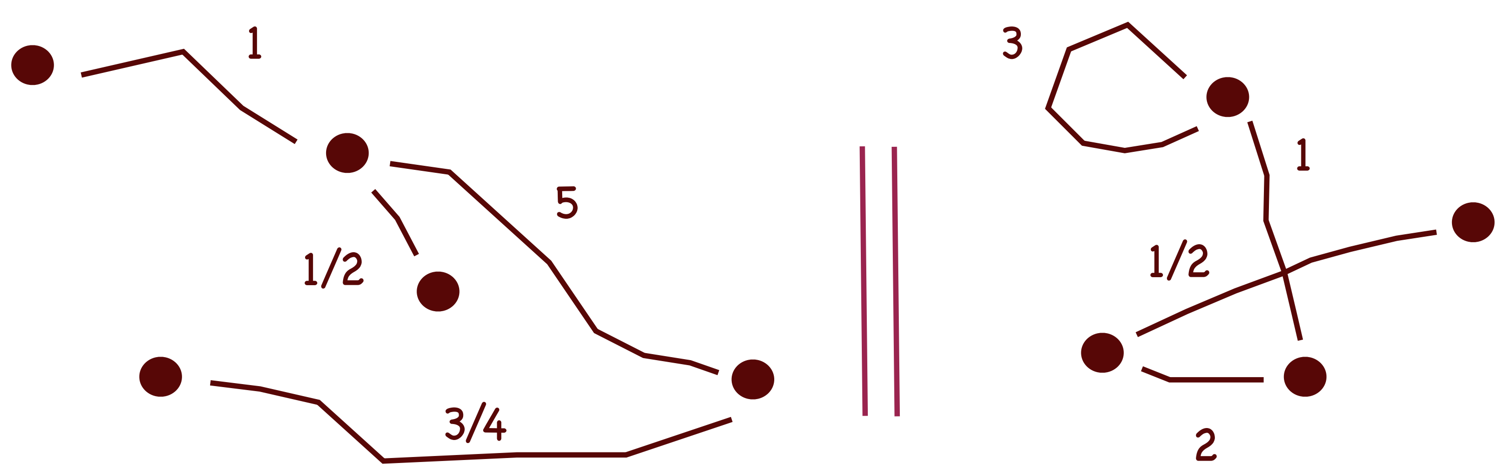
# Probabilistic Systems Semantics via Coalgebra

Ana Sokolova UNIVERSITY of SALZBURG




TbiLLC'17 tutorial, part 3

Plan:

**Part 1.** Modelling probabilistic systems for branching-time semantics

bisimilarity

**Part 2.** Traces, linear-time semantics

trace equivalence

**Part 3.** Belief-state-transformer semantics via convexity

distribution bisimilarity

Mathematical framework based on category theory for state-based systems semantics

all with help of coalgebra

Plan:

we still have something to discuss here

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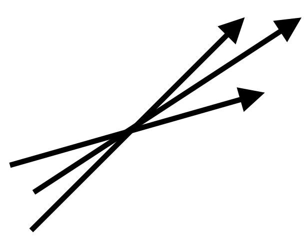

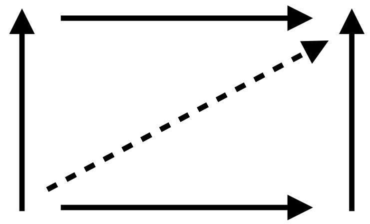



# Probabilistic systems
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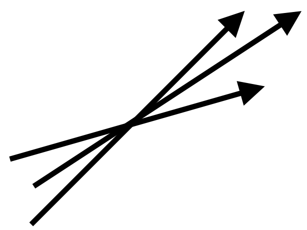
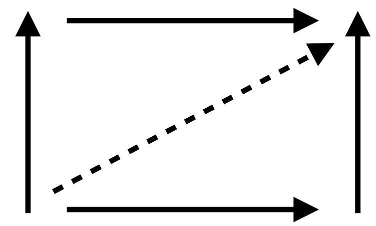
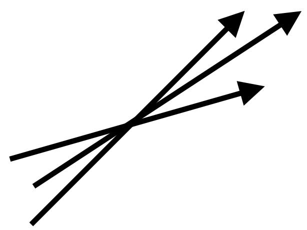


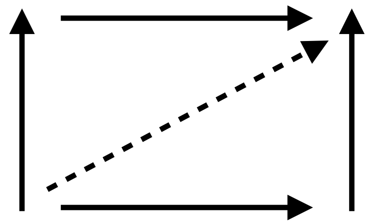


Source Wikipedia, by Diacritica - Own work


# Probabilistic systems are coalgebras

$\mathcal{D}$ on **Sets**







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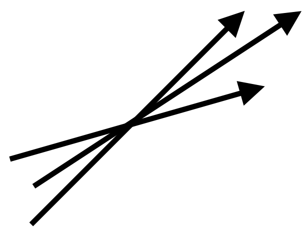
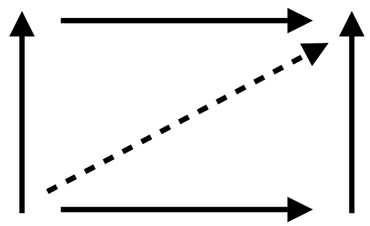
$\mathcal{D}$ on **Sets**

