).
\end{aligned}
$$

$\square$

It is worth to explicitly note the particular case of this proposition when $|I| = 1$.

Corollary 5.10 *Let* $\Pi_1 \subseteq \text{FPaths}(s)$, $\Pi_2 \subseteq \text{FPaths}$. *Let* $\Pi_{2,t} = \{\pi_2 \in \Pi_2 \mid \text{first}(\pi_2) = t\}$. *Then, if* $\min(\Pi_1)$, $\min(\Pi_2)$ *and* $\min(\Pi_1 \cdot \Pi_2)$, *and if for any* $t', t'' \in \text{first}(\Pi_2)$, $\text{Prob}(\Pi_{2,t'}) = \text{Prob}(\Pi_{2,t''})$, *we have that*

$$\text{Prob}(\Pi_1 \cdot \Pi_2) = \text{Prob}(\Pi_1) \cdot \text{Prob}(\Pi_{2,t})$$

for arbitrary $t \in \text{first}(\Pi_2)$. $\square$

For further reference, we state the following simple property.

Proposition 5.11 *Consider a generative system $\langle S, A, P \rangle$. Let $s \in S$, $W \subseteq A^*$ and $S' \subseteq S$ such that it partitions as $S' = \sqcup_{i \in I} S_i$. Then*

$$\text{Prob}(s, W, S') = \sum_{i \in I} \text{Prob}(s, W, S_i, \neg S').$$

Proof The result follows from the observation $s \xrightarrow{W} S' = \biguplus_{i \in I} s \xrightarrow{W}_{\neg S'} S_i$. $\square$

5.3 The concrete weak bisimulation

In this subsection we recall the original definition of weak bisimulation and branching bisimulation for generative systems proposed by Baier and Hermanns and we establish some properties of these relations that are essential for the correspondence result in Section 5.5 below.

Definition 5.12 *[7, 6, 8] Let $\langle S, A, P \rangle$ be a generative system. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, P \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for all actions $a \in A \setminus \{\tau\}$ and for all equivalence classes $C \in S/R$:*

$$\text{Prob}(s, \tau^* a \tau^*, C) = \text{Prob}(t, \tau^* a \tau^*, C) \tag{25}$$

and for all $C \in S/R$:

$$\text{Prob}(s, \tau^*, C) = \text{Prob}(t, \tau^*, C). \tag{26}$$

Two states s and t are weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_g t$.

Note the analogy between the transfer conditions (25), (26) and (4). The definition of branching bisimulation for generative systems is given below.

Definition 5.13 *[7, 6, 8] Let $\langle S, A, P \rangle$ be a generative system. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a branching bisimulation on $\langle S, A, P \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for all actions $a \in A \setminus \{\tau\}$ and for all equivalence classes $C \in S/R$:*

$$\text{Prob}(s, \tau^* a, C) = \text{Prob}(t, \tau^* a, C) \tag{27}$$

and for all $C \in S/R$:

$$\text{Prob}(s, \tau^*, C) = \text{Prob}(t, \tau^*, C). \tag{28}$$

Two states s and t are branching bisimilar if and only if they are related by some branching bisimulation relation. Notation $s \approx_g^{br} t$.

Baier and Hermanns have shown [6, 8] the following correspondence result for finite systems, i.e. systems with finite set of states.

Proposition 5.14 *Any weak bisimulation on a finite generative system is a branching bisimulation and vice versa. Hence, branching bisimilarity and weak bisimilarity coincide on finite systems.* □

Also for arbitrary generative systems branching bisimilarity implies weak bisimilarity, i.e., the proof of this direction of Proposition 5.14 does not require finiteness, as shown below.

Proposition 5.15 *Any branching bisimulation on a generative system is a weak bisimulation as well.*

Proof The property follows since we have $s \xrightarrow{\tau^* a \tau^*} C = \biguplus_{C' \in S/R} s \xrightarrow{\tau^* a} C' \cdot C' \xrightarrow{\tau^*} C$ given a branching bisimulation R, $s \in S$, $a \in A$ and $C \in S/R$. □

Whether a coincidence result as in Proposition 5.14 holds for arbitrary systems is an open question. The proof for finite systems can not be extended to arbitrary systems - in particular in Lemma 7.5.4 of [6] we can not obtain regularity for arbitrary matrices. On the other hand, up to now, an example showing the difference between weak and branching bisimilarity for arbitrary systems is not known to us. Therefore, we distinguish between the two notions.

Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Define a relation $\rightarrow$ on S/R by
$$C_1 \rightarrow C_2 \iff \mathrm{Prob}(C_1, \tau^*, C_2) = 1$$
and denote by $\leftrightarrow$ the equivalence closure of $\rightarrow$, i.e., $\leftrightarrow = (\rightarrow \cup \leftarrow)^*$.

A weak or branching bisimulation on $\langle S, A, P \rangle$ is called *complete*, if

$$\mathrm{Prob}(C_1, \tau^*, C_2) = 1 \iff C_1 = C_2$$

for all classes $C_1, C_2 \in S/R$. Hence, if R is a complete weak or branching bisimulation then for any two different classes $C_1, C_2 \in S/R$ it holds that $\mathrm{Prob}(C_1, \tau^*, C_2) < 1$.

A similar result to our next property is also stated in [8, 6] without a proof. It is essential for the correspondence result below and non-trivial, so we provide a detailed proof. To this we devote the remaining part of this subsection.

Proposition 5.16 *Let $\langle S, A, P \rangle$ be a generative system and let $s \approx_g t$ or $s \approx_g^{br} t$. Then there exists a complete weak or a complete branching bisimulation R, respectively, relating s and t.*

We will gradually build up the proof of Proposition 5.16, by a sequence of lemmas showing properties of the $\rightarrow$ relation.

Lemma 5.17 *The relation $\to$ corresponding to a weak or branching bisimulation R is reflexive and transitive.*

Proof Reflexivity follows since $s \xrightarrow{\tau^*} C = \{\epsilon\}$ for any class C, state $s \in C$, and hence $\text{Prob}(C, \tau^*, C) = \text{Prob}(s, \tau^*, C) = 1$, i.e. $C \to C$ for any class C.

Assume $C_1 \to C_2$, $C_2 \to C_3$, and fix a state $s \in C_1$. Using Corollary 5.10 and (23), since the set $s \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3$ is minimal, we get

$$
\begin{aligned}
1 &= \text{Prob}(s \xrightarrow{\tau^*} C_2) \cdot \text{Prob}(C_2 \xrightarrow{\tau^*} C_3) \\
&= \text{Prob}(s \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3) \\
&\leq \text{Prob}(\{\pi \in \text{FPaths}(s) \mid \text{trace}(\pi) \in \tau^*, \text{last}(\pi) \in C_3\}) \\
&= \text{Prob}(C_1 \xrightarrow{\tau^*} C_3) \\
&\leq 1.
\end{aligned}
$$

Hence $\text{Prob}(C_1 \xrightarrow{\tau^*} C_3) = 1$. $\qquad\square$

We next investigate in more detail the behavior of the $\to$ relation.

Lemma 5.18 *Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Let C_1, C_2, C_3 be different elements of S/R and assume $C_1 \to C_2$. Then either (i) or (ii) holds.*

(i) $\forall \pi \in C_1 \xrightarrow{\tau^*} C_3, \exists \pi' \in C_1 \xrightarrow{\tau^*} C_2 : \pi' \prec \pi$,
i.e. all τ^* paths from C_1 to C_3 pass C_2.

(ii) $C_3 \to C_2$

Proof Assume $C_1 \to C_2$ and not (i). Let $\pi \in C_1 \xrightarrow{\tau^*} C_3$ be a path that does not pass C_2. Let $s = \text{first}(\pi)$. Since $\text{Prob}(s, \tau^*, C_2) = 1$, also

$$
\text{Prob}(\pi{\uparrow} \cup \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi}{\uparrow}) = 1
$$

implying that, by additivity and $\text{Prob}(\pi{\uparrow}) > 0$,

$$
\pi{\uparrow} \cap \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi}{\uparrow} \neq \emptyset
$$

i.e., there exists $\bar{\pi} \in s \xrightarrow{\tau^*} C_2$ such that $\pi{\uparrow} \cap \bar{\pi}{\uparrow} \neq \emptyset$ which implies that $\pi \prec \bar{\pi}$ or $\bar{\pi} \prec \pi$. Note that $\pi \neq \bar{\pi}$ since C_2 and C_3 are different. Also the case $\bar{\pi} \prec \pi$ is excluded by assumption. Now,

$$
\pi{\uparrow} \cup \biguplus_{\bar{\pi} \in s \xrightarrow{\tau^*} C_2} \bar{\pi}{\uparrow} = \left(\pi{\uparrow} \cup \biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar{\pi}{\uparrow} \right) \sqcup \biguplus_{\substack{\bar{\pi} \in s \xrightarrow{\tau^*} C_2 \\ \bar{\pi}{\uparrow} \cap \pi{\uparrow} = \emptyset}} \bar{\pi}{\uparrow}.
$$

Hence,

$$\mathrm{Prob}(\pi{\uparrow} \cup \biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}) + \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} = \emptyset}} \bar\pi{\uparrow}) = 1$$

and, on the other hand, since $\mathrm{Prob}(s, \tau^*, C_2) = 1$,

$$\mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}) + \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} = \emptyset}} \bar\pi{\uparrow}) = 1$$

implying

$$\mathrm{Prob}(\pi{\uparrow} \cup \biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}) = \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow})$$

and, since for any $\bar\pi \in s \xrightarrow{\tau^*} C_2$ with $\bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset$ we have (as before) $\pi \prec \bar\pi$, i.e. $\bar\pi{\uparrow} \subseteq \pi{\uparrow}$, we get that

$$\mathrm{Prob}(\pi{\uparrow}) = \mathrm{Prob}(\pi{\uparrow} \cup \biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}) = \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}). \tag{29}$$

Consider the set of paths that extend π to a path in $s \xrightarrow{\tau^*} C_2$

$$\Pi = \{\hat\pi \mid \pi \cdot \hat\pi \in s \xrightarrow{\tau^*} C_2\}.$$

Recall that $\mathrm{last}(\pi) \in C_3$. Then

$$\Pi \subseteq \mathrm{last}(\pi) \xrightarrow{\tau^*} C_2 \tag{30}$$

and therefore the set Π is minimal and $\Pi \subseteq C_3 \xrightarrow{\tau^*} C_2$. For any $\hat\pi \in \Pi$ such that $\pi \cdot \hat\pi = \bar\pi$, we have $\mathrm{Prob}(\hat\pi) = \frac{\mathrm{Prob}(\bar\pi)}{\mathrm{Prob}(\pi)}$. Therefore

$$
\begin{aligned}
\mathrm{Prob}(\Pi) \quad &= \quad \sum_{\hat\pi \in \Pi} \mathrm{Prob}(\hat\pi) \\
&= \quad \frac{1}{\mathrm{Prob}(\pi)} \cdot \sum_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \mathrm{Prob}(\bar\pi) \\
&\overset{(*)}{=} \quad \frac{1}{\mathrm{Prob}(\pi{\uparrow})} \cdot \mathrm{Prob}(\biguplus_{\substack{\bar\pi \in s \xrightarrow{\tau^*} C_2 \\ \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset}} \bar\pi{\uparrow}) \\
&\overset{(29)}{=} \quad 1
\end{aligned}
$$

where $(*)$ holds by the minimality of the set $\{\bar\pi \mid \bar\pi \in s \xrightarrow{\tau^*} C_2, \bar\pi{\uparrow} \cap \pi{\uparrow} \neq \emptyset\}$. Hence, by (30),

$$\mathrm{Prob}(C_3 \xrightarrow{\tau^*} C_2) \geq \mathrm{Prob}(\Pi) = 1,$$

i.e. $C_3 \to C_2$. $\qquad\qquad\square$

The next lemma states that, given that $C_1 \to C_2$, if a path leaves a class C_1 with a trace that does not consist entirely of τ's, then this path must pass C_2 after performing a τ-trace.

Lemma 5.19 *Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Let C_1, C_2 be two elements of S/R and assume $C_1 \to C_2$. If for $s \in C_1$, $\pi \in s \xrightarrow{\tau^* a} S$, then there exists $\pi' \in C_1 \xrightarrow{\tau^*} C_2$ such that $\pi' \prec \pi$.*

Proof A similar argument as for Lemma 5.18 applies here as well. Assume $\pi \in s \xrightarrow{\tau^* a} S$. Since $\mathrm{Prob}(s, \tau^*, C_2) = 1$, also

$$\mathrm{Prob}(\pi\!\!\uparrow \cup \biguplus_{\bar\pi \in s \xrightarrow{\tau^*} C_2} \bar\pi\!\!\uparrow) = 1$$

implying that

$$\pi\!\!\uparrow \cap \biguplus_{\bar\pi \in s \xrightarrow{\tau^*} C_2} \bar\pi\!\!\uparrow \neq \emptyset$$

i.e., there exists $\bar\pi \in s \xrightarrow{\tau^*} C_2$ such that $\pi\!\!\uparrow \cap \bar\pi\!\!\uparrow \neq \emptyset$ which implies that $\bar\pi \prec \pi$ (since $\pi \preceq \bar\pi$ is excluded by the form of the traces). $\qquad\square$

Our next lemma shows a semi-Euclidean property of the $\to$ relation.

