Weak bisimulation for action-type coalgebras

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Outline

- Introduction and motivation

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  - LTS

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  - generative probabilistic systems

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- Conclusions

Labelled transition systems

LTS is a triple $\langle\ S\ ,\ A\ ,\ \rightarrow\ \subseteq\ S \times A \times S\rangle$

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$s_1 \xrightarrow{a} s_2$ for $\langle s_1, a, s_2 \rangle \in \rightarrow$

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Example:

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\begin{tikzcd}
 & s_2 \arrow[r, "a"] \arrow[dr, "a"] & s_4 & & \\
s_1 \arrow[ur, "b"] \arrow[dr, "a"'] & & s_5 \arrow[r, "b"] & s_6 & \cdots \\
 & s_3 & & &
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Strong bisimulation for LTS

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Weak bisimulation as strong ...

$$\langle S, A, \rightarrow \rangle \quad \rightsquigarrow \quad \langle S, A, \Rightarrow \rangle$$

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- weak boils down to strong !

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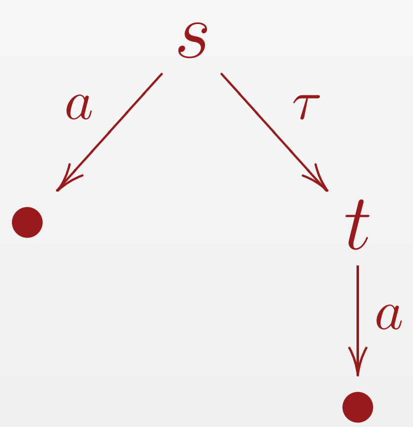

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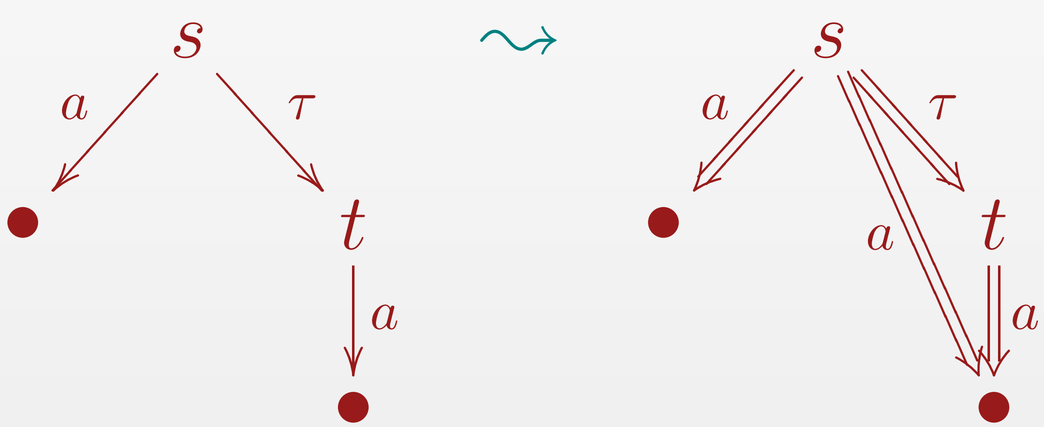

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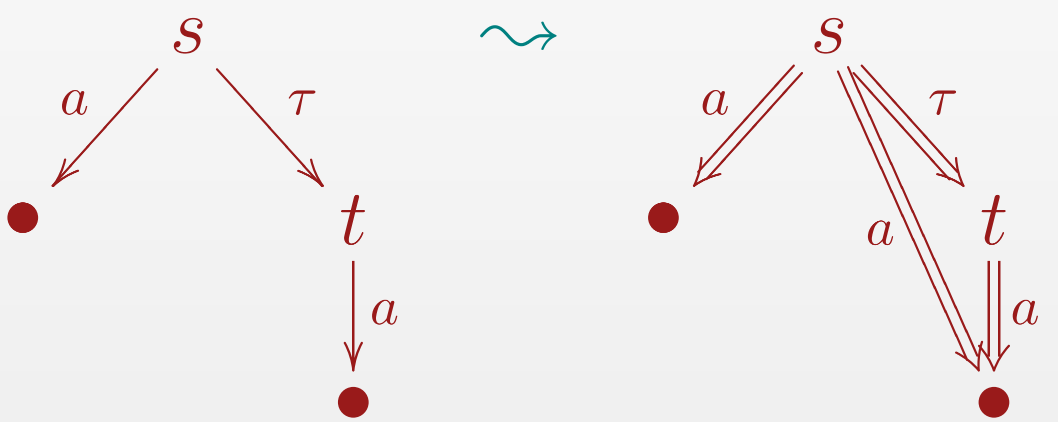

since $s \sim_{\text{LTS}} t$ in the "double-arrow" system

Generative probabilistic systems

Generative system is a triple

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- $\mathcal{F}$ - bifunctor $\Rightarrow \mathcal{F}_A$ - Set endofunctor, A - fixed set,

$$\mathcal{F}_A(S) = \mathcal{F}(A, S),$$

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* an action-type coalgebra of type $\mathcal{F}_A$ is a triple

$$\langle S, A, \alpha : S \to \mathcal{F}_A(S) \rangle$$

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having

$$\langle a, s' \rangle \in \alpha(s) \iff s \xrightarrow{a} s'$$

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where $\mathcal{D}$ - distribution functor

$$\alpha(s)(a, s') = P(s, a, s')$$

Coalgebraic bisimulation

A bisimulation between two $\mathcal{F}_A$-coalgebras $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ is a relation