$\mathcal{G}$ on **Meas**







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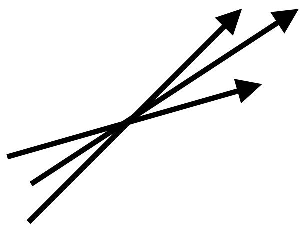
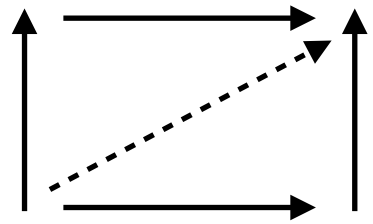
generic notion of behavioural equivalence







# Probabilistic systems are coalgebras

$\mathcal{D}$ on **Sets**

$\mathcal{G}$ on **Meas**

generic notion of behavioural equivalence

strong, branching-time semantics

# Trace semantics coalgebraically

we need to move out of **Sets**

trace equivalence is behaviour equivalence

# Trace semantics coalgebraically

we need to move out of **Sets**

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category

of a monad $T$

there is a way to connect (1) and (2)

trace equivalence is behaviour equivalence

# Main ideas in both approaches

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NFA / LTS

Two ideas:

(1) unfold branching + transitions on words

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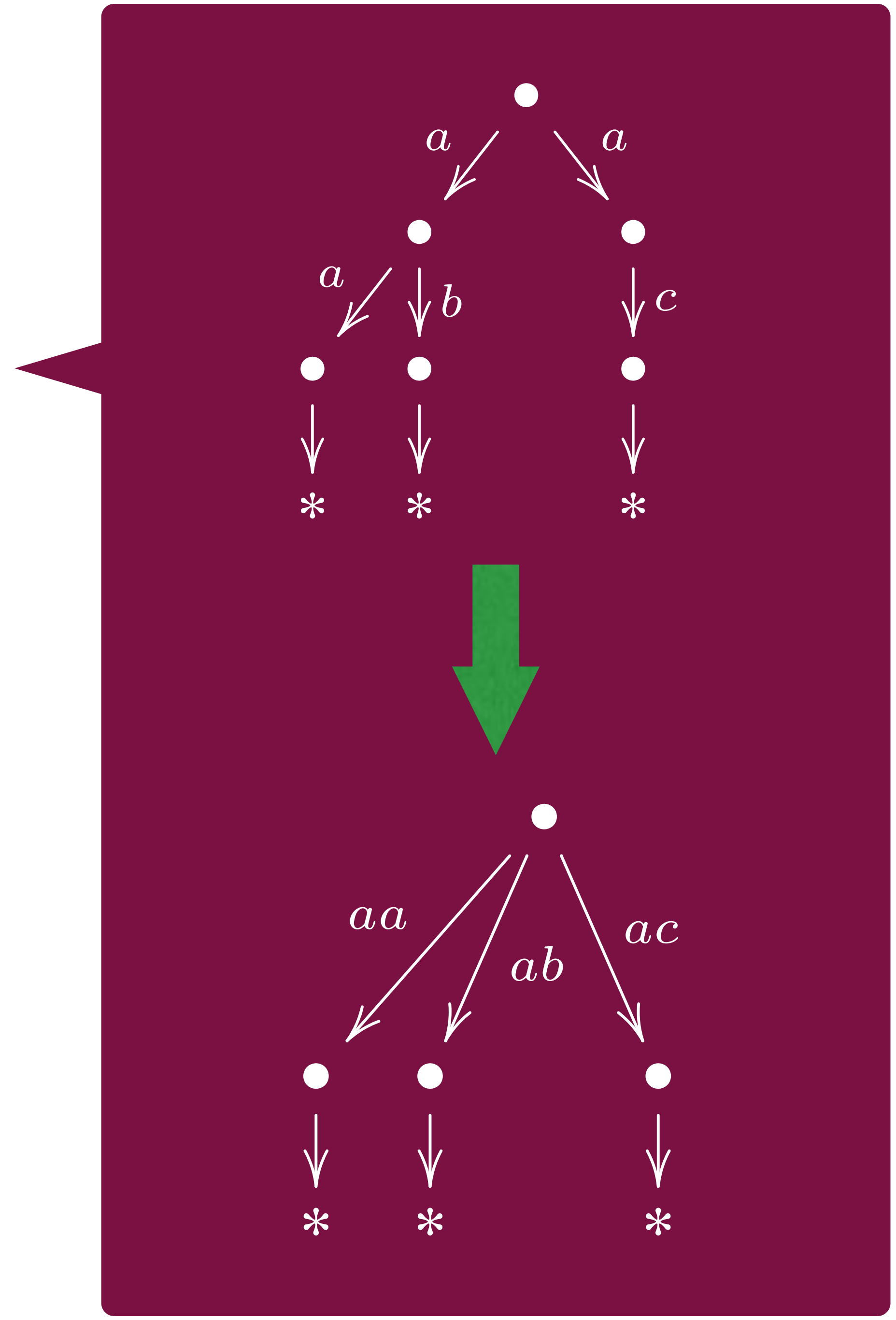