Lemma 5.20 *Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$ and let $C_1, C_2, C_3 \in S/R$. If $C_1 \to C_2$ and $C_1 \to C_3$, then $C_2 \to C_3$ or $C_3 \to C_2$.*

Proof From Lemma 5.18 we get that either $C_3 \to C_2$, or each path from C_1 to C_3 with a trace in τ^* passes C_2. Hence, in the latter case, we have

$$C_1 \xrightarrow{\tau^*} C_3 \subseteq C_1 \xrightarrow{\tau^*} C_2 \cdot C_2 \xrightarrow{\tau^*} C_3,$$

thus from Corollary 5.10,

$$\mathrm{Prob}(C_1, \tau^*, C_3) \leq \mathrm{Prob}(C_1, \tau^*, C_2) \cdot \mathrm{Prob}(C_2, \tau^*, C_3)$$

which leads to $1 \leq \mathrm{Prob}(C_2, \tau^*, C_3)$ i.e. $C_2 \to C_3$. $\qquad\square$

Next we establish a "sink" property for two $\to$ connected classes.

Lemma 5.21 *Let R be a weak bisimulation or a branching bisimulation on $\langle S, A, P \rangle$. If $C_1 \leftrightarrow C_2$, then there exists C such that $C_1 \to C$ and $C_2 \to C$.*

Proof We prove this by induction on the length of the sequence of $\rightarrow$ and $\leftarrow$ connecting C_1 and C_2. For a sequence of length 0, we have $C_1 = C_2$ and the statement holds trivially, by reflexivity, with $C = C_1 = C_2$. Assume $C_1 \leftrightarrow C_2$ via a sequence of $\rightarrow$ and $\leftarrow$ of length $k + 1$. Then there is a C_3 such that $C_1 \leftrightarrow C_3$ via a sequence of $\rightarrow$ and $\leftarrow$ of length k, and, $C_2 \rightarrow C_3$ or $C_3 \rightarrow C_2$. By the inductive hypothesis, there exists C such that $C_1 \rightarrow C$ and $C_3 \rightarrow C$. Now, if $C_2 \rightarrow C_3$, then also, by transitivity, $C_2 \rightarrow C$. If, on the other hand, $C_3 \rightarrow C_2$, then since also $C_3 \rightarrow C$, by Lemma 5.20, we get either $C \rightarrow C_2$ implying $C_1 \rightarrow C_2$ which gives the result with $C = C_2$, or $C_2 \rightarrow C$. $\square$

Lemma 5.21, by a straightforward induction on the number of elements extends to any finite set of $\rightarrow$ connected classes.

Lemma 5.22 *Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Let $F \subseteq S/R$ be a finite set of classes, with the property that for all $C_1, C_2 \in F$, $C_1 \leftrightarrow C_2$. Then there exists a class $C \in S/R$ such that for all $C' \in F$, $C' \rightarrow C$.*
$\square$

The next result shows that we can join $\rightarrow$ connected classes of a weak or branching bisimulation and still have a weak or branching bisimulation, respectively. In the sequel by $[C]_\leftrightarrow$ we denote the $\leftrightarrow$ - equivalence class of C.

Lemma 5.23 *Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Let $C_0 \in S/R$ be a fixed class such that $U = [C_0]_\leftrightarrow \neq \{C_0\}$. Define an equivalence R' on S by*

$$\langle s, t \rangle \in R' \iff \langle s, t \rangle \in R \vee \{s, t\} \subseteq \bigcup_{C \in U} C.$$

Then R' is a weak or branching bisimulation, respectively, and $R \subset R'$.

Proof We prove only the case of weak bisimulation. For branching bisimulation the proof is almost the same, only simpler at several points. We need to prove that for all $a \in A$, all $K_1, K_2 \in S/R'$ and for all $s, t \in K_1$

$$\text{Prob}(s, \tau^* \hat{a} \tau^*, K_2) = \text{Prob}(t, \tau^* \hat{a} \tau^*, K_2)$$

where $\hat{a} = a$ if $a \neq \tau$ and $\hat{\tau} = \varepsilon$, the empty word. There are several cases:

Case 1. $K_1, K_2 \in S/R$.
The statement holds since R is a weak bisimulation relation.

Case 2. $K_1 \in S/R, K_2 = \cup_{C \in U} C$.
If $U = [C_0]_\leftrightarrow$ contains a sink C for U, i.e. for all $C' \in U$ we have $C' \rightarrow C$, we can write

$$s \xrightarrow{\tau^* \hat{a} \tau^*} C = s \xrightarrow{\tau^* \hat{a} \tau^*}_{\neg K_2} C \ \uplus \ \biguplus_{C' \in \ U - \{C\}} s \xrightarrow{\tau^* \hat{a} \tau^*}_{\neg K_2} C' \cdot C' \xrightarrow{\tau^*} C$$

32

and since there are at most countably many R-classes $C' \in U - \{C\}$ for which $s \xrightarrow{\tau^*\hat{a}\tau^*} {}_{\neg K_2} C' \neq \emptyset$, we get

$$
\begin{aligned}
\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C) &= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C, \neg K_2) \\
&\quad + \sum_{C' \in U - \{C\}} \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C', \neg K_2) \\
&= \sum_{C' \in U} \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C', \neg K_2) \\
&= \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2).
\end{aligned}
$$

The last equation holds since

$$
s \xrightarrow{\tau^*\hat{a}\tau^*} K_2 = \biguplus_{C' \in U} s \xrightarrow{\tau^*\hat{a}\tau^*} {}_{\neg K_2} C'.
$$

In the same way we get $\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2)$, thus

$$
\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2).
$$

Note that we only used that U has a sink, and not that it is a whole class of the equivalence relation $\leftrightarrow$.

On the other hand, if U does not contain an R-class which is a sink (and this can only happen for infinite U because of Lemma 5.22), we use an approximation argument. Since there are at most countably many paths outgoing from s, there exists a countable set $U_s \subseteq U$ such that $\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U_s} C) = \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U} C)$. For the same reason, there exists $U_t \subseteq U$, a countable set with the property $\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U_t} C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U} C)$. Taking $U' = U_s \cup U_t$ we get a countable set, such that both

$$
\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{C \in U'} C) = \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, K_2) \tag{31}
$$

and

$$
\mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{C \in U'} C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, K_2). \tag{32}
$$

Let $\{C_i \mid i \in \mathbb{N}\}$ be an enumeration of U'. We will define a chain of subsets of U in the following way. Put $U_1 = \{C_1\}$ and

$$
U_{n+1} = U_n \cup \{C_{n+1}\} \cup \{C^{n+1}\}
$$

where $C^{n+1} \in S/R$ is a sink for $U_n \cup \{C_{n+1}\}$. Such a sink exists by Lemma 5.22, and it belongs to U, since U is a $\leftrightarrow$ equivalence class. We have $U_n \subseteq U_{n+1}$ for every natural number n, and also

$$
U' \subseteq \bigcup_{n \in \mathbb{N}} U_n \subseteq U.
$$

Next we denote some sets of finite paths. Let

$$
\begin{aligned}
\Pi_s^n &= \{\pi \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) \in \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in \cup_{C \in U_n} C\} \\
\Pi_s^U &= \{\pi \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) \in \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in \cup_{C \in U} C\} \\
\Pi_s^{U'} &= \{\pi \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) \in \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in \cup_{C \in U'} C\}
\end{aligned}
$$

and similarly we use $\Pi_t^n, \Pi_t^U, \Pi_t^{U'}$. We have

$$
\Pi_s^{U'} \subseteq \bigcup_{n \in \mathbb{N}} \Pi_s^n \subseteq \Pi_s^U
$$

and similar holds for t in place of s. Furthermore, by (31) and (23) we have $\mathrm{Prob}(\Pi_s^{U'}) = \mathrm{Prob}(\Pi_s^U)$, hence

$$
\mathrm{Prob}(\cup_{n \in \mathbb{N}} \Pi_s^n) = \mathrm{Prob}(s, \tau^* \hat{a} \tau^*, K_2).
$$

Also, by (32),
$$
\mathrm{Prob}(\cup_{n \in \mathbb{N}} \Pi_t^n) = \mathrm{Prob}(t, \tau^* \hat{a} \tau^*, K_2).
$$

Now since $\Pi_s^n \subseteq \Pi_s^{n+1}$ and $\Pi_t^n \subseteq \Pi_t^{n+1}$ we get that

$$
\begin{aligned}
\mathrm{Prob}(\cup_{n \in \mathbb{N}} \Pi_s^n) &= \lim_{n \to \infty} \mathrm{Prob}(\Pi_s^n) \\
&= \lim_{n \to \infty} \mathrm{Prob}(s, \tau^* \hat{a} \tau^*, \cup_{C \in U_n} C) \\
&\overset{(*)}{=} \lim_{n \to \infty} \mathrm{Prob}(t, \tau^* \hat{a} \tau^*, \cup_{C \in U_n} C) \\
&= \mathrm{Prob}(\cup_{n \in \mathbb{N}} \Pi_t^n)
\end{aligned}
$$

where $(*)$ holds since each U_n is a set of R-classes that contains a sink, which completes the proof of this case.

Case 3. $K_1 = \cup_{C \in U} C, \; K_2 \in S/R'$

Consider $s, t \in K_1$. There exist R-classes C_1 and C_2 such that $s \in C_1$ and $t \in C_2$. We have $C_1 \leftrightarrow C_2$. By Lemma 5.21, there also exists an R-class C such that $C_1 \to C$ and $C_2 \to C$, and moreover $C \in U$, again since U is a $\leftrightarrow$ equivalence class.

If $K_2 = K_1$, then we have

$$
\mathrm{Prob}(s, \tau^*, K_2) = \mathrm{Prob}(t, \tau^*, K_2) = 1.
$$

If $K_2 \neq K_1$ then $K_2 \in S/R$ and $C \not\leftrightarrow K_2$. So, by Lemma 5.18 any τ^* path from C_i to K_2 must pass C, for $i \in \{1, 2\}$. Hence,

$$
C_i \xrightarrow{\tau^*} K_2 \subseteq C_i \xrightarrow{\tau^*} C \cdot C \xrightarrow{\tau^*} K_2. \tag{33}
$$

This implies $C_i \xrightarrow{\tau^*} C = C_i \xrightarrow{\tau^*}_{\neg(K_2 \cup C)} C$ since, if a τ^* path from C_i to C passes K_2 on the way, then either it was not minimal, i.e. it has a prefix that is also a

τ^* path from C_i to C, or $K_2 \to C$ which is not possible, since $K_2 \neq K_1$. Note that in (33) also equality holds. Hence, in this case we have

$$
\begin{aligned}
\mathrm{Prob}(s, \tau^*, K_2) &= \mathrm{Prob}(s \xrightarrow{\tau^*} K_2) \\
&= \mathrm{Prob}(C_1, \tau^*, C) \cdot \mathrm{Prob}(C, \tau^*, K_2) \\
&= \mathrm{Prob}(C, \tau^*, K_2) \\
&= \mathrm{Prob}(C_2, \tau^*, C) \cdot \mathrm{Prob}(C, \tau^*, K_2) \\
&= \mathrm{Prob}(t, \tau^*, K_2).
\end{aligned}
$$

Next we consider paths with traces in $\tau^* a \tau^*$. For $i \in \{1, 2\}$, and $K_2 \in S/R'$ arbitrary ($K_2 = K_1$ is also possible), by Lemma 5.19 we have

$$
C_i \xrightarrow{\tau^* a \tau^*} K_2 \subseteq C_i \xrightarrow{\tau^*} C \cdot C \xrightarrow{\tau^* a \tau^*} K_2.
$$

Here also equality holds, since no path on the right hand side can have a proper prefix in $C_i \xrightarrow{\tau^* a \tau^*} K_2$. Hence, similar as before,

$$
\mathrm{Prob}(s, \tau^* a \tau^*, K_2) = \mathrm{Prob}(C, \tau^* a \tau^*, K_2) = \mathrm{Prob}(t, \tau^* a \tau^*, K_2).
$$

The notation $\mathrm{Prob}(C, \tau^* a \tau^*, K_2)$ if $K_2 = K_1$ is justified by Case 2. $\qquad\square$

We need one more property in order to prove Proposition 5.16.

Lemma 5.24 *Let R be a weak (respectively branching) bisimulation on $\langle S, A, P \rangle$. Consider the set*

$$\{ R' \mid R' \text{ is a weak (resp. branching) bisimulation on } \langle S, A, P \rangle, \ R' \supseteq R \}$$

ordered by inclusion. Every chain of this ordered set has an upper bound.

Proof We present the proof for weak bisimulation. The branching case is completely analogous. Let $\{ R_i \mid i \in I \}$ be a chain of elements of $\mathcal{W}$, where I is also a chain of indices, and $R_i \subseteq R_j$ for $i \leq j$. We show that $\cup_{i \in I} R_i \in \mathcal{W}$. Note that if $C \in S/ \cup_{i \in I} R_i$ is a class, then $C = \cup_{i \in I} C_i$ where $C_i \in S/R_i$, and $C_i \subseteq C_j$ for $i \leq j$.

The simplest case is when the chain has a largest element, say R_m and hence also $C = C_m$ and the property $\mathrm{Prob}(s, \tau^* \hat{a} \tau^*, C) = \mathrm{Prob}(t, \tau^* \hat{a} \tau^*, C)$ for $\langle s, t \rangle \in \cup_{i \in I} R_i$ holds for R_m is a weak bisimulation.

