Weak bisimulation for action-type coalgebras

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Outline

- Introduction and motivation
  * coalgebras (action-type)
  * LTS
  * bisimulation, strong and weak
  * coalgebraic vs. concrete bisimulation
- Weak bisimulation for coalgebras
- Examples and correspondence results
  * LTS
  * generative probabilistic systems
- Conclusions

Action-type coalgebras

A coalgebra (on Set, of type $\mathcal{F}$) is a pair

$$\langle S, \alpha : S \to \mathcal{F}S \rangle$$

for $\mathcal{F}$ a Set endofunctor

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- $\mathcal{F}$ - bifunctor, A - fixed set
  $\Rightarrow \mathcal{F}_A$ - Set endofunctor,

$$\mathcal{F}_A(S) = \mathcal{F}(A, S),$$

$$\mathcal{F}_A f = \mathcal{F}\langle \mathrm{id}_A, f \rangle, \; f : S \to T$$

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* an action-type coalgebra of type $\mathcal{F}_A$ is a triple

$$\langle S, A, \alpha : S \to \mathcal{F}_A(S) \rangle$$

Coalgebraic bisimulation

A bisimulation between two $\mathcal{F}_A$-coalgebras $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ is a relation

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$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
\mathcal{F}_A S & \mathcal{F}_A R \arrow[l, "\mathcal{F}_A \pi_1"'] \arrow[r, "\mathcal{F}_A \pi_2"] & \mathcal{F}_A T
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$s \sim t$ - bisimilarity, as usual ...

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$s_1 \xrightarrow{a} s_2$ for $\langle s_1, a, s_2 \rangle \in \rightarrow$

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Example:

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Note: LTS $\langle S, A, \rightarrow \rangle = \mathcal{L}_A$ coalgebra $\langle S, A, \alpha \rangle$ for the bifunctor $\mathcal{L} = \mathcal{P}(\mathcal{I} \times \mathcal{I})$ with

$$\langle a, s' \rangle \in \alpha(s) \iff s \xrightarrow{a} s'$$

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$\sim_{\text{LTS}}$ - strong bisimilarity

Characterizing bisimulation

* $R \subseteq S \times T$, $\mathcal{F}$ - functor, $\equiv_{\mathcal{F},R}$ - lifting

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R\colon \mathcal{F}\pi_1(z) = x,\ \mathcal{F}\pi_2(z) = y$$

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- $R \subseteq S \times T$ - bisimulation between $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ iff

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- if $\mathcal{F}$ w.p. total pullbacks and R - equivalence, then $\equiv_{\mathcal{F},R}$ is the pullback of

$$\mathcal{F}S \xrightarrow{\mathcal{F}c} \mathcal{F}(S/R) \xleftarrow{\mathcal{F}c} \mathcal{F}S$$

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$\approx_{\mathrm{LTS}}$ - weak bisimilarity for LTS

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by

$$\langle a_1 \dots a_k, s' \rangle \in \alpha'(s) \iff s \xrightarrow{a_1} \circ \dots \circ \xrightarrow{a_k} s'$$

and

$$\langle a_1 \dots a_k, s' \rangle \in \alpha''(s) \iff \langle w, s' \rangle \in \alpha'(s)$$

for some $w \in \tau^* a_1 \tau^* \dots \tau^* a_k \tau^*$

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then: $s \approx_{\mathsf{LTS}} t$ in $\langle S, A, \alpha \rangle$

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- weak boils down to strong !

Weak bisimulation - $\mathcal{F}_A$ coalgebras

two stages approach:

1. transform any $\mathcal{F}_A$ coalgebra into $\mathcal{G}_{A^*}$ coalgebra, faithfully.
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weak bisimulation = bisimulation for the ”double-arrow” coalgebra

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$$
\begin{tikzcd}
\langle S, A, \alpha \rangle \arrow[r, squiggly, "\Phi"] \arrow[d] & \langle S, A^*, \alpha' \rangle \arrow[r, squiggly, "\Psi_\tau"] \arrow[dd] & \langle S, (A \setminus \tau)^*, \alpha'' \rangle \arrow[ddd] \\
\mathcal{F}_A\text{- coalgebra} & & \\
& \mathcal{G}_{A^*}\text{- coalgebra} & \\
& & \mathcal{G}_{(A\setminus\tau)^*}\text{- coalgebra}
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"double-arrow coalgebra": $\Psi_\tau \circ \Phi(\langle S, A, \alpha \rangle)$

Weak bisimilarity, properties

Given $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ and $\tau \subseteq A$

$s \approx_\tau t$ in an $\mathcal{F}_A$ coalgebra

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Properties:

- $\sim \subseteq \approx_\tau$
- $\sim = \approx_\emptyset$
- $\tau_1 \subseteq \tau_2 \Rightarrow \approx_{\tau_1} \subseteq \approx_{\tau_2}$

Generative probabilistic systems

Generative system is a triple

$$\langle\; S \;,\; A \;,\; P \;: S \times A \times S \to [0,1] \rangle$$

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Example:

$$\begin{tikzcd}
 & s_2 \arrow[r, "{a[\frac{1}{2}]}"] \arrow[dr, "{a[\frac{1}{2}]}"'] & s_4 & & \\
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Note: g. s. $\langle S, A, P \rangle = \mathcal{G}_A$ coalgebra $\langle S, A, \alpha \rangle$ for the bifunctor $\mathcal{G} = \mathcal{D}(\mathcal{I} \times \mathcal{I}) + 1$ with

$$\alpha(s)(a, s') = P(s, a, s')$$

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$\mathcal{G} = \mathcal{D}(\mathcal{I} \times \mathcal{I}) + 1$: $\mathcal{D}$ - distribution functor

$\mathcal{D}S = \{\mu : S \to [0,1], \mu[S] = 1\}, \quad \mu[X] = \sum_{s \in X} \mu(x)$

$\mathcal{D}f : \mathcal{D}S \to \mathcal{D}T,\ \mathcal{D}f(\mu)(t) = \mu[f^{-1}(\{t\})]$

Strong and weak - generative

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R - equivalence strong bisimulaiton
transfer condition:

$$P(s, a, C) = P(t, a, C)$$

for $C \in S/R$ and $P(s, a, C) = \sum_{s' \in C} P(s, a, s')$

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transfer condition:

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R - equivalence weak bisimulaiton
transfer condition:

$$\mathrm{Prob}(s, \tau^*\hat{a}\tau^*, C) = \mathrm{Prob}(t, \tau^*\hat{a}\tau^*, C)$$

for $C \in S/R$ and $\hat{a} = a$ if $a \in A \setminus \{\tau\}$ and $\hat{\tau} = \varepsilon$

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$\sim_{\mathrm{GEN}}, \approx_{\mathrm{GEN}}$ - strong, weak bisimilarity

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bifunctor $\mathcal{G}^*$

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for

$$f = \langle f_1, f_2 \rangle \colon A \times S \to B \times T$$

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Properties:

- $\mathcal{G}^*_A$ w.p. total pullbacks
- it does not w.p. pullbacks

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generative system $\langle S, A, \alpha \rangle$ i.e. $\langle S, A, P \rangle$

sets $\mathrm{Paths}(s), \mathrm{FPaths}(s), \mathrm{CPaths}(s)$

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Γ - set of cones,

$$\Gamma \subseteq \mathcal{P}(\mathrm{CPaths}(s))$$

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then: Prob is a pre-measure, and
it extends to a probability measure !

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then: Φ is indeed a *-translation

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for $B_i = h_\tau^{-1}(w_i),\ w_i \in A_\tau$ and $C_j \in S/R$

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proof: difficult for $\Rightarrow$

Conclusion

- general notion of weak bisimulation
  * two phase approach
  * - extension and hiding internal actions
  * weak bisimulation is strong
- from coalgebraic bisimulation to transfer conditions
- correspondence results for
  * LTS
  * generative probabilistic systems