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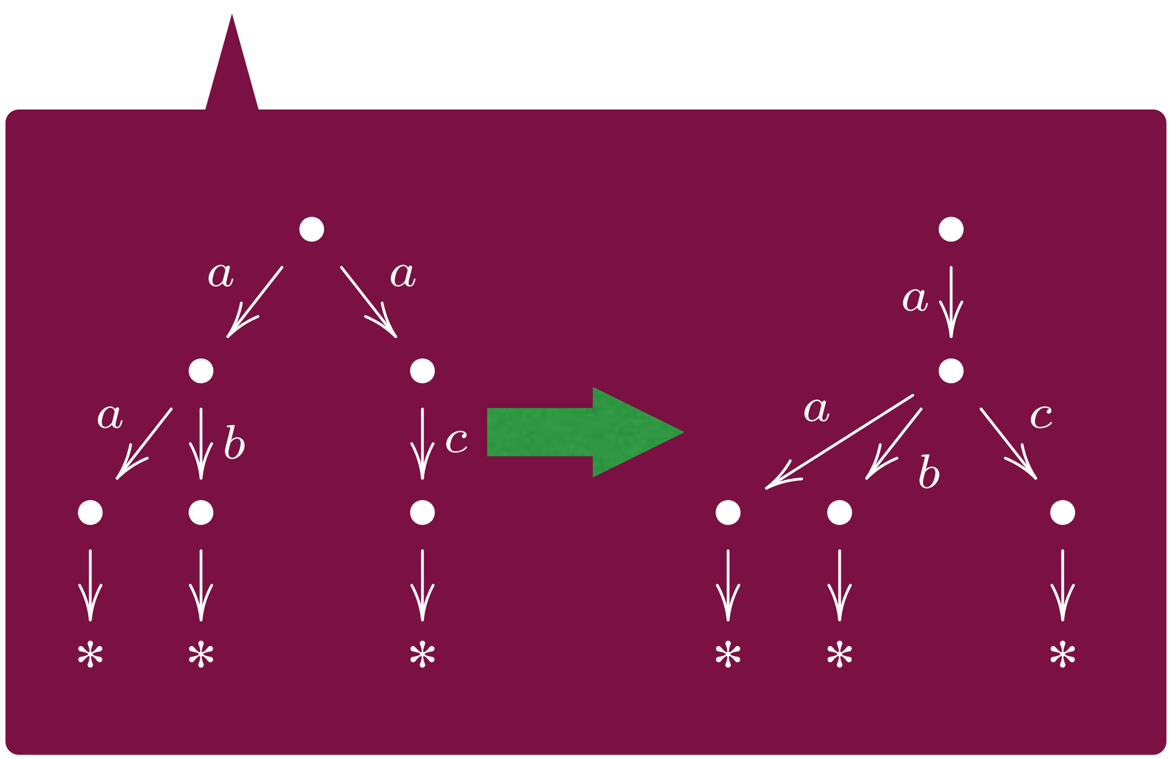


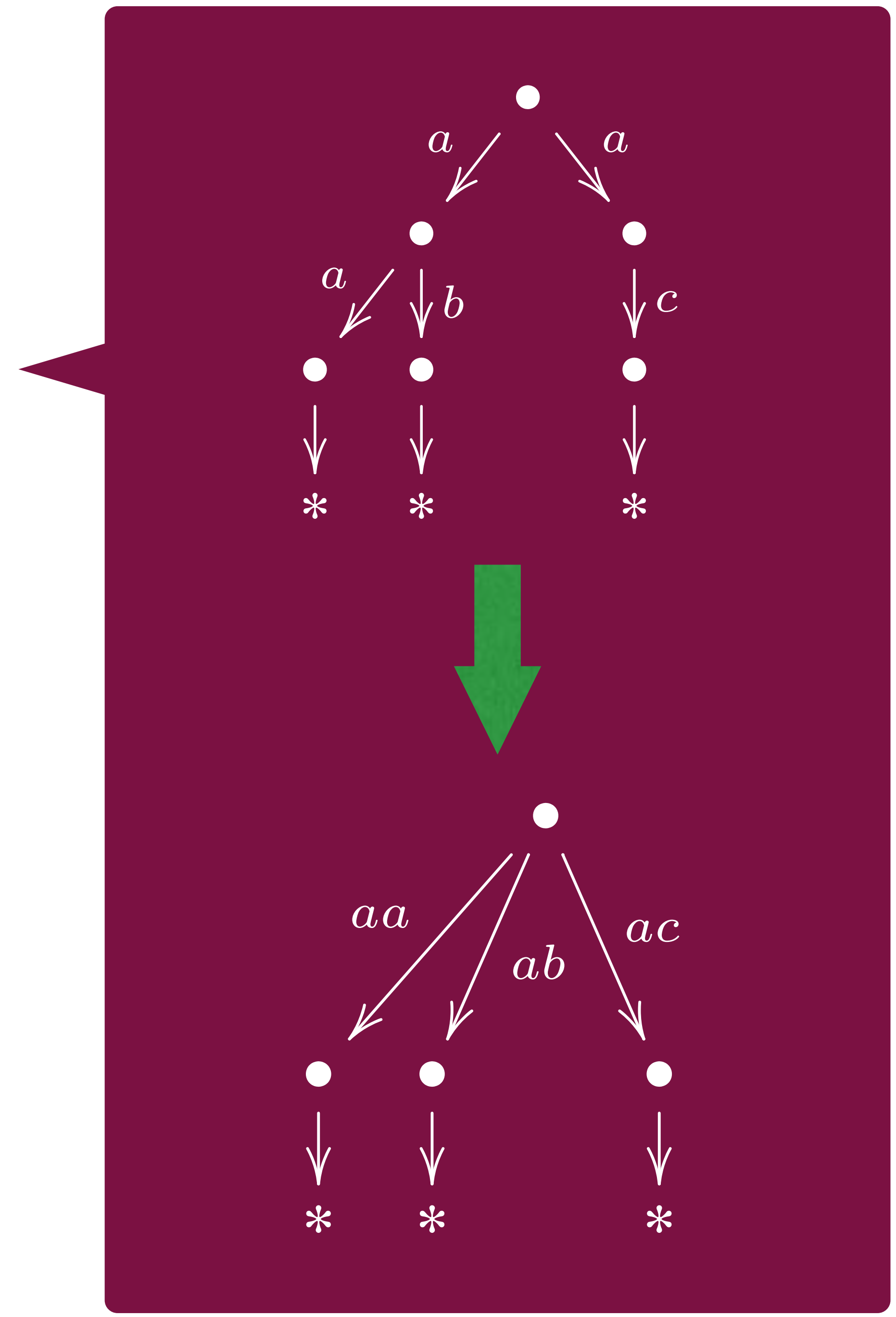

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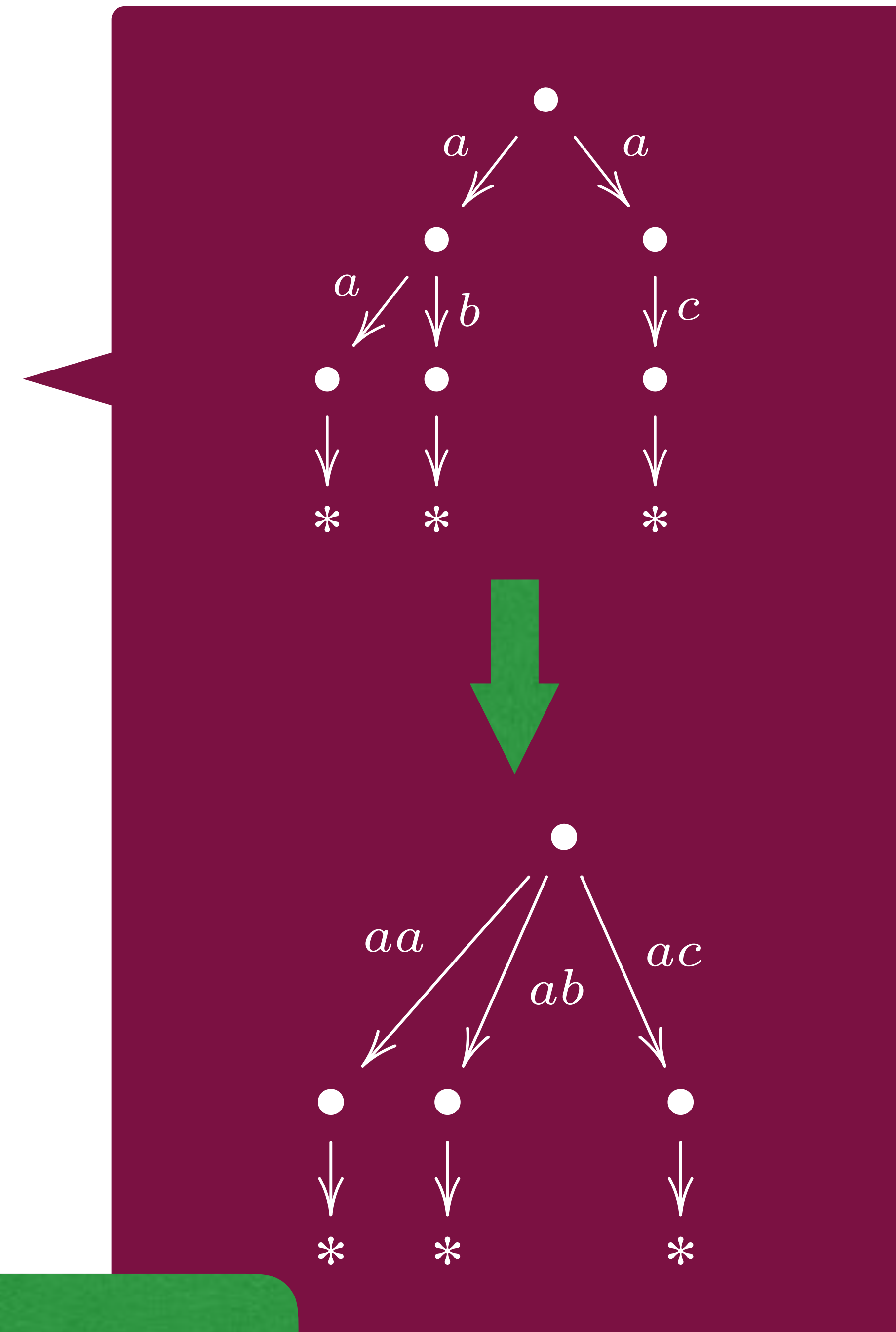