We next treat the case when I is a countable set, ordered as the natural numbers, $I = \mathbb{N}$, i.e., $\{ R_i \mid i \in \mathbb{N} \}$ is a countable chain, with $R_i \subseteq R_{i+1}$. Let $\langle s, t \rangle \in \cup_{i \in \mathbb{N}} R_i$. Then there exists j such that $\langle s, t \rangle \in R_j$, but also $\langle s, t \rangle \in R_n$ for all $n \geq j$. Consider the sets of paths

$$
\begin{aligned}
\Pi_s &= \cup \{ \pi \uparrow \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) = \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in C \} \\
\Pi_s^i &= \cup \{ \pi \uparrow \mid \mathrm{first}(\pi) = s, \mathrm{trace}(\pi) = \tau^* \hat{a} \tau^*, \mathrm{last}(\pi) \in C_i \}, \ i \in \mathbb{N}
\end{aligned}
$$

Similarly, we use Π_t and Π_t^i. We have $\Pi_s = \cup_{i \in \mathbb{N}} \Pi_s^i$ and $\Pi_s^i \subseteq \Pi_s^{i+1}$ for all i. Hence,

$$
\begin{aligned}
\text{Prob}(s, \tau^* \hat{a} \tau^*, C) &= \text{Prob}(\Pi_s) \\
&= \text{Prob}(\cup_{i \in \mathbb{N}} \Pi_s^i) \\
&\overset{(a)}{=} \lim_{n \to \infty} \text{Prob}(\Pi_s^n) \\
&= \lim_{n \to \infty} \text{Prob}(s, \tau^* \hat{a} \tau^*, C_n) \\
&\overset{(b)}{=} \lim_{n \to \infty} \text{Prob}(t, \tau^* \hat{a} \tau^*, C_n) \\
&= \text{Prob}(t, \tau^* \hat{a} \tau^*, C)
\end{aligned}
$$

where (a) holds since Prob is a measure, and (b) holds since for $n \geq j$ we have: $\langle s, t \rangle \in R_n$, C_n is an R_n-class, and R_n is a weak bisimulation.

We further show that if I is a countable chain of sets $\{C_i \mid i \in I\}$, then there exists a sub-chain I' of I with $\cup_{i \in I} C_i = \cup_{i \in I'} C_i$ and I' is either finite or isomorphic to ω, the order type of the natural numbers. We give the construction of I'. Given a countable chain I, denote by $f : \mathbb{N} \to I$ the bijection that exists since I is countable. Define a sequence of finite sub-chains of I by $I_0 = \{f(0)\}$ and

$$
I_{n+1} = \begin{cases} I_n \cup \{f(n+1)\} & \forall i \in I_n : f(n+1) > i \\ I_n & \text{otherwise.} \end{cases}
$$

Put

$$
I' = \bigcup_{n \in \mathbb{N}} I_n.
$$

It is straightforward to see that either I' is a finite chain, or I' is isomorphic to ω and in any case

$$
\bigcup_{i \in I} C_i = \bigcup_{i \in I'} C_i.
$$

Assume now that $\{R_i \mid i \in I\}$ is an arbitrary chain in $\mathcal{W}$. Let $\langle s, t \rangle \in \cup_{i \in I} R_i$, and let $C \in S/ \cup_{i \in I} R_i$. Then $C = \cup_{i \in I} C_i$. Let

$$
\begin{aligned}
\Pi_s &= \{\pi \mid \text{first}(\pi) = s, \text{trace}(\pi) = \tau^* \hat{a} \tau^*, \text{last}(\pi) \in C = \cup_{i \in I} C_i\} \\
\Pi_t &= \{\pi \mid \text{first}(\pi) = t, \text{trace}(\pi) = \tau^* \hat{a} \tau^*, \text{last}(\pi) \in C = \cup_{i \in I} C_i\}
\end{aligned}
$$

Let in be a function, $in : \Pi_s \cup \Pi_t \to I$ such that $\text{last}(\pi) \in C_{in(\pi)}$. Such a function exists by the definition of Π_s and Π_t. Then the set $I' = in(\Pi_s \cup \Pi_t) \subseteq I$ is at most countable since such are Π_s and Π_t. Furthermore, let

$$
\begin{aligned}
\Pi_s' &= \{\pi \mid \text{first}(\pi) = s, \text{trace}(\pi) = \tau^* \hat{a} \tau^*, \text{last}(\pi) \in C = \cup_{i \in I'} C_i\} \\
\Pi_t' &= \{\pi \mid \text{first}(\pi) = t, \text{trace}(\pi) = \tau^* \hat{a} \tau^*, \text{last}(\pi) \in C = \cup_{i \in I'} C_i\}
\end{aligned}
$$

By the construction of I' we have that $\Pi_s = \Pi_s'$ and $\Pi_t = \Pi_t'$ and

$$
\text{Prob}(s, \tau^* \hat{a} \tau^*, C) = \text{Prob}(\Pi_s) = \text{Prob}(\Pi_s') \overset{(*)}{=} \text{Prob}(\Pi_t') = \text{Prob}(t, \tau^* \hat{a} \tau^*, C).
$$

The equality marked by $(*)$ holds since $\mathrm{Prob}(\Pi'_s) = \mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{i\in I'}C_i)$ and $\mathrm{Prob}(\Pi'_t) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{i\in I'}C_i)$, and as proved above, in the case of a finite chain of classes or a countable chain of classes of order type ω, we have $\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, \cup_{i\in I'}C_i) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, \cup_{i\in I'}C_i)$. $\qquad\square$

Finally, Proposition 5.16 follows from the lemmas 5.17-5.24.

Proof [of Proposition 5.16]The set

$$\{R' \mid R' \text{ is a weak (resp. branching) bisimulation on } \langle S, A, P\rangle,\ R' \supseteq R\}$$

is nonempty, as it contains R. By Lemma 5.24 we can apply Zorn's Lemma and obtain that this set has a maximal element. Let it be $\tilde{R}$. Assume $\tilde{R}$ is not complete, i.e. there exists two different classes $C_1, C_2 \in S/\tilde{R}$ such that $C_1 \to C_2$. Then by Lemma 5.23 we can construct a weak or branching bisimulation $\tilde{R}' \supset \tilde{R}$, respectively, which contradicts the maximality of $\tilde{R}$. Hence $\tilde{R}$ is complete i.e. for any two different $C_1, C_2 \in S/\tilde{R}$ we have $\mathrm{Prob}(C_1, \tau^*, C_2) < 1$, and since $R \subseteq \tilde{R}$ it relates s and t which completes the proof. $\qquad\square$

5.4 Weak coalgebraic bisimulation for generative systems

In this subsection we provide a coalgebraic definition of weak bisimulation for generative systems, according to the approach from Section 3. For this we need a $*$-translation that will transform the generative systems with action set A into systems with action set A^*. Unlike for LTSs, the $*$-translation employed will yield coalgebras of a different type.

Let $\mathcal{G}^*$ be the bifunctor defined by

$$\mathcal{G}^*(A, S) = \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$$

on objects $\langle A, S\rangle$ and for morphisms $\langle f_1, f_2\rangle \colon \langle A, S\rangle \to \langle B, T\rangle$ by

$$\mathcal{G}^*\langle f_1, f_2\rangle = (\nu \mapsto \nu \circ (f_1^{-1} \times f_2^{-1}) \mid \nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]).$$

Consider the Set functor $\mathcal{G}^*_A$ corresponding to $\mathcal{G}^*$, so that

$$\mathcal{G}^*_A(S) = (\mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1])$$

and for a mapping $f \colon S \to T$,

$$\mathcal{G}^*_A f(\nu) = \nu \circ (id_A^{-1} \times f^{-1})$$

for $\nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$.

We will use the functor $\mathcal{G}^*_A$ to model the $*$-translation of generative systems. Therefore we are interested in characterizing equivalence bisimulations for this functor. In order to apply the results from Section 2 we need the following proposition. We dedicate Appendix C to its proof.

Proposition 5.25 *The functor $\mathcal{G}_A^*$ weakly preserves total pullbacks, but it does not preserve weak pullbacks.* □

Let R be an equivalence relation on a set S. A subset $M \subseteq S$ is an *R-saturated set* if for all $s \in M$ the whole equivalence class of s is contained in M. We denote by $\mathrm{Sat}(R)$ the set of all R-saturated sets, $\mathrm{Sat}(R) \subseteq \mathcal{P}(S)$. Actually, M is a saturated set if and only if $M = \cup_{i \in I} C_i$ for $C_i \in S/R$. Hence there is a one-to-one correspondence between the R-saturated sets and the elements of $\mathcal{P}(S/R)$.

The next lemma contains a transfer condition for equivalence bisimulations for systems of type $\mathcal{G}_A^*$. Its proof follows the approach discussed in Section 2 (see Lemma 2.11 and Lemma 2.12).

Lemma 5.26 *An equivalence relation R on a set S is a bisimulation on the $\mathcal{G}_A^*$ system $\langle S, A, \alpha \rangle$ if and only if*

$$\langle s, t \rangle \in R \implies \forall A' \subseteq A, \forall M \in \mathrm{Sat}(R) \colon \alpha(s)(A', M) = \alpha(t)(A', M).$$

Proof Consider the pullback P of the cospan

$$\mathcal{G}_A^* S \xrightarrow{\mathcal{G}_A^* c} \mathcal{G}_A^*(S/R) \xleftarrow{\mathcal{G}_A^* c} \mathcal{G}_A^* S$$

where c is the canonical projection of S onto S/R. We have $\langle \mu, \nu \rangle \in P$ if and only if $\mathcal{G}_A^* c(\mu) = \mathcal{G}_A^* c(\nu)$, i.e. $\mu \circ (id_A^{-1} \times c^{-1}) = \nu \circ (id_A^{-1} \times c^{-1})$. This is equivalent to

$$\forall A' \subseteq A, \forall M \subseteq S/R \colon \mu(A', c^{-1}(M)) = \nu(A', c^{-1}(M))$$

and, since $c^{-1} \colon \mathcal{P}(S/R) \to \mathrm{Sat}(R)$ is a bijection, we get an equivalent condition

$$\forall A' \subseteq A, \forall M \in \mathrm{Sat}(R) \colon \mu(A', M) = \nu(A', M).$$

Now, using Lemma 2.11, and Proposition 5.25, we obtain the stated characterization. □

We proceed by presenting a suitable $*$-translation for generative systems. The translation will yield a system of type $\mathcal{G}_{A^*}^*$. Recall that generative systems are coalgebras of the functor $\mathcal{G}_A = \mathcal{D}(A \times \mathcal{I}d) + 1$.

Definition 5.27 *Let Φ^g assign to every generative system $\langle S, A, P \rangle$, i.e. any $\mathcal{G}_A$-coalgebra $\langle S, A, \alpha \rangle$, the $\mathcal{G}_{A^*}^*$-coalgebra $\langle S, A^*, \alpha' \rangle$, where for $W \subseteq A^*$ and $S' \subseteq S$, $\alpha'(s)(W, S') = \mathrm{Prob}(s, W, S')$.*

In order to show that the translation defined above is indeed a $*$-translation we need the following property.

Lemma 5.28 *Let $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, be a $\mathcal{G}_A$ system, R a bisimulation equivalence on $\langle S, A, \alpha \rangle$ and $\langle s, t \rangle \in R$. For $k \in \mathbb{N}, C_i \in S/R$ and $a_i \in A$, $i \in \{1, \dots, k\}$, let $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ denote the set of paths*

$$s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k = \{s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \mid s_i \in C_i, i = 1, \dots, k\}.$$

Then $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ is minimal and

$$\text{Prob}(s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) = \text{Prob}(t \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) \qquad (34)$$

Proof The fact that $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ is minimal is clear, since all paths in this set have the same length. We use induction on k to establish (34). For $k = 1$ the statement is $\sum_{s' \in C_1} P(s, a_1, s') = \sum_{s' \in C_1} P(t, a_1, s')$ and it holds since R is a bisimulation relation and $\langle s, t \rangle \in R$. Consider

$$s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_{k+1}} C_{k+1} = s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k \cdot C_k \xrightarrow{a_{k+1}} C_{k+1}.$$

By the inductive hypothesis,

$$\text{Prob}(s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) = \text{Prob}(t \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k).$$

By the bisimulation condition for generative systems,

$$\text{Prob}(t' \xrightarrow{a_{k+1}} C_{k+1}) = \text{Prob}(t'' \xrightarrow{a_{k+1}} C_{k+1})$$

for all $t', t'' \in C_k$. Hence, by Corollary 5.10, we get

$$\begin{aligned}
&\text{Prob}(s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k \cdot C_k \xrightarrow{a_{k+1}} C_{k+1}) \\
&= \text{Prob}(s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) \cdot \text{Prob}(C_k \xrightarrow{a_{k+1}} C_{k+1}) \\
&= \text{Prob}(t \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) \cdot \text{Prob}(C_k \xrightarrow{a_{k+1}} C_{k+1}) \\
&= \text{Prob}(t \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k \cdot C_k \xrightarrow{a_{k+1}} C_{k+1}).
\end{aligned}$$

$\square$

We can now show that the defined map is a $*$-translation.

Proposition 5.29 *The assignment Φ^g from Definition 5.27 is a $*$-translation.*

Proof We need to check that Φ^g is injective and preserves and reflects bisimilarity. For injectivity, assume $\Phi^g(\langle S, A, \alpha \rangle) = \Phi^g(\langle S, A, \beta \rangle) = \langle S, A^*, \alpha' \rangle$. Then, by the definition of Prob, cf. (24), we get that for any $s, t \in S$ and any $a \in A$, $\alpha(s)(\langle a, t \rangle) = P(s, a, t) = \text{Prob}(s, \{a\}, \{t\}) = \alpha'(s)(\{a\}, \{t\}) = \beta(s)(\langle a, t \rangle)$.

Reflection of bisimilarity is direct from Lemma 5.26: Assume $s \sim t$ in $\Phi^g(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$ and assume that R is an equivalence bisimulation on $\langle S, A^*, \alpha' \rangle$ such that $\langle s, t \rangle \in R$. By Lemma 5.26, we get that for $W \subseteq A^*$ and for $M \in \text{Sat}(R)$,

$$\alpha'(s)(W, M) = \alpha'(t)(W, M). \qquad (35)$$

In particular, for all $a \in A$ and all $C \in S/R$, we have

$$\alpha'(s)(\{a\}, C) = \alpha'(t)(\{a\}, C). \tag{36}$$

By the definition of α' and Prob we have

$$\alpha'(s)(\{a\}, C) = \text{Prob}(s, \{a\}, C) = \sum_{s' \in C} P(s, a, s') = \sum_{s' \in C} \alpha(s)(\langle a, s' \rangle)$$

and therefore, for all $a \in A$ and all $C \in S/R$,

$$\sum_{s' \in C} \alpha(s)(\langle a, s' \rangle) = \sum_{s' \in C} \alpha(t)(\langle a, s' \rangle) \tag{37}$$

which means that R is a bisimulation equivalence on the generative system $\langle S, A, \alpha \rangle$, i.e. $s \sim t$ in the original system.

The proof of preservation of bisimilarity uses Lemma 5.28. Let $s \sim t$ in the generative system $\langle S, A, \alpha \rangle$. Then there exists an equivalence bisimulation R with $\langle s, t \rangle \in R$. The relation R induces an equivalence R_s on $\text{FPaths}(s)$ defined by

$$\langle s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \ , \ s \xrightarrow{a'_1} s'_1 \cdots \xrightarrow{a'_{k'}} s'_{k'} \rangle \in R_s$$

if and only if $k = k'$, $a_i = a'_i$ and $\langle s_i, s'_i \rangle \in R$ for $i = 1, \ldots, k$. The classes of R_s are exactly the sets $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ for $C_i \in S/R$ and $a_i \in A$.

Assume $M \in \text{Sat}(R)$ and $W \subseteq A^*$. We show that the set $s \xrightarrow{W} M$ is saturated with respect to R_s. Namely, let $\pi \equiv s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \in s \xrightarrow{W} M$ and let $\pi' \equiv s \xrightarrow{a_1} s'_1 \cdots \xrightarrow{a_k} s'_k$ be a path such that $\langle \pi, \pi' \rangle \in R_s$. Then $\text{trace}(\pi) = \text{trace}(\pi')$, $\text{first}(\pi) = \text{first}(\pi')$ and $\langle \text{last}(\pi), \text{last}(\pi') \rangle \in R$. Since M is saturated, $\text{last}(\pi') \in M$ for $\text{last}(\pi) \in M$. Furthermore, π' does not have a proper prefix with trace in W and last in M, since this would imply that π has such a prefix, contradicting $\pi \in s \xrightarrow{W} M$. Hence, $\pi' \in s \xrightarrow{W} M$.

Therefore, the set $s \xrightarrow{W} M$ is a disjoint union of some R_s classes and, since $s \xrightarrow{W} M$ is minimal, we can write

$$s \xrightarrow{W} M = \biguplus_{i \in I} s \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik_i}} C_{ik_i} \, ,$$

and it follows that $\text{Prob}(s, W, M) = \sum_{i \in I} \text{Prob}(s \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik}} C_{ik})$. Similarly, $t \xrightarrow{W} M$ is a disjoint union of some R_t classes, for R_t being an equivalence on $\text{FPaths}(t)$, defined as R_s with t instead of s. Using that R is a bisimulation and $\langle s, t \rangle \in R$, it is not difficult to see that actually

$$t \xrightarrow{W} M = \biguplus_{i \in I} t \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik_i}} C_{ik_i} \, .$$

By Lemma 5.28, we get that $\text{Prob}(s, W, M) = \text{Prob}(t, W, M)$, i.e. $\alpha'(s)(W, M) = \alpha'(t)(W, M)$ proving that R is a bisimulation on $\langle S, A^*, \alpha' \rangle$ and $s \sim t$ in the $*$-extension $\langle S, A^*, \alpha' \rangle$. $\qquad\square$

The $*$-translation Φ^g is also not induced by a natural transformation, as the systems of Example 4.4 (Section 4) when each transition is considered as probabilistic with probability 1 show.

Remark 5.30 The $*$-translation Φ^g together with a subset $\tau \subseteq A$ determines a weak-τ-bisimulation. For a generative system $\langle S, A, \alpha \rangle$, the weak-$\tau$-system is

$$\Psi_\tau \circ \Phi^g(\langle S, A, \alpha \rangle) = \Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \alpha'' \rangle$$

where $\alpha''(s) : \mathcal{P}(A_\tau) \times \mathcal{P}(S) \to [0,1]$ is given by

$$\alpha''(s) = \eta_S^\tau(\alpha'(s)) = \mathcal{G}^* \langle h_\tau, id_S \rangle (\alpha'(s)) = \alpha'(s) \circ (h_\tau^{-1} \times id_S^{-1}).$$

Hence for $X \subseteq A_\tau$ and $S' \subseteq S$,

$$\alpha''(s)(X, S') = \alpha'(s)(h_\tau^{-1}(X), S') = \alpha'(s)(\bigcup_{w \in X} B_w, S') = \mathrm{Prob}(s, \bigcup_{w \in X} B_w, S'),$$

where, B_w is the block $B_w = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* = h_\tau^{-1}(\{w\})$, for a word $w = a_1 \ldots a_k \in A_\tau$.

Therefore, from Lemma 5.26 we get that an equivalence relation R is a weak-τ-bisimulation w.r.t. $\langle \Phi^g, \tau \rangle$ on the generative system $\langle S, A, \alpha \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for any collection $(B_i)_{i \in I}$ of blocks writing B_i as a shorthand for B_{w_i} for some word $w_i \in A^*$, and any collection $(C_j)_{j \in J}$ of classes $C_j \in S/R$,

$$\mathrm{Prob}(s, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j) = \mathrm{Prob}(t, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j). \tag{38}$$

Sets of the form $\cup_{i \in I} B_i$ will be called *saturated blocks*.

5.5 Correspondence results

We are now able to state and prove the correspondence results for generative systems. The first statement is obvious from the definitions.

Theorem 5.31 *Let $\langle S, A, \alpha \rangle$ be a generative system. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 3.3 with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$ implies $s \approx_g t$ according to Definition 5.12.*

Proof The statement holds trivially, having in mind Definition 5.12 and Remark 5.30, equation (38), since τ^* as well as $\tau^* a \tau^*$, for any $a \in A \setminus \{\tau\}$ is a saturated block and also each R-equivalence class is an R saturated set. Hence $\approx_{\{\tau\}}$ is at least as strong as $\approx_g$, $\approx_{\{\tau\}} \subseteq \approx_g$. $\qquad \square$

In the opposite direction we have that coalgebraic weak bisimilarity is implied by branching bisimilarity.

Theorem 5.32 *Let $\langle S, A, \alpha \rangle$ be a generative system. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_g^{br} t$ according to Definition 5.13 implies $s \approx_{\{\tau\}} t$ according to Definition 3.3 with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$.*

In order to build the proof of the theorem, we present a sequence of lemmas.

Lemma 5.33 *Let $\langle S, A, P \rangle$ be a generative system and let $s \approx_g^{br} t$. If R is a branching bisimulation relating s and t, then for all $a_1, \ldots, a_k \in A \setminus \{\tau\}$ and for all classes $C \in S/R$*

$$\mathrm{Prob}(s, \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*, C) = \mathrm{Prob}(t, \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*, C).$$

Proof Let R be a branching bisimulation on $\langle S, A, P \rangle$ such that $\langle s, t \rangle \in R$. We prove, by induction on k, that

$$\mathrm{Prob}(s, \tau^* a_1 \tau^* \ldots \tau^* a_k, C) = \mathrm{Prob}(t, \tau^* a_1 \tau^* \ldots \tau^* a_k, C).$$

For $k \in \{0, 1\}$ the property holds by Definition 5.13. Let $B = \tau^* a_1 \tau^* \ldots \tau^* a_k$. Assume $\mathrm{Prob}(s, B, C) = \mathrm{Prob}(t, B, C)$ for all $C \in S/R$ and let $B' = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* a_{k+1}$. We have

$$s \xrightarrow{B'} C = \biguplus_{C' \in S/R} s \xrightarrow{B} C' \cdot C' \xrightarrow{\tau^* a_{k+1}} C$$

and, since R is a branching bisimulation, for any class $C' \in S/R$ and for any $t', t'' \in C'$ we have $\mathrm{Prob}(t', \tau^* a_{k+1}, C) = \mathrm{Prob}(t'', \tau^* a_{k+1}, C)$ and we may write this common value as $\mathrm{Prob}(C', \tau^* a_{k+1}, C)$. Hence, we may apply Corollary 5.10 and we get,

$$
\begin{aligned}
\mathrm{Prob}(s, B', C) \quad &= \quad \sum_{C' \in S/R} \mathrm{Prob}(s, B, C') \cdot \mathrm{Prob}(C', \tau^* a_{k+1}, C) \\
&\stackrel{(IH)}{=} \quad \sum_{C' \in S/R} \mathrm{Prob}(t, B, C') \cdot \mathrm{Prob}(C', \tau^* a_{k+1}, C) \\
&= \quad \mathrm{Prob}(t, B', C).
\end{aligned}
$$

Finally, the property holds since we have, for $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$ and $B' = \tau^* a_1 \tau^* \ldots \tau^* a_k$,

$$s \xrightarrow{B} C = \biguplus_{C' \in S/R} s \xrightarrow{B'} C' \cdot C' \xrightarrow{\tau^*} C.$$

$\square$

Lemma 5.34 *Assume that R is a complete branching bisimulation on a generative system $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, with $\langle s, t \rangle \in R$. For any saturated set $M = \sqcup_{i=1}^n C_i$ consisting of finitely many classes $C_i \in S/R$, for any block $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$ where $a_1, \ldots, a_k \in A \setminus \{\tau\}$ and for any $i \in \{1, \ldots, n\}$,*

$$\mathrm{Prob}(s, B, C_i, \neg M) = \mathrm{Prob}(t, B, C_i, \neg M).$$

Proof We use induction on n, the number of classes that M contains. For $n = 1$ the property is simply Lemma 5.33. Assume $\text{Prob}(s, B, C_i, \neg M) = \text{Prob}(t, B, C_i, \neg M)$ for any R-saturated set M being a union of less than n classes, and each class $C_i \subseteq M$. Let M be an R-saturated set which is a union of n classes, i.e. $M = \sqcup_{i=1}^{n} C_i$ for some $C_i \in S/R$. We use the following notation, for $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$:

$$V_i = \text{Prob}(s, B, C_i) \overset{\text{Lem.5.33}}{=} \text{Prob}(t, B, C_i)$$

$$G_i^j = \text{Prob}(s, B, C_j, \neg \sqcup_{k=1, k \neq i}^{n} C_k) \overset{IH}{=} \text{Prob}(t, B, C_j, \neg \sqcup_{k=1, k \neq i}^{n} C_k).$$

Therefore, it is justified to put

$$T_i^j = \text{Prob}(C_j, \tau^*, C_i)$$

$$H_i^j = \text{Prob}(C_i, \tau^*, C_j, \neg \sqcup_{k=1, k \neq i}^{n} C_k).$$

We define a function $\omega : s \overset{B}{\to} S \to \{1, 2\}^*$. The function ω will, in a sense, trace the classes that a path visits with a word in B. Two auxiliary functions $\tilde{\omega}$ and d will be needed for the definition of ω. We can explain the definition of the maps d, $\tilde{\omega}$ and ω as follows: The map $\tilde{\omega}$ takes a path with a trace in B and encodes the sequence of the classes that are visited by the path, after a word in B has already been performed. The encoding is 1 if the class under consideration, C_i, has been visited and 2 if any other class from M has been visited, there is no record of classes outside M. Then the map d removes adjacent multiple occurrences of 1 and 2 in the word obtained by $\tilde{\omega}$, except for the multiple occurrences at the end of the word. The intuition is that multiple occurrences at the end of the word indicate that the paths mapped are of a different nature. For example, $\omega^{-1}(\{11\})$ is not a minimal set of paths whereas $\omega^{-1}(\{1\})$ is and we need to distinguish between them. Basically, the map d is computed by the normal algorithm $\{112 \to 12, 221 \to 21\}$. We put $\omega = d \circ \tilde{\omega}$.

More precisely, $\tilde{\omega} : \left(s \overset{B}{\to} S \right) \to \{1, 2\}^*$ is defined by

$$\tilde{\omega}(\pi \cdot \text{last}(\pi) \overset{a}{\to} r) = \begin{cases} 1 & r \in C_i, \pi \notin s \overset{B}{\to} S \\ 2 & r \in M \setminus C_i, \pi \notin s \overset{B}{\to} S \\ \varepsilon & r \notin M, \pi \notin s \overset{B}{\to} S \\ \tilde{\omega}(\pi) \cdot 1 & r \in C_i, \pi \in s \overset{B}{\to} S \\ \tilde{\omega}(\pi) \cdot 2 & r \in M \setminus C_i, \pi \in s \overset{B}{\to} S \\ \tilde{\omega}(\pi) & r \notin M, \pi \in s \overset{B}{\to} S \end{cases}$$

and if $\varepsilon \in s \overset{B}{\to} S$, then $\tilde{\omega}(\varepsilon) = \varepsilon$.

Let $d : \{1,2\}^* \to \{1,2\}^*$ and $d' : \{1,2\}^* \to \{1,2\}^*$ be defined in the following way, for $u, v \in \{1,2\}^*$ and $x, y \in \{1,2\}$:

$$d(u \cdot x) = \begin{cases} d(u) \cdot x & u = v \cdot x \\ d'(u) \cdot x & u = v \cdot y, y \neq x \end{cases}$$

$$d'(u \cdot x) = \begin{cases} d'(u) & u = v \cdot x \\ d'(u) \cdot x & u = v \cdot y, y \neq x \end{cases}$$

and $d(\varepsilon) = d'(\varepsilon) = \varepsilon$. It is important to note that

$$\omega^{-1}(\{1\}) = s \xrightarrow[\neg M]{B} C_i.$$

Hence, we need to calculate $\mathrm{Prob}(\omega^{-1}(\{1\}))$. By the definition of ω we easily get that

$$\omega^{-1}(\{1, 21\}) = \omega^{-1}(\{1\}) \uplus \omega^{-1}(\{21\}).$$

Therefore, we try to express $\mathrm{Prob}(\omega^{-1}(\{1,21\}))$ and $\mathrm{Prob}(\omega^{-1}(\{21\}))$ in terms of V_i, G_i^j, T_i^j and H_i^j. It is obvious that $\mathrm{Prob}(\omega^{-1}(\{1,21\})) = \mathrm{Prob}(s, B, C_i) = V_i$. A more careful inspection shows that

$$\omega^{-1}(\{21\}) \uplus \left(\biguplus_{j=1, j \neq i}^{n} \omega^{-1}(\{1\}) \cdot C_i \xrightarrow[\neg M \backslash C_i]{\tau^*} C_j \xrightarrow{\tau^*} C_i \right)$$

$$= \biguplus_{j=1, j \neq i}^{n} s \xrightarrow[\neg M \backslash C_i]{B} C_j \xrightarrow{\tau^*} C_i.$$

This, together with Proposition 5.9 and Corollary 5.10, implies that

$$\mathrm{Prob}(\omega^{-1}(\{21\})) = \sum_{j=1, j \neq i}^{n} G_i^j \cdot T_i^j - \mathrm{Prob}(\omega^{-1}(\{1\})) \cdot \sum_{j=1, j \neq i}^{n} H_i^j \cdot T_i^j$$

and we get

$$\mathrm{Prob}(\omega^{-1}(\{1\}))$$
$$= \mathrm{Prob}(\omega^{-1}(\{1, 21\})) - \mathrm{Prob}(\omega^{-1}(\{21\}))$$
$$= V_i - \left(\sum_{j=1, j \neq i}^{n} G_i^j \cdot T_i^j - \mathrm{Prob}(\omega^{-1}(\{1\})) \cdot \sum_{j=1, j \neq i}^{n} H_i^j \cdot T_i^j \right).$$

Let $\rho = \sum_{j=1, j \neq i}^{n} H_i^j \cdot T_i^j$. Let $T_i = \max_{j=1, j \neq i}^{n} T_i^j$. By the completeness of R, $T_i^j < 1$ for all $j \neq i$ and therefore $T_i < 1$. Furthermore, by Proposition 5.11,

$$\sum_{j=1, j \neq i}^{n} H_i^j = \mathrm{Prob}(C_i, \tau^*, \bigsqcup_{j=1, j \neq i}^{n} C_j) \leq 1.$$

Hence,

$$\rho = \sum_{j=1, j \neq i}^{n} H_i^j \cdot T_i^j \leq T_i \cdot \sum_{j=1, j \neq i} H_i^j \leq T_i < 1.$$

We have

$$\text{Prob}(s, B, C_i, \neg M) = V_i - \sum_{j=1, j \neq i}^{n} G_i^j \cdot T_i^j + \text{Prob}(s, B, C_i, \neg M) \cdot \rho$$

and, since $\rho < 1$, we obtain

$$\text{Prob}(s, B, C_i, \neg M) = \frac{V_i - \sum_{j=1, j \neq i}^{n} G_i^j \cdot T_i^j}{1 - \rho}.$$

The expression on the right side does not depend on the starting state s and we get

$$\text{Prob}(s, B, C_i, \neg M) = \text{Prob}(t, B, C_i, \neg M)$$

which completes the proof. $\qquad \square$

Next we extend the property captured by Lemma 5.34 to arbitrary R-saturated sets.

Lemma 5.35 *Assume that R is a complete branching bisimulation on a generative system $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, with $\langle s, t \rangle \in R$. For any R-saturated set M, for any block $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$ where $a_1, \ldots, a_k \in A \setminus \{\tau\}$ and for any class $C \subseteq M$*

$$\text{Prob}(s, B, C, \neg M) = \text{Prob}(t, B, C, \neg M).$$

Proof

Let $C \subseteq M$. We will show that we can assume that M contains at most countably many classes. Let S' be the set of states that are reachable from s by a finite path. This set is at most countable since each finite path contributes to S' with finitely many states, and there are at most countably many paths starting in s according to Lemma 5.1. Let M_s be the smallest R-saturated set containing $S' \cap M$ and C. Since $S' \cap M$ is at most countable, the set M_s contains at most countably many classes and $\text{Prob}(s, B, C, \neg M) = \text{Prob}(s, B, C, \neg M_s)$. In the same way we get a saturated set M_t containing at most countably many classes such that $\text{Prob}(t, B, C, \neg M) = \text{Prob}(t, B, C, \neg M_t)$. Then $M' = M_s \cup M_t$ is a saturated set containing at most countably many classes. Moreover,

$$\text{Prob}(s, B, C, \neg M') = \text{Prob}(s, B, C, \neg M),$$

$$\text{Prob}(t, B, C, \neg M') = \text{Prob}(t, B, C, \neg M).$$

So, assume $M = \sqcup_{i \geq 0} C_i$, and $C = C_{i_0}$. Note that

$$s \xrightarrow[\neg M]{B} C = \bigcap_{k \geq i_0} s \xrightarrow[\neg U_k]{B} C$$

for $U_k = C_0 \cup \cdots \cup C_k$, and the intersection is clearly countable. Moreover, let $J = \{I \mid I \subseteq \mathbb{N} \setminus \{0, \ldots, i_0 - 1\}, I \text{ is finite}\}$. If $I \in J$ with $m = \max(I)$, then

$$\bigcap_{i \in I} s \xrightarrow[\neg U_i]{B} C = s \xrightarrow[\neg U_m]{B} C.$$

We use the following simple property from measure theory: If μ is a probability measure on some set and if $A = \cap_{n \in \mathbb{N}} A_n$ is a measurable set which is a countable intersection of measurable sets, then $\mu(A) = \inf\{\mu(\cap_{i \in I} A_i) \mid I \subseteq \mathbb{N}, I \text{ finite}\}$. Hence,

$$
\begin{aligned}
\text{Prob}&(s, B, C, \neg M) \\
&= \quad \inf\{\text{Prob}(\cap_{i \in I} s \xrightarrow[\neg U_i]{B} C) \mid I \in J\} \\
&= \quad \inf\{\text{Prob}(s, B, C, \neg U_m) \mid I \in J, m = \max(I)\} \\
&\overset{\text{Lem.5.34}}{=} \quad \inf\{\text{Prob}(t, B, C, \neg U_m) \mid I \in J, m = \max(I)\} \\
&= \quad \text{Prob}(t, B, C, \neg M).
\end{aligned}
$$

$\square$

By Lemma 5.35, noting that $\text{Prob}(s, B, M) = \text{Prob}(s, B, \sqcup_{i \in I} C_i) = \sum_{i \in I} \text{Prob}(s, B, C_i, \neg M)$, we get the following property.

Corollary 5.36 *Assume that R is a complete branching bisimulation on a generative system $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, with $\langle s, t \rangle \in R$. For any R-saturated set M it holds that*
$$\text{Prob}(s, B, M) = \text{Prob}(t, B, M),$$
for any block $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$ with $a_1, \ldots, a_k \in A \setminus \{\tau\}$. $\square$

We proceed to saturated blocks. Again, we first treat saturated blocks containing finitely many blocks and then extend to arbitrary saturated blocks.

Lemma 5.37 *Assume that R is a complete branching bisimulation on a generative system $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, with $\langle s, t \rangle \in R$. For any R-saturated set M and for any saturated block $W = \sqcup_{i=1}^n B_i$ containing finitely many blocks, it holds that*
$$\text{Prob}(s, W, M) = \text{Prob}(t, W, M).$$

Proof Note that

$$\text{Prob}(s, W, M) = \sum_{i=1}^n \text{Prob}(s, B_i, \neg W, M)$$

since

$$s \xrightarrow{W} M = \biguplus_{i=1}^n s \xrightarrow[\neg W]{B_i} M,$$

and also

$$\text{Prob}(s, B_i, \neg W, M) = \sum_{j:\, C_j \subseteq M} \text{Prob}(s, B_i, \neg W, C_j, \neg M)$$

since

$$s \xrightarrow{B_i}_{\neg W} M = \biguplus_{C_j \subseteq M} s \xrightarrow[\neg M]{B_i}_{\neg W} C_j,$$

as in Proposition 5.11. (Here, C_j stands for an equivalence class of R and M is a disjoint union of classes.) The summation is possible since all but at most countably many summands are empty. Hence it suffices to prove that

$$\text{Prob}(s, B_i, \neg W, C_j, \neg M) = \text{Prob}(t, B_i, \neg W, C_j, \neg M)$$

for any B_i, $i \in \{1, \ldots, n\}$ and any class $C_j \subseteq M$. For any i, let $w_i \in A \setminus \{\tau\}^*$, $w_i = a_{i1} \ldots a_{ik_i}$ be the word such that $B_i = B_{w_i} = \tau^* a_{i1} \tau^* \cdots \tau^* a_{ik_i} \tau^*$. The prefix ordering on the set of words $\{w_1, \ldots, w_n\}$ induces an ordering on the set of blocks $\{B_1, \ldots, B_n\}$ given by $B_i \prec B_j$ if and only if $w_i \prec w_j$. If $B_i \prec B_j$, by B_{j-i} we denote the block corresponding to w_{j-i}, the unique word satisfying $w_i \cdot w_{j-i} = w_j$. We are going to prove, by induction on the number of elements in the set $\{i \in \{1, \ldots, n\} \mid B_i \prec B_j\}$, that

$$s \xrightarrow{B_j}_{\neg M} C = s \xrightarrow[\neg M]{B_j}_{\neg W} C \uplus \left(\biguplus_{B_i \prec B_j} \biguplus_{C' \subseteq M} s \xrightarrow[\neg M]{B_i}_{\neg W} C' \xrightarrow{B_{j-i}}_{\neg M} C \right) \qquad (39)$$

where $C' \subseteq M$ is a class. First of all we have to make sure that the right hand side of the equation is well defined, i.e. that the unions are really disjoint and minimal. By the definition of the involved sets of paths a careful inspection shows that it is indeed the case. It is rather obvious that the right hand side is contained in the left hand side since all the paths of the right hand side do start in s, have a trace in B_j and end up in C, without reaching M before with a prefix whose trace is also in B_j. For the opposite inclusion we use an inductive argument. Assume B_j has no (strict) prefixes in $\{B_1, \ldots, B_n\}$. Then the equation becomes $s \xrightarrow{B_j}_{\neg M} C = s \xrightarrow[\neg M]{B_j}_{\neg W} C$ and it holds since, by assumption, no path which has a trace in B_j can have a strict prefix with a trace in W which does not belong to B_j. For the inductive step, assume $\pi \in s \xrightarrow{B_j}_{\neg M} C$ and $\pi \notin s \xrightarrow[\neg M]{B_j}_{\neg W} C$. This means that π has a prefix that has a trace in $\cup_{i=1, i \neq j}^n B_i$ and ends in M. So, $\pi \in s \xrightarrow{B_k} C' \xrightarrow{B_{j-k}}_{\neg M} C$ for some k and for some class $C' \subseteq M$. We want to show that $\pi \in \uplus_{B_i \prec B_j} \uplus_{C' \subseteq M} s \xrightarrow[\neg M]{B_i}_{\neg W} C' \xrightarrow{B_{j-i}}_{\neg M} C$.

We can assume that $\pi \in s \xrightarrow{B_k}_{\neg M} C' \xrightarrow{B_{j-k}}_{\neg M} C$ by taking C' to be the first class of M that π hits after having performed a trace in B_k. Now B_k, being a proper prefix of B_j, has less prefixes than B_j and therefore, by the inductive hypothesis, either

$$\pi \in s \xrightarrow[\neg M]{B_k}_{\neg W} C' \xrightarrow{B_{j-k}}_{\neg M} C$$

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or there exist $r \in \{1, \ldots, n\}$ and a class $C'' \subseteq M$ such that

$$\pi \in s \xrightarrow[\neg M]{B_r}{}_{\neg W} C'' \xrightarrow{B_{k-r}}{}_{\neg M} C' \xrightarrow{B_{j-k}}{}_{\neg M} C,$$

i.e. $\pi \in s \xrightarrow[\neg M]{B_r}{}_{\neg W} C'' \xrightarrow{B_{j-r}}{}_{\neg M} C$, which completes the proof of equation (39). Now, by the same inductive argument, if B_j has no proper prefixes, then

$$
\begin{aligned}
\mathrm{Prob}(s, B_j, \neg W, C, \neg M) &= \mathrm{Prob}(s, B_j, C, \neg M) \\
&\overset{\mathrm{Lem.5.35}}{=} \mathrm{Prob}(t, B_j, C, \neg M) \\
&= \mathrm{Prob}(t, B_j, \neg W, C, \neg M).
\end{aligned}
$$

Assume that $\mathrm{Prob}(s, B_i, \neg W, C, \neg M) = \mathrm{Prob}(t, B_i, \neg W, C, \neg M)$ for all $B_i \prec B_j$. Then by (39), by Proposition 5.9 and by Lemma 5.35, we get

$$
\begin{aligned}
\mathrm{Prob}&(s, B_j, \neg W, C, \neg M) \\
&= \mathrm{Prob}(s, B_j, C, \neg M) \\
&\quad - \sum_{B_i \prec B_j} \sum_{C' \subseteq M} \mathrm{Prob}(s, B_i, \neg W, C', \neg M) \cdot \mathrm{Prob}(C', B_{j-i}, C, \neg M) \\
&\overset{(IH)}{=} \mathrm{Prob}(t, B_j, C, \neg M) \\
&\quad - \sum_{B_i \prec B_j} \sum_{C' \subseteq M} \mathrm{Prob}(t, B_i, \neg W, C', \neg M) \cdot \mathrm{Prob}(C', B_{j-i}, C, \neg M) \\
&= \mathrm{Prob}(t, B_j, \neg W, C, \neg M)
\end{aligned}
$$

which completes the proof. $\qquad\square$

We next extend the last property to arbitrary saturated blocks.

Lemma 5.38 *Assume that R is a complete branching bisimulation on a generative system $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, with $\langle s, t \rangle \in R$. For any R-saturated set M and for any saturated block W*

$$\mathrm{Prob}(s, W, M) = \mathrm{Prob}(t, W, M).$$

Proof We first consider the countable case. Let $W = \sqcup_{n \in \mathbb{N}} B_n$. Let

$$\Pi_n^s = \{\pi \mid \mathrm{first}(\pi) = s, \mathrm{last}(\pi) \in M, \mathrm{trace}(\pi) \in B_n\}$$

$$\Pi_n^t = \{\pi \mid \mathrm{first}(\pi) = t, \mathrm{last}(\pi) \in M, \mathrm{trace}(\pi) \in B_n\}.$$

Then

$$
\begin{aligned}
\mathrm{Prob}(s, W, M) &= \mathrm{Prob}(s, \sqcup_{n \in \mathbb{N}} B_n, M) \\
&= \mathrm{Prob}((\cup_{n \in \mathbb{N}} \Pi_n^s) \downarrow) \\
&= \mathrm{Prob}(\cup_{n \in \mathbb{N}} \Pi_n^s) \\
&\stackrel{(*)}{=} \sup\{\mathrm{Prob}(\cup_{i \in I} \Pi_i^s) \mid I \subseteq \mathbb{N}, I \text{ finite }\} \\
&= \sup\{\mathrm{Prob}(s, W_I, M) \mid W_I = \sqcup_{i \in I} B_i, I \text{ finite }\} \\
&= \sup\{\mathrm{Prob}(t, W_I, M) \mid W_I = \sqcup_{i \in I} B_i, I \text{ finite }\} \\
&= \mathrm{Prob}(t, W, M),
\end{aligned}
$$

where the equality $(*)$ holds because of the following elementary property from measure theory: Let μ be a measure on some set, and let $A = \cup_{n \in \mathbb{N}} A_n$ be a measurable set which is a countable union of measurable sets. Then $\mu(A) = \sup\{\mu(\cup_{i \in I} A_i) \mid I \subseteq \mathbb{N}, I \text{ finite}\}$.

If $W = \sqcup_{i \in I} B_i$ contains arbitrary many blocks then there exists a countable index set $I_s \subseteq I$ and a saturated set $W_s = \sqcup_{i \in I_s} B_i$ such that $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(s, W_s, M)$ using Lemma 5.1. For the same reason, there exists a countable index set $I_t \subseteq I$ and a corresponding saturated set $W_t = \sqcup_{i \in I_t} B_i$ with $\mathrm{Prob}(t, W, M) = \mathrm{Prob}(t, W_t, M)$. Hence $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(s, W_s \cup W_t, M) = \mathrm{Prob}(t, W_s \cup W_t, M) = \mathrm{Prob}(t, W, M)$ since $W_s \cup W_t$ is countable. $\square$

Finally, we can prove Theorem 5.32.

Proof [of Theorem 5.32] Assume $s \approx_g^{br} t$ in a system $\langle S, A, \alpha \rangle$. Let R be a branching bisimulation according to Definition 5.13 such that $\langle s, t \rangle \in R$. By Proposition 5.16, we can assume that R is complete. By Lemma 5.38, we get that the transfer condition (38) of Remark 5.30 holds, and hence R is a coalgebraic weak bisimulation witnessing that $s \approx_{\{\tau\}} t$. $\square$

By Theorem 5.31, Theorem 5.32, and Proposition 5.14 we get the following corollary which gives us the correspondence result for finite systems.

Corollary 5.39 *For finite generative systems, coalgebraic weak bisimilarity $\approx_{\{\tau\}}$ according to Definition 3.3, with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$, coincides with concrete weak bisimilarity $\approx_g$ according to Definition 5.12.* $\square$

6 Concluding remarks

In this paper, we have proposed a coalgebraic definition of weak bisimulation for action-type systems. For its justification we have considered the case of familiar labelled transition systems and of generative probabilistic systems, and we have compared our notion to the concrete definitions. In particular, we have obtained that the coalgebraic definition of weak bisimulation (for a suitably chosen $*$-extension) for LTSs coincides with the standard definition of weak bisimulation.

For generative probabilistic systems, the situation is more complex. Most of the work and technical difficulties of this paper are related to the correspondence results for generative probabilistic systems. As the standard notion of concrete weak bisimulation we have adopted the one proposed by Baier and Hermanns. The same authors also propose a notion of branching bisimulation. Their investigations and results are limited to finite systems where, as the authors show, the concrete notions of weak and branching bisimulation coincide. On the other hand, we provide a coalgebraic definition of weak bisimulation for generative systems that is not limited to finite systems. The situation is as follows:

$$\text{concrete branching} \subseteq \text{coalgebraic weak} \subseteq \text{concrete weak}$$

As mentioned before, in case of finite systems, we have

$$\text{concrete branching} = \text{concrete weak.}$$

So, in the finite case, that was considered for the concrete notions, all three notions: coalgebraic weak, concrete weak, and concrete branching coincide. The situation for the infinite case remains to be unravelled, although it seems that the coincidence of concrete branching and concrete weak bisimulation will carry over to a wide class of well-behaved infinite systems.

It is clear that the present approach is not the final word to the weak bisimulation problem for coalgebras. In particular, the main issue here is that one has to come up with a suitable definition of a $*$-translation oneself, in order to obtain a weak bisimulation for a class of coalgebras of a given type. Ideally, a coalgebraic construction would automatically induce the $*$-translation. A method for systematically obtaining $*$-translations is a topic for further research.

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A (Weak) Pullbacks and their preservation

A *span* $\langle S, s_1, s_2 \rangle$, between X and Y, is a diagram of the form $X \xleftarrow{s_1} S \xrightarrow{s_2} Y$. It is *jointly injective* if the mapping $\langle s_1, s_2 \rangle : S \to X \times Y$, defined by $\langle s_1, s_2 \rangle(s) = \langle s_1(s), s_2(s) \rangle$ is injective. A relation $R \subseteq X \times Y$ gives rise to the jointly injective span $\langle R, \pi_1, \pi_2 \rangle$ between X and Y. Dually, a cospan $\langle C, c_1, c_2 \rangle$ is a diagram of the form $X \xrightarrow{c_1} C \xleftarrow{c_2} Y$.

A pullback, of a cospan $\langle C, c_1, c_2 \rangle$, is a span $\langle P, p_1, p_2 \rangle$ as in the diagram below satisfying $c_1 \circ p_1 = c_2 \circ p_2$ and such that for every span $\langle S, s_1, s_2 \rangle$ with $c_1 \circ s_1 = c_2 \circ s_2$ there exists a unique mediating map $m : S \to P$ satisfying $s_1 = p_1 \circ m$ and $s_2 = p_2 \circ m$. A *weak pullback* is a pullback for which the mediating arrow m need not be unique.
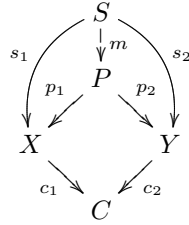
A pullback of a cospan $\langle C, c_1, c_2 \rangle$ between sets X and Y is the span arising from the relation

$$Q := \{ \langle x, y \rangle \in X \times Y \mid c_1(x) = c_2(y) \}.$$

A weak pullback arising from a relation $R \subseteq X \times Y$ is also an ordinary pullback, as one can derive from the joint injectivity of the two projections.

A functor $\mathcal{F}$ is said to *preserve* a (weak) pullback $\langle P, p_1, p_2 \rangle$ of a cospan $\langle C, c_1, c_2 \rangle$, if $\langle \mathcal{F}P, \mathcal{F}p_1, \mathcal{F}p_2 \rangle$ is again a (weak) pullback of $\langle \mathcal{F}C, \mathcal{F}c_1, \mathcal{F}c_2 \rangle$, i.e. if it transforms a (weak) pullback of a cospan into a (weak) pullback of the transformed cospan. The functor $\mathcal{F}$ *weakly preserves* a pullback of a cospan if it transforms it into a weak pullback of the transformed cospan. We note the following two properties taken from [19, 18].

Lemma A.1 *Let $\mathcal{F}$ be a* Set *endofunctor. Then*

(i) $\mathcal{F}$ *preserves weak pullbacks if and only if it weakly preserves pullbacks.*

(ii) $\mathcal{F}$ *preserves weak pullbacks if and only if for any cospan $\langle C, c_1, c_2 \rangle$ we have: Given u and v with $\mathcal{F}c_1(u) = \mathcal{F}c_2(v)$ then there exists a $w \in \mathcal{F}\{\langle x, y \rangle \mid c_1(x) = c_2(y)\}$ with $\mathcal{F}\pi_1(w) = u$ and $\mathcal{F}\pi_2(w) = v$.* □

We end this section by mentioning a special type of pullback. A (weak) pullback $\langle P, p_1, p_2 \rangle$ is said to be *total* if its canonical morphisms, or legs, p_1 and

p_2 are epi. In Set a pullback of a cospan $\langle C, c_1, c_2 \rangle$ where $c_1 : X \to C$ and $c_2 : Y \to C$ are surjective, is a total pullback. Moreover, it is easy to see the following.

Lemma A.2 *In* Set, *the pullback of a cospan $\langle C, c_1 : X \to C, c_2 : Y \to C \rangle$ is total if and only if the images of X and Y under c_1 and c_2, respectively, are equal, i.e. $c_1(X) = c_2(Y)$.* □

We say that a functor weakly preserves total pullbacks if it transforms any total pullback into a weak pullback. According to Lemma A.2, weakly preserving total pullbacks is the same as weakly preserving pullbacks of cospans $\langle C, c_1, c_2 \rangle$ with $c_1(X) = c_2(Y)$. Clearly, if a functor preserves weak pullbacks, then it weakly preserves total pullbacks. We shall see in Appendix C that weak preservation of total pullbacks is a strictly weaker notion, i.e., there exists a functor that weakly preserves total pullbacks but does not preserve weak pullbacks.

B Weak pullback preservation of the distribution functor

Here we establish the weak pullback preservation of $\mathcal{G}_A$, the functor defining generative probabilistic systems. Actually, we show weak pullback preservation of the probability distribution functor $\mathcal{D}$. For the probability distribution functor with finite support weak pullback preservation was proven by De Vink and Rutten [43], using the graph-theoretic min cut - max flow theorem, and by Moss [30], using an elementary matrix fill-in property. Following Moss [30] we show that the needed matrix fill-in property can be used and holds for arbitrary, infinite, matrices as well.

We start with a simple auxiliary property, that is also needed for the proof of Lemma 5.1 (Section 5.1). This property also justifies the name "discrete" probability distributions.

Lemma B.1 *Let $f : S \to \mathbb{R}_{\geq 0}$ be a function with the property $\sum_{s \in S} f(s) < \infty$. Then the support set of this function, $\mathrm{supp}(f) = \{s \in S \mid f(s) > 0\}$ is at most countable.*

Proof Let $s \in \mathrm{supp}(f)$. Then $f(s) > 0$ and therefore there exists a natural number n such that $f(s) > 1/n$. So we have, $\mathrm{supp}(f) \subseteq \cup_{n \in \mathbb{N}} \mathrm{supp}_n(\mu)$ where $\mathrm{supp}_n(\mu) = \{s \in \mathrm{supp}(\mu) \mid f(s) > 1/n\}$. Now, since $\sum_{s \in \mathrm{supp}(f)} f(s) = r < \infty$, the set $\mathrm{supp}_n(f)$ has less than n/r elements, i.e., it is finite , for all $n \in \mathbb{N}$. Therefore the set $\mathrm{supp}(f)$ is at most countable, being a countable union of finite sets. □

Next we present the matrix fill-in property for countable matrices.

Lemma B.2 *For any two infinite sequences of non-negative real numbers* $(x_i)_{i \in \mathbb{N}}$ *and* $(y_j)_{j \in \mathbb{N}}$ *such that*

$$\sum_{i \in \mathbb{N}} x_i = \sum_{j \in \mathbb{N}} y_j < \infty,$$

there exist non-negative real numbers $(z_{i,j})_{i,j \in \mathbb{N}}$ *such that*

$$\sum_{j \in \mathbb{N}} z_{i,j} = x_i \quad and \quad \sum_{i \in \mathbb{N}} z_{i,j} = y_j,$$

for all $i \in \mathbb{N}$ *and* $j \in \mathbb{N}$, *respectively.*

Before we present the rather technical proof, let us discuss the idea, also used in [30], on a finite example. Let two finite sequences x and y be given by $x_1 = 2, x_2 = 1, x_3 = 3$ and $y_1 = 1, y_2 = 3, y_3 = 0, y_4 = 2$. Since

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 + y_4$$

the statement claims that there exists a matrix Z, in this case of order 3×4, such that x_i is the sum of the i-th row and y_j the sum of the j-th column. The matrix

$$Z = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

satisfies that property. We have constructed it in the following way. For $z_{1,1}$ we take the minimum $\min\{x_1, y_1\}$, hence $z_{1,1} = y_1 = 1$. Since the first column sum has already been achieved we fill-in $z_{2,1} = z_{3,1} = 0$ and the next element to be filled-in is $z_{1,2}$. We fill it with the value $\min\{x_1 - z_{1,1}, y_2\} = x_1 - z_{1,1} = 1$. Since the first row-sum has been achieved, we put $z_{1,3} = z_{1,4} = 0$, and continue with $z_{2,2}$. It gets the value $\min\{x_2 - z_{2,1}, y_2 - z_{1,2}\} = x_2 - z_{2,1} = 1$. Hence, $z_{2,3} = z_{2,4} = 0$ and the next element to be filled-in is $z_{3,2}$. Its value is then $\min\{x_3 - z_{3,1}, y_2 - z_{1,2} - z_{2,2}\} = y_2 - z_{1,2} - z_{2,2} = 1$, which completes the second column. Next is $z_{3,3} = \min\{x_3 - z_{3,1} - z_{3,2}, y_3 - z_{1,3} - z_{2,3}\} = y_3 - z_{1,3} - z_{2,3} = 0$. We fill-in the last element $z_{3,4}$ with the remaining value $x_3 - z_{3,1} - z_{3,2} - z_{3,3} = y_4 - z_{1,4} - z_{2,4} - z_{3,4} = 2$.

Proof [of Lemma B.2] Define, for $n \in \mathbb{N}$, inductively, non-negative numbers $(z_{i,j}^n)_{i,j \in \mathbb{N}}$ and indices i_n, j_n as follows. We put $z_{i,j}^0 = 0$ for all i, j and $i_0 = 0$, $j_0 = 0$. Next, assume, for some $n \in \mathbb{N}$, the numbers $z_{i,j}^n$ and indices i_n, j_n are defined. Put

$$\xi_n = \textstyle\sum_{j < j_n} z_{i_n,j}^n \text{ and } \eta_n = \textstyle\sum_{i < i_n} z_{i,j_n}^n.$$

We distinguish three cases.

(i) $x_{i_n} - \xi_n < y_{j_n} - \eta_n$:
   Then we define $z_{i_n,j_n}^{n+1} = x_{i_n} - \xi_n$ and $z_{i,j}^{n+1} = z_{i,j}^n$ if $i \neq i_n$ or $j \neq j_n$. Additionally, we put $i_{n+1} = i_n + 1$ and $j_{n+1} = j_n$.

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(ii) $x_{i_n} - \xi_n = y_{j_n} - \eta_n$:
Then we define $z^{n+1}_{i_n,j_n} = x_{i_n} - \xi_n = y_{j_n} - \eta_n$ and $z^{n+1}_{i,j} = z^n_{i,j}$ if $i \neq i_n$ or $j \neq j_n$ and we set $i_{n+1} = i_n + 1$ and $j_{n+1} = j_n + 1$.

(iii) $x_{i_n} - \xi_n > y_{j_n} - \eta_n$:
Then we define $z^{n+1}_{i_n,j_n} = y_{j_n} - \eta_n$ and $z^{n+1}_{i,j} = z^n_{i,j}$ if $i \neq i_n$ or $j \neq j_n$. We also put $i_{n+1} = i_n$ and $j_{n+1} = j_n + 1$.

Note that in any case $z^{n+1}_{i_n,j_n} = \min\{x_{i_n} - \xi_n, y_{j_n} - \eta_n\}$.

We claim that for all n, if $i > i_n$ or $j > j_n$, then

$$z^n_{i,j} = 0, \tag{40}$$

and, for all $i, j \in \mathbb{N}$,

$$\sum_j z^n_{i,j} \leq x_i \quad \text{and} \quad \sum_i z^n_{i,j} \leq y_j. \tag{41}$$

This can be verified by induction on n. The base case, $n = 0$, is clear, as $z^0_{i,j} = 0$ for all i, j. As to the induction step, suppose equations (40) and (41) hold for n. Note that $i_{n+1} \geq i_n$ and $j_{n+1} \geq j_n$. Hence, if $i > i_{n+1}$ or $j > j_{n+1}$ we have that $z^{n+1}_{i,j} = z^n_{i,j} = 0$. Further, for $i \neq i_n$, $\sum_j z^{n+1}_{i,j} = \sum_j z^n_{i,j} \leq x_i$. Also,

$$
\begin{aligned}
\sum_j z^{n+1}_{i_n,j} &= \Big(\sum_{j<j_n} z^{n+1}_{i_n,j}\Big) + z^{n+1}_{i_n,j_n} + \Big(\sum_{j>j_n} z^{n+1}_{i_n,j}\Big) \\
&\overset{(a)}{=} \Big(\sum_{j<j_n} z^n_{i_n,j}\Big) + z^{n+1}_{i_n,j_n} + 0 \\
&\leq \Big(\sum_{j<j_n} z^n_{i_n,j}\Big) + x_{i_n} - \sum_{j<j_n} z^n_{i,j} \\
&= x_{i_n}.
\end{aligned}
$$

where the equality (a) holds by the definition of $z^{n+1}_{i,j}$ for $j \neq j_n$ and the inductive hypothesis, and the inequality holds by the definition of $z^{n+1}_{i_n,j_n}$ and ξ_n.

Hence, $\sum_j z^{n+1}_{i,j} \leq x_i$. Similarly, $\sum_i z^{n+1}_{i,j} \leq y_j$. This proves validity of the equations (40) and (41).

We next prove that

$$\sum_j z^n_{i,j} = x_i \tag{42}$$

for any n and i such that $i < i_n$, by induction on n. For $n = 0$ this is trivial, since $i_0 = 0$. Suppose $\sum_j z^n_{i,j} = x_i$. We need to show $\sum_j z^{n+1}_{i,j} = x_i$ for $i < i_{n+1}$. We distinguish two cases.

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(i) $x_{i_n} - \xi_n \le y_{j_n} - \eta_n$. Note $i_{n+1} = i_n + 1$. For $i < i_n$ we have $z_{i,j}^{n+1} = z_{i,j}^n$, so equation (42) also holds for $n+1$. For the index i_n we have, as before,

$$\sum_j z_{i_n,j}^{n+1} = \sum_{j<j_n} z_{i_n,j}^n + \left(x_{i_n} - \sum_{j<j_n} z_{i_n,j}^n\right) + 0 = x_{i_n},$$

as required.

(ii) $x_{i_n} - \xi_n > y_{j_n} - \eta_n$. Note $i_{n+1} = i_n$ in this case. So, if $i < i_{n+1}$ then also $i < i_n$. Therefore,

$$\sum_j z_{i,j}^{n+1} = \sum_j z_{i,j}^n = x_i$$

by the induction hypothesis.

Symmetrically, we obtain $\sum_i z_{i,j}^n = y_j$, for any n and j such that $j < j_n$.

Next, we check

$$z_{i,j}^n \ge 0$$

for all i,j by induction on n. For $n = 0$ this is clear by definition. Consider $z_{i,j}^{n+1}$. If $i \ne i_n$ or $j \ne j_n$, then $z_{i,j}^{n+1} = z_{i,j}^n$. So, by induction hypothesis, $z_{i,j}^n \ge 0$ in that case. Regarding z_{i_n,j_n}^{n+1}, we have, by equation (41) and the induction hypothesis

$$
\begin{aligned}
x_{i_n} - \xi_n &= x_{i_n} - \sum_{j<j_n} z_{i_n,j}^n &\ge\; x_{i_n} - \sum_j z_{i_n,j}^n &\ge\; 0 \\
y_{j_n} - \eta_n &= y_{j_n} - \sum_{i<i_n} z_{i,j_n}^n &\ge\; y_{j_n} - \sum_i z_{i,j_n}^n &\ge\; 0.
\end{aligned}
$$

So, also $z_{i_n,j_n}^{n+1} = \min\{\, x_{i_n} - \xi_n, y_{j_n} - \eta_n \,\} \ge 0$.

Note that, since $i_n \le i_{n+1}$, $j_n \le j_{n+1}$ and $i_n + j_n < i_{n+1} + j_{n+1}$, the sequence $(z_{i,j}^n)_{n\in\mathbb{N}}$ is either constantly 0, which happens if $\langle i,j \rangle \notin \{\langle i_n, j_n \rangle \mid n \in \mathbb{N}\}$ or

$$z_{i,j}^n = \begin{cases} 0 & n \le n_0 \\ z_{i,j}^{n_0+1} & n > n_0 \end{cases}$$

in case $\langle i,j \rangle = \langle i_{n_0}, j_{n_0} \rangle$. In particular, we have established

$$z_{i,j}^n \le z_{i,j}^{n+1}, \; n \in \mathbb{N}. \tag{43}$$

Now, we define, for $i, j \in \mathbb{N}$,

$$z_{i,j} = \lim_{n\to\infty} z_{i,j}^n.$$

We show that $z_{i,j}$ satisfy the properties required in the assertion of the lemma. Since $i_n + j_n \to \infty$ if $n \to \infty$, either $i_n \to \infty$ or $j_n \to \infty$. Suppose, without loss of generality, $i_n \to \infty$ for $n \to \infty$. Let $i \in \mathbb{N}$ be fixed and let $n \in \mathbb{N}$ be such that $i < i_n$ (then also $i < i_{n+1}$). Then for all $m > n$

$$z_{i,j}^m = z_{i,j}^{n+1} = z_{i,j}$$

and thus
$$\sum_j z_{i,j} = \sum_j z_{i,j}^{n+1} = x_i$$
proving the first part of our property.

Now pick any j. By equation (41) we have that
$$\sum_i z_{i,j} = \sum_i \lim_{n\to\infty} z_{i,j}^n = \lim_{n\to\infty} \sum_i z_{i,j}^n \le y_j, \tag{44}$$
where the change of the limit and the sum is allowed since $(z_{i,j}^n)_{n\in\mathbb{N}}$ is a non-negative, monotone sequence. In order to show $\sum_i z_{i,j} = y_j$, we reason as follows. By assumption $\sum_i x_i = \sum_j y_j$. Hence,
$$\sum_j y_j = \sum_i x_i = \sum_i \sum_j z_{i,j} = \sum_j \sum_i z_{i,j}.$$

Changing the order of summation is allowed, since we are dealing with non-negative numbers only. For the same reason, this together with (44) implies that for all $j \in \mathbb{N}$, $\sum_i z_{i,j} = y_j$. This completes the proof. $\qquad\square$

We next show that such a matrix fill-in property holds for arbitrary (not necessarily countable) matrices as well.

Lemma B.3 *Let I and J be arbitrary sets. For any two sets $\{x_i \mid i \in I\}$ and $\{y_j \mid j \in J\}$ of non-negative real numbers such that*
$$\sum_{i\in I} x_i = \sum_{j\in J} y_j < \infty,$$
there exist non-negative real numbers $\{z_{i,j} \mid i \in I, j \in J\}$ such that
$$\sum_{j\in J} z_{i,j} = x_i \quad and \quad \sum_{i\in I} z_{i,j} = y_j$$
for all $i \in I, j \in J$.

Proof We first consider the case when I and J are at most countable. If they are both countable, then the property holds by Lemma B.2. It may be that one of them, or both, are finite.

Write $I = \{i_k \mid k \in \mathbb{N}, k < |I|\}$ and $J = \{j_\ell \mid \ell \in \mathbb{N}, \ell < |J|\}$ and define x'_k, y'_ℓ, for $k, \ell \in \mathbb{N}$, by
$$x'_k = \begin{cases} x_{i_k} & k < |I| \\ 0 & \text{otherwise} \end{cases} \qquad y'_\ell = \begin{cases} y_{j_\ell} & \ell < |J| \\ 0 & \text{otherwise} \end{cases}$$

By Lemma B.2, we obtain $z'_{k,\ell}$ for $k, \ell \in \mathbb{N}$ with $\sum_{\ell\in\mathbb{N}} z'_{k,\ell} = x'_k$ and $\sum_{k\in\mathbb{N}} z'_{k,\ell} = y'_\ell$ for all $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, respectively. If $k \ge |I|$ then $x'_k = 0$ and hence $z'_{k,\ell} = 0$ for any $\ell \in \mathbb{N}$. Similarly, for $\ell \ge |J|$, $z'_{k,\ell} = 0$ for any $k \in \mathbb{N}$. Thus
$$z_{i_k,j_\ell} = z'_{k,l}, \quad \text{for} \quad k < |I|, \ \ell < |J|$$

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satisfy the requirements of the lemma.

Now consider arbitrary I and J. Let $I' = \{i \in I \mid x_i > 0\}$, $J' = \{j \in J \mid y_j > 0\}$. Then I' and J' are at most countable, by Lemma B.1. Let $x'_i = x_i$ for $i \in I'$ and $y'_j = y_j$ for $j \in J'$. Let $\{z'_{i,j} \mid i \in I', j \in J'\}$ be non-negative numbers such that for any $i \in I'$ and $j \in J'$

$$\sum_{j \in J'} z'_{i,j} = x'_i, \text{ and } \sum_{i \in I'} z'_{i,j} = y'_j.$$

Such numbers exist by the first part of the proof. Define, for any $i \in I, j \in J$ non-negative real numbers

$$z_{i,j} = \begin{cases} z'_{i,j} & i \in I', j \in J' \\ 0 & \text{otherwise} \end{cases}$$

These numbers fulfill the requirements of the lemma. $\qquad\square$

Lemma B.4 *The functor $\mathcal{D}$ preserves weak pullbacks.*

Proof It suffices to show that a pullback diagram

will be transformed to a weak pullback diagram (Lemma A.1). Let P' be the pullback of the cospan $\mathcal{D}X \xrightarrow{\mathcal{D}f} \mathcal{D}Z \xleftarrow{\mathcal{D}g} \mathcal{D}Y$. Since $\mathcal{D}f \circ \mathcal{D}\pi_1 = \mathcal{D}g \circ \mathcal{D}\pi_2$, there exists $\gamma : \mathcal{D}P \to P'$ such that the next diagram commutes

and it is enough to show that γ is surjective in order to get a mediating morphism from P' to $\mathcal{D}P$. Let $\langle u, v \rangle \in P'$ be given. If $\mu \in \mathcal{D}P$ is such that

$$(\mathcal{D}\pi_1)(\mu) = u, \qquad (\mathcal{D}\pi_2)(\mu) = v \tag{45}$$

then $\gamma(\mu) = \langle u, v \rangle$ since π_1 and π_2 are jointly injective i.e. $\pi_1 \times \pi_2$ is injective. Hence the task is to find a function $\mu \in \mathcal{D}P$ which satisfies (45). More explicitely we have to find $\mu : P \to [0, 1]$ such that for all $x_0 \in X, y_0 \in Y$

$$\sum_{y \in Y : \langle x_0, y \rangle \in P} \mu(x_0, y) = u(x_0), \qquad \sum_{x \in X : \langle x, y_0 \rangle \in P} \mu(x, y_0) = v(y_0) \qquad (46)$$

For if $\mu : P \to [0, 1]$ satisfies (46), then $\mu \in \mathcal{D}P$ and (45) holds.

The set P can be written as the union

$$P = \bigcup_{z \in Z} f^{-1}(\{z\}) \times g^{-1}(\{z\})$$

of disjoint rectangles, in fact rectangles with non-overlapping edges. Therefore, the existence of a map μ which satisfies condition (46) is equivalent to the condition that for all $z \in Z$ there exists a function $\mu_z : f^{-1}(\{z\}) \times g^{-1}(\{z\}) \to [0, 1]$ such that for all $x_0 \in f^{-1}(\{z\})$, and all $y_0 \in g^{-1}(\{z\})$,

$$\sum_{y \in g^{-1}(\{z\})} \mu_z(x_0, y) = u(x_0), \qquad \sum_{x \in f^{-1}(\{z\})} \mu_z(x, y_0) = v(y_0). \qquad (47)$$

Since $\langle u, v \rangle \in P'$, we have

$$\sum_{x \in f^{-1}(\{z\})} u(x) = (\mathcal{D}f)(u)(z) = (\mathcal{D}g)(v)(z) = \sum_{y \in g^{-1}(\{z\})} v(y). \qquad (48)$$
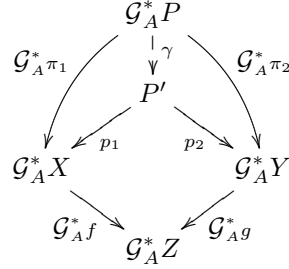
Thus we may apply the matrix fill-in property, Lemma B.3. $\qquad \square$

C Weak pullback preservation of the functor $\mathcal{G}_A^*$

In this part we investigate the weak pullback preservation of the functor $\mathcal{G}_A^*$. We establish that the functor preserves total weak pullbacks, but does not preserve weak pullbacks, i.e. we give a proof of Proposition 5.25.

Lemma C.1 *The functor $\mathcal{G}_A^*$ weakly preserves total pullbacks.*

Proof Let $\langle P, \pi_1, \pi_2 \rangle$ be a total pullback in Set of the cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$, i.e. $P = \{\langle x, y \rangle \mid f(x) = g(y)\}$ and π_1, π_2 surjective. Then the outer square of the diagram below commutes. Moreover, there exists a mediating morphism $\gamma : \mathcal{G}_A^* P \to P'$ from the candidate pullback $\langle \mathcal{G}_A^* P, \mathcal{G}_A^* \pi_1, \mathcal{G}_A^* \pi_2 \rangle$ to the pullback $\langle P', p_1, p_2 \rangle$ of the cospan $\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* Y$.

$$\begin{array}{ccc}
& \mathcal{G}_A^* P & \\
\mathcal{G}_A^*\pi_1 \swarrow & \downarrow \gamma & \searrow \mathcal{G}_A^*\pi_2 \\
& P' & \\
& p_1 \swarrow \quad \searrow p_2 & \\
\mathcal{G}_A^* X & & \mathcal{G}_A^* Y \\
\mathcal{G}_A^* f \searrow & & \swarrow \mathcal{G}_A^* g \\
& \mathcal{G}_A^* Z &
\end{array}$$

It is enough to prove that γ is surjective (Lemma A.1(ii)). So, we show that for every $\langle u,v\rangle \in P'$ there exists $w \in \mathcal{G}_A^* P$ with $\mathcal{G}_A^*\pi_1(w) = u$ and $\mathcal{G}_A^*\pi_2(w) = v$ which is equivalent to $w \circ (id_A^{-1} \times \pi_1^{-1}) = u$ and $w \circ (id_A^{-1} \times \pi_2^{-1}) = v$. Fix $\langle u,v\rangle \in P'$. We have

$$\langle u,v\rangle \in P' \implies \forall A' \subseteq A, \forall Z' \subseteq Z \colon u(A', f^{-1}(Z')) = v(A', g^{-1}(Z')). \quad (49)$$

Let $X' \subseteq X, Y' \subseteq Y$ and assume $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$. Then

(i) $f^{-1}(f(X')) = X'$:
 Clearly $X' \subseteq f^{-1}(f(X'))$. Let $x \in f^{-1}(f(X'))$ such that $f(x) = f(x')$ for some $x' \in X'$. Since π_1 is surjective, there exists $y \in Y$ with $\langle x,y\rangle \in P$ i.e. $f(x) = g(y)$, and hence also $f(x') = g(y)$, i.e. $\langle x',y\rangle \in P$. Thus $\langle x',y\rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ implying $y \in Y'$. Hence $\langle x,y\rangle \in \pi_2^{-1}(Y') = \pi_1^{-1}(X')$ i.e. $x \in X'$.

(ii) $g^{-1}(g(Y')) = Y'$: similar as (i).

(iii) $f(X') = g(Y')$:
 Let $z \in f(X')$, i.e. $z = f(x')$ for $x' \in X'$. Since π_1 is surjective there exists $y \in Y$ with $\langle x',y\rangle \in P$, i.e. $f(x') = g(y)$. Now, $\langle x',y\rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ and therefore $y \in Y'$, i.e. $z = f(x') = g(y) \in g(Y')$. We have shown $f(X') \subseteq g(Y')$. Similarly, $g(Y') \subseteq f(X')$.

Hence, if $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$ for $X' \subseteq X, Y' \subseteq Y$ we get, for any $A' \subseteq A$,

$$u(A', X') \overset{(i)}{=} u(A', f^{-1}(f(X'))) \overset{(49)}{=} v(A', g^{-1}(f(X'))) \overset{(iii)}{=}$$

$$v(A', g^{-1}(g(Y'))) \overset{(ii)}{=} v(A', Y').$$

Since π_1 and π_2 are surjective,

$$\pi_1^{-1}(X') = \pi_1^{-1}(X'') \implies X' = X''$$

and

$$\pi_2^{-1}(Y') = \pi_2^{-1}(Y'') \implies Y' = Y''$$

for any $X', X'' \subseteq X$ and any $Y', Y'' \subseteq Y$. So the function $w \colon \mathcal{P}(A) \times \mathcal{P}(P) \to [0, 1]$ given by

$$w(A', Q) = \begin{cases} u(A', X') & Q = \pi_1^{-1}(X') \\ v(A', Y') & Q = \pi_2^{-1}(Y') \\ 0 & \text{otherwise} \end{cases}$$

is well defined. Clearly, $w \circ (id_A^{-1} \times \pi_1^{-1}) = u$ and $w \circ (id_A^{-1} \times \pi_2^{-1}) = v$. Thus the functor $\mathcal{G}_A^*$ weakly preserves total pullbacks. $\square$

However, note that although $\mathcal{G}_A^*$ weakly preserves total pullbacks, it does not preserve weak pullbacks, as shown by the next example.

Example C.2 $\mathcal{G}_A^*$ does not preserve weak pullbacks.

Choose X with $|X| \geq 3$. Fix $x_0 \in X$. Let $Z = \{1, 2, 3\}$ and consider the cospan $X \xrightarrow{f} Z \xleftarrow{g} X$ for the maps

$$f(x) = \begin{cases} 2 & x = x_0 \\ 1 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 2 & x = x_0 \\ 3 & \text{otherwise.} \end{cases}$$

The Set pullback of this cospan is then $P = \{\langle x_0, x_0 \rangle\}$. On the other hand, let P' be the pullback of the cospan

$$\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* X \;.$$

We have $\langle \mu, \nu \rangle \in P'$ if and only if

$$\mathcal{G}_A^* f(\mu) = \mathcal{G}_A^* g(\nu),$$

i.e.

$$\mu(A', f^{-1}(Z')) = \nu(A', g^{-1}(Z'))$$

for all $A' \subseteq A, Z' \subseteq Z$. Therefore, every pair $\langle \mu, \nu \rangle \in \mathcal{G}_A^* X \times \mathcal{G}_A^* X$ with the property

$$\mu(A', \emptyset) = \mu(A', \{x_0\}) = \mu(A', X \setminus \{x_0\}) = \mu(A', X) =$$

$$= \nu(A', \emptyset) = \nu(A', \{x_0\}) = \nu(A', X \setminus \{x_0\}) = \nu(A', X)$$

belongs to P', since $\emptyset, \{x_0\}, X \setminus \{x_0\}$ and X are the only subsets of X that are inverse images of subsets of Z under f and g.

Now we consider $\mathcal{G}_A^* P = \mathcal{P}(A) \times \mathcal{P}(P) \to [0, 1]$. If $\mu \in \mathcal{G}_A^* X$ is such that $\mu = (\mathcal{G}_A^* \pi_1)(\chi)$ for some $\chi \in \mathcal{G}_A^* P$, then $\mu = \chi \circ (id_A^{-1} \times \pi_1^{-1})$. Hence, for $A' \subseteq A, X' \subseteq X$ we have

$$\mu(A', X') = \begin{cases} \chi(A', \emptyset) & x_0 \notin X' \\ \chi(A', \{\langle x_0, x_0 \rangle\}) & x_0 \in X'. \end{cases}$$

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Choose $x_1 \in X$, $x_1 \neq x_0$. Since $|X| \geq 3$ we have $\{x_0, x_1\} \notin \{\emptyset, \{x_0\}, X \setminus \{x_0\}, X\}$. Define $\xi \colon \mathcal{P}(A) \times \mathcal{P}(X) \to [0, 1]$ by

$$\xi(A', X') = \begin{cases} 1 & X' = \{x_0, x_1\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\xi \in \mathcal{G}_A^*(X)$ and the pair $\langle \xi, \xi \rangle$ belongs to P', since for every $A' \subseteq A$,

$$\xi(A', \emptyset) = \xi(A', \{x_0\}) = \xi(A', X \setminus \{x_0\}) = \xi(A', X) = 0.$$

However, ξ can not be written as $(\mathcal{G}_A^* \pi_1)(\chi)$ for any $\chi \in \mathcal{G}_A^* P$, since

$$\xi(A', \{x_0, x_1\}) \neq \xi(A', \{x_0\}),$$

while, as noted above,

$$(\mathcal{G}_A^* \pi_1)(\chi)(A', \{x_0, x_1\}) = \chi(A', \{\langle x_0, x_0 \rangle\}) = (\mathcal{G}_A^* \pi_1)(\chi)(A', \{x_0\}).$$

Hence, for the pair $\langle \xi, \xi \rangle \in P'$ there does not exist an element $\chi \in \mathcal{G}_A^* P$ such that $\mathcal{G}_A^* \pi_1(\chi) = \xi$ and $\mathcal{G}_A^* \pi_2(\chi) = \xi$, which by Lemma A.1 shows that $\mathcal{G}_A^*$ does not preserve weak pullbacks.