The Microcosm Principle and Concurrency in Coalgebras

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Our questions. *Compositionality* is an important property in modular verification of complex component-based systems. It is usually expressed as follows.

$$x \sim x' \quad \text{and} \quad y \sim y' \quad \Longrightarrow \quad (x \parallel y) \;\sim\; (x' \parallel y')$$

When we take *final coalgebra semantics* as behavior of systems

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ {\scriptstyle c}\uparrow & & \cong\uparrow{\scriptstyle \text{final}} \\ X & \underset{\mathsf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

the following comes natural as a "coalgebraic presentation of compositionality".

$$\mathsf{beh}\left(\begin{array}{c} FX \\ {\scriptstyle c}\uparrow \\ X \end{array} \Bigg\| \begin{array}{c} FY \\ {\scriptstyle d}\uparrow \\ Y \end{array}\right) \quad = \quad \mathsf{beh}\left(\begin{array}{c} FX \\ {\scriptstyle c}\uparrow \\ X \end{array}\right) \Bigg\| \; \mathsf{beh}\left(\begin{array}{c} FY \\ {\scriptstyle d}\uparrow \\ Y \end{array}\right) \qquad (1)$$

Here arise some questions which we believe are important for our mathematical understanding of parallel composition, or *concurrency*, of systems.

- The operator $\parallel$ on the left gives us composition of *systems*, being an operation on the category $\mathbf{Coalg}_F$. When is it available?
- The other $\parallel$ appearing on the right has a different domain: it is an operation on the final coalgebra Z! In this way we observe
  - the same algebraic structure (or algebraic "theory") which concerns the operation $\parallel$ and possibly some axioms like associativity,
  - interpreted on two different levels—on a category $\mathbf{Coalg}_F$ and on its object $Z \in \mathbf{Coalg}_F$—in a nested manner.

  What is the mathematical principle behind this?

In the sequel we shall sketch our first answers given in our preprint [2]. Our title refers to the phenomenon of nested algebraic structures which is called the *microcosm principle* [1].

Parallel composition of coalgebras. The "outer" composition (of coalgebras) is described as a bifunctor $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \to \mathbf{Coalg}_F$. Such an operation is usually denoted by $\otimes$ (rather than $\parallel$) and called a *tensor product*: we follow this tradition. Composition of systems is most of the time *associative*—$(c \otimes d) \otimes e \cong c \otimes (d \otimes e)$—making $\otimes$ an *associative tensor*.

Theorem 1 *Let $\mathbb{C}$ be a base category equipped with an associative tensor $\otimes$; and a functor $F : \mathbb{C} \to \mathbb{C}$ be equipped with the "synchronization" natural transformation $FX \otimes FY \xrightarrow{\mathsf{sync}_{X,Y}} F(X \otimes Y)$ compatible with associativity in $\mathbb{C}$. They induce a canonical associative tensor $\otimes$ on $\mathbf{Coalg}_F$.*

The final coalgebra carries an "inner associative tensor" $\|$ on Z, induced on the following left. It is associative in the sense of the diagram on the right.

$$\begin{array}{ccc} F(Z \otimes Z) & \dashrightarrow & FZ \\ {\scriptstyle \zeta\otimes\zeta\,=\,\mathsf{sync}_{Z,Z}\circ(\zeta\otimes_{\mathbb{C}}\zeta)}\uparrow & & \uparrow{\scriptstyle \cong\,\zeta} \\ Z \otimes Z & \underset{\|}{\dashrightarrow} & Z \end{array} \qquad \begin{array}{ccccc} (Z \otimes Z) \otimes Z & \xrightarrow{\cong} & Z \otimes (Z \otimes Z) & \xrightarrow{Z\otimes\|} & Z \otimes Z \\ {\scriptstyle \|\otimes Z}\downarrow & & & & \downarrow{\scriptstyle \|} \\ Z \otimes Z & & \xrightarrow[\|]{\qquad\qquad} & & Z \end{array}$$

For such compositions we have the compositionality result (1) for free. □

When $F = \mathcal{P}_\omega(A \times \_)$ for which a coalgebra is a finitely-branching LTS, we can realize all of the ACP/CCS/CSP-style synchronizations by taking different sync.

The microcosm principle. The *microcosm principle* is exemplified by the sentence: "a monoid is defined in a monoidal category". What we saw above is one instance of such phenomena. We pursue a mathematical formulation of this principle for general algebraic theories. In its course we use 2-categorical notions since a 2-category ("categories in a category") well accommodates microcosm phenomena.

For our purpose, a *Lawvere theory* $\mathbb{L}$ is an appropriate categorical presentation of an algebraic theory. An $\mathbb{L}$-*category*—a category with the $\mathbb{L}$-structure—is a product-preserving pseudo-functor $\mathbb{L} \xrightarrow{\mathbb{C}} \mathbf{Cat}$. It is "pseudo" because equations hold only up to isomorphism. Now an object in $X \in \mathbb{C}$ which has the "inner" $\mathbb{L}$-structure is defined to be a lax natural transformation $\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbf{1}}{\Downarrow \chi}} \mathbf{Cat}$. The component $\chi_1 : \mathbf{1} \to \mathbb{C}$ specifies the object X; the operations in $\mathbb{L}$ are interpreted by the mediating 2-cells of lax naturality.

In this general setting we can state the compositionality (1) as follows. It subsumes Theorem 1.

Theorem 2 *For an $\mathbb{L}$-category $\mathbb{C}$ and $F : \mathbb{C} \to \mathbb{C}$ being a lax $\mathbb{L}$-functor, the functor $\mathbf{Coalg}_F \xrightarrow{\mathsf{beh}} \mathbb{C}/Z$ as a morphism of $\mathbb{L}$-categories.* □

In [2] we also present this framework in less categorical terms, presenting an algebraic theory concretely by a pair (Σ, E) of operations and equations.

References

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2. I. Hasuo, B. Jacobs, and A. Sokolova. The microcosm principle and concurrency in coalgebras, 2007. Preprint, available from `http://www.cs.ru.nl/~ichiro/papers`.