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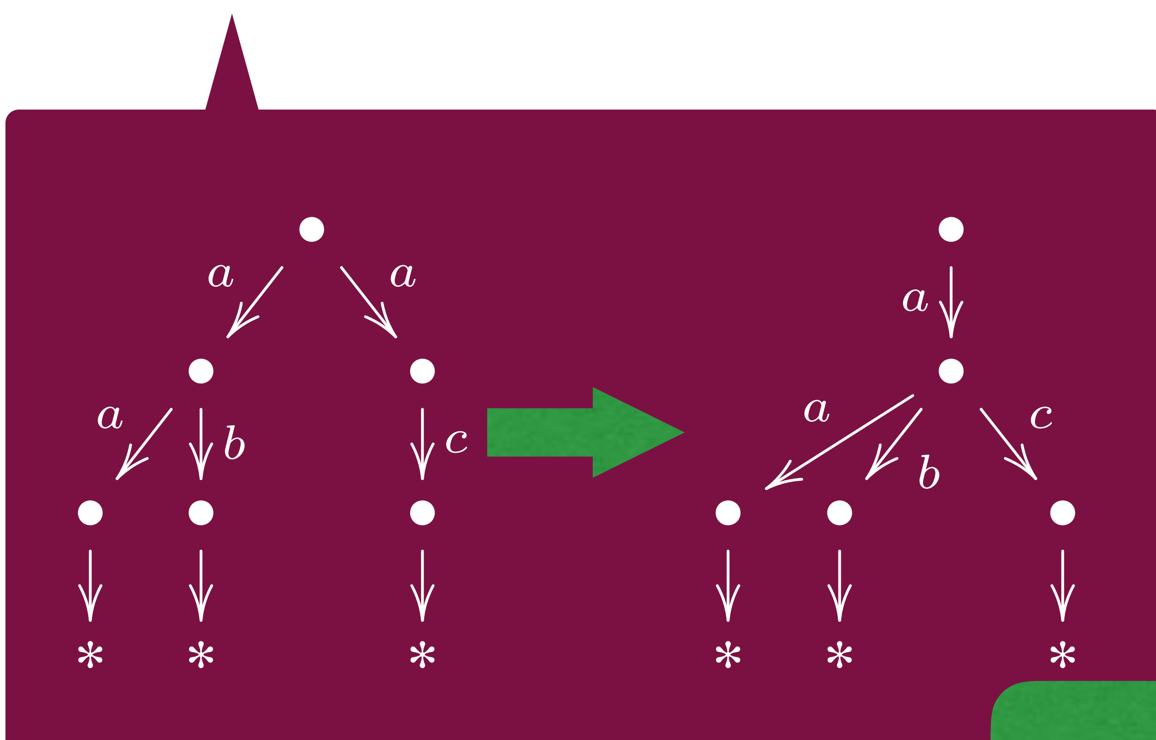


monads !

# Eilenberg-Moore Algebras

convex algebras

$\mathcal{EM}(\mathcal{D})$

finitely supported

- objects

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array}$$

satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{D}A \\ & {\scriptstyle a} \searrow\!\!\!\!\searrow & \downarrow a \\ & & A \end{array} \qquad \begin{array}{ccc} \mathcal{D}\mathcal{D}A & \xrightarrow{\mu} & \mathcal{D}A \\ \mathcal{D}a \downarrow & & \downarrow a \\ \mathcal{D}A & \xrightarrow{a} & A \end{array}$$

- morphisms

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array} \xrightarrow{h} \begin{array}{c} \mathcal{D}B \\ \downarrow b \\ B \end{array}$$

$$\begin{array}{ccc} \mathcal{D}A & \xrightarrow{\mathcal{D}h} & \mathcal{D}B \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{h} & B \end{array}$$

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- algebras
- convex (affine) maps

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satisfying

- Projection $\sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

# Trace axioms for generative PTS

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Silva, S.
MFPS '11

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Axioms for bisimilarity

+

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$$p \cdot a \cdot (p_1 \mathsf{E}_1 \oplus p_2 \mathsf{E}_2) \quad \equiv \quad p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \qquad (D)$$

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soundness and
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soundness and completeness

Happens in $\mathcal{EM}(\mathcal{D})$

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positive convex algebras

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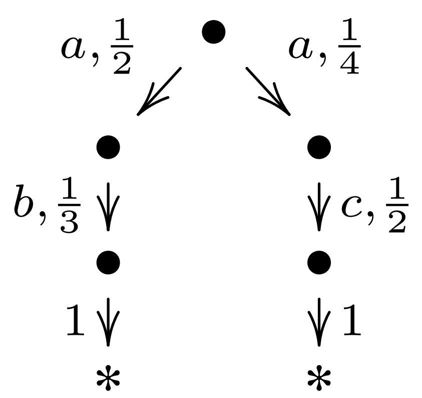
$\mathcal{D}\,(1 + A \times (-))$



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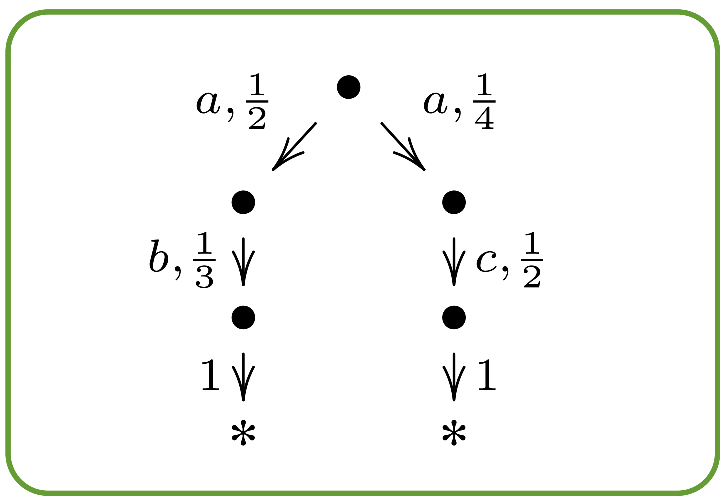


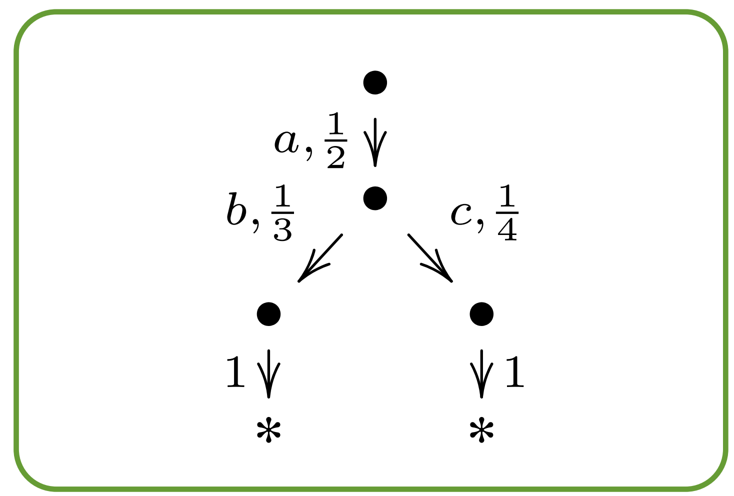

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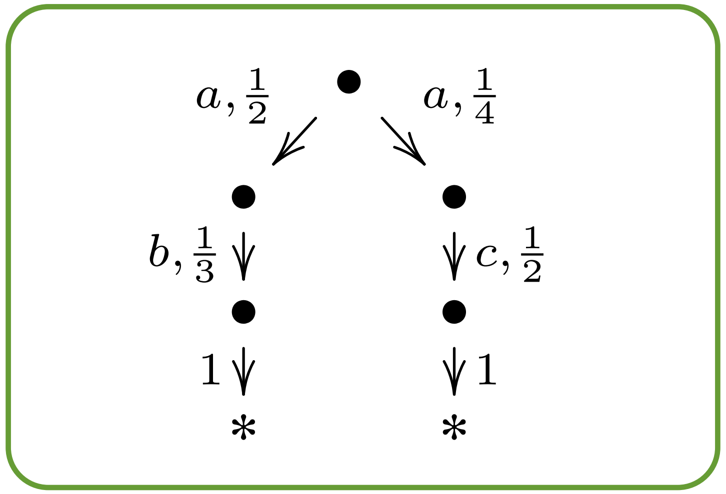

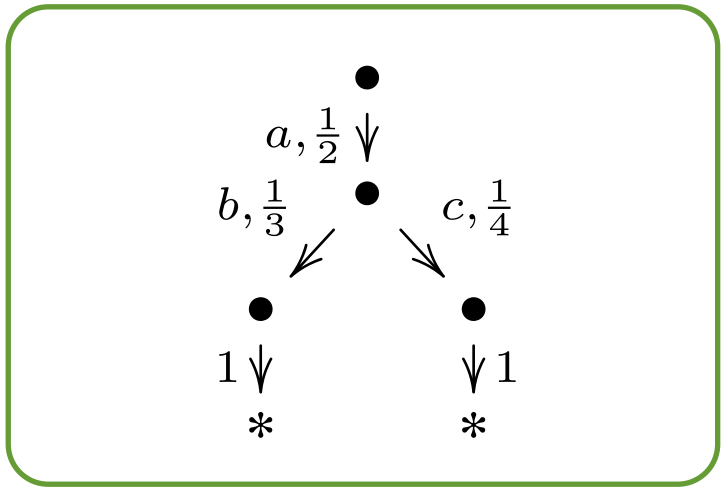

$$\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right)\overset{(Cong)}{\equiv}\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{2}\cdot a\cdot\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

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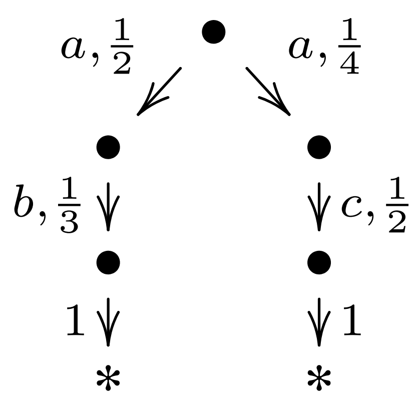

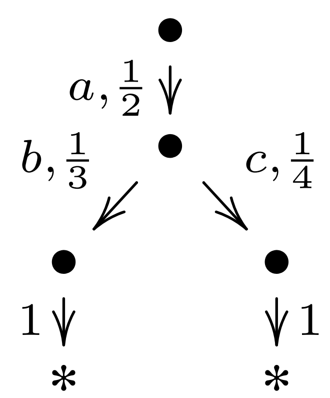

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Inspired lots of new research:

- Congruences of convex algebras

- Proper functors

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if f.p. = f.g. and ★

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Woracek, S.'17

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NEW!

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need a move out of **Sets**

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Trace semantics

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$\mathcal{EM}(\mathcal{D})$ convex algebras

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Trace semantics

$\mathcal{EM}(\mathcal{D})$
convex algebras

$\mathcal{EM}(\mathcal{P})$
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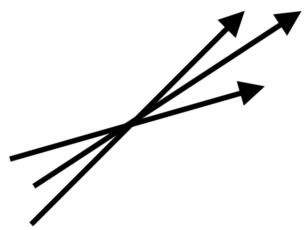
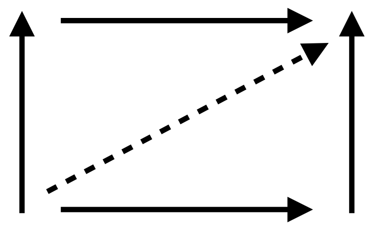
no concrete proof of completeness of the axiomatisation found





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# Part III

## Modelling probabilistic systems
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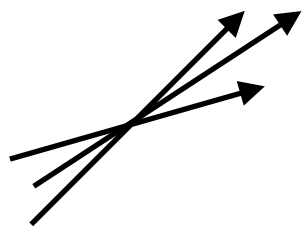

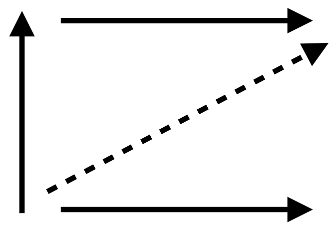



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## Modelling probabilistic systems
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coalgebraically

probabilistic automata

The true nature of probabilistic systems

as transformers of belief states

probabilistic automata

The true nature of probabilistic systems as transformers of belief states

Bonchi, Silva, S.
CONCUR '17

# Determinisations

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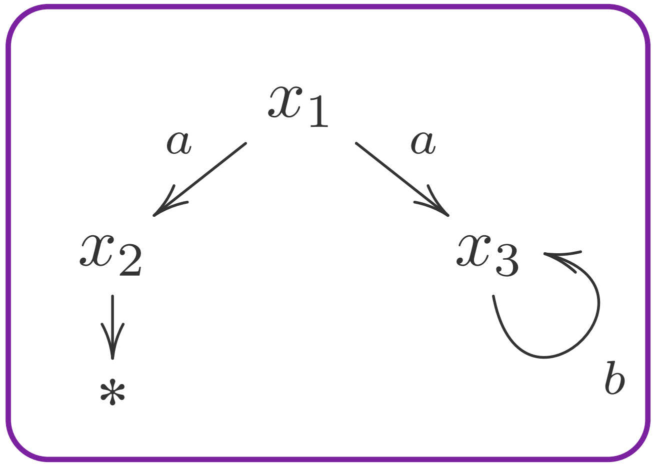
NFA

$X \rightarrow 2 \times (\mathcal{P}(X))^A$



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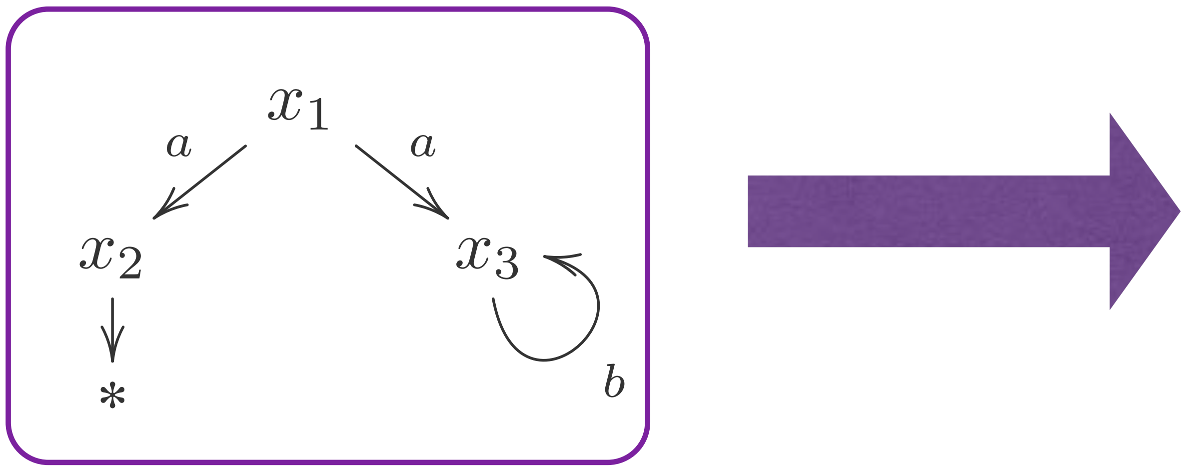

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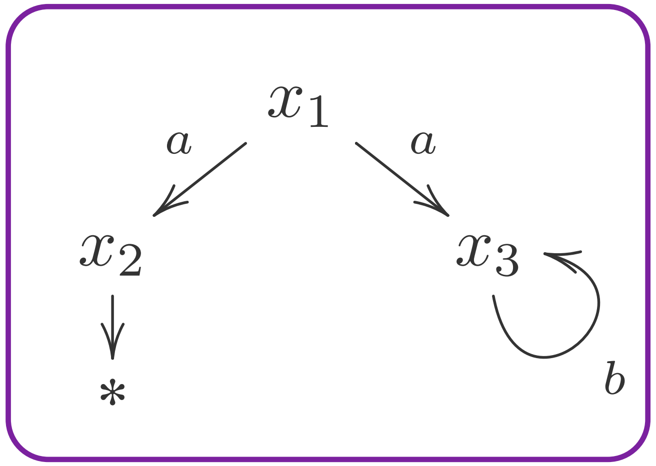


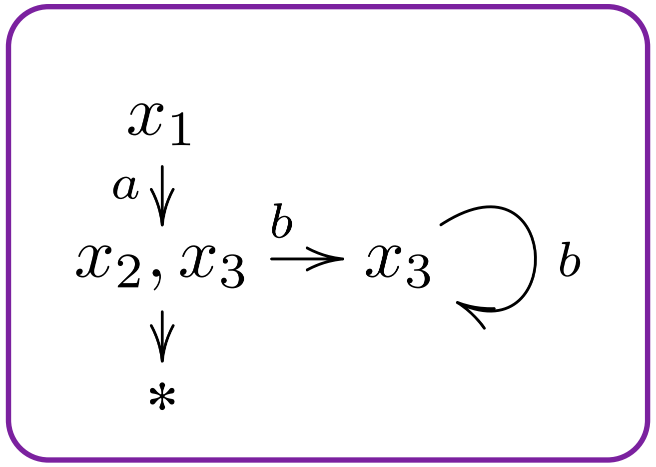

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