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$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
\mathcal{F}_A S & \mathcal{F}_A R \arrow[l, "\mathcal{F}_A \pi_1"'] \arrow[r, "\mathcal{F}_A \pi_2"] & \mathcal{F}_A T
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$s \sim t$ - bisimilarity, as usual ...

Characterizing bisimulation

* $R \subseteq S \times T,\ \mathcal{F}$ - functor, $\equiv_{\mathcal{F},R}$ - lifting

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R\colon \mathcal{F}\pi_1(z) = x,\ \mathcal{F}\pi_2(z) = y$$

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- $R \subseteq S \times T$ - bisimulation between $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ iff

  $\langle s, t \rangle \in R \Rightarrow \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t)$

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- if $\mathcal{F}$ w.p. total pullbacks and R - equivalence, then $\equiv_{\mathcal{F},R}$ is the pullback of

$$\mathcal{F}S \xrightarrow{\mathcal{F}c} \mathcal{F}(S/R) \xleftarrow{\mathcal{F}c} \mathcal{F}S$$

Weak bisimulation - $\mathcal{F}_A$ coalgebras

two stages approach:

1. transform any $\mathcal{F}_A$ coalgebra into $\mathcal{G}_{A^*}$ coalgebra, faithfully.
2. fix a set $\tau \subseteq A$ of invisible actions, and hide them in the $\mathcal{G}_{A^*}$ coalgebra.

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weak bisimulation = bisimulation for the ”double-arrow” coalgebra

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$$\begin{tikzcd}
\langle S, A, \alpha \rangle \arrow[r, rightsquigarrow, "\Phi"] \arrow[d] & \langle S, A^*, \alpha' \rangle \arrow[r, rightsquigarrow, "\Psi_\tau"] \arrow[dd] & \langle S, A_\tau, \alpha'' \rangle \arrow[ddd] \\
\mathcal{F}_A\text{- coalgebra} & & \\
& \mathcal{G}_{A^*}\text{- coalgebra} & \\
& & \mathcal{G}_{A_\tau}\text{- coalgebra}
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\begin{aligned}
A_\tau &= A^* / \ker(h_\tau) \\
&= \{\tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* \mid a_1 \ldots a_k \in A \setminus \tau\}
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the canonical projection $c_\tau : A^* \to A_\tau$ induces a natural transformation

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”double-arrow coalgebra”: $\Psi_\tau \circ \Phi(\langle S, A, \alpha \rangle)$

Properties of weak bisimilarity

Given $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ and $\tau \subseteq A$

$s \approx_\tau t$ for $\mathcal{F}_A$ coalgebras

iff

$s \sim t$ for $\Psi_\tau \circ \Phi$ of $\mathcal{F}_A$ coalgebras

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Properties:

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- $\sim = \approx_\emptyset$
- $\tau_1 \subseteq \tau_2 \Rightarrow \approx_{\tau_1} \subseteq \approx_{\tau_2}$

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$$\Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \eta_S^\tau \circ \alpha' \rangle$$

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$$\begin{aligned}
(\eta_S^\tau \circ \alpha')(s) &= \eta_S^\tau(\alpha'(s)) \\
&= \mathcal{P}(< c_\tau, \mathrm{id}_S >)(\alpha'(s)) \\
&= \{< c_\tau(w), s' > | < w, s' > \in \alpha'(s)\} \\
&= \{< \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*, s' > | \; \exists w \in \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* : s \overset{w}{\Rightarrow} s'\}
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\end{aligned}$$

hence for any $B = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^*$

$$s \overset{B}{\Rightarrow}_\tau s' \iff \exists w \in B : s \overset{w}{\Rightarrow} s'$$

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Then:

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Example - generative systems

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and for $f = \langle f_1, f_2 \rangle \colon A \times S \to B \times T$

$$\mathcal{G}^* f \colon \nu \mapsto \nu \circ \langle f_1^{-1}, f_2^{-1} \rangle$$

where $\nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0,1]$

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hence: bisimilarity is an equivalence
and can be characterized
as before...

Transfer condition for $\mathcal{G}^*_A$

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Γ - set of cones,

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then: Prob is a pre-measure, and
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proof: difficult for $\Leftarrow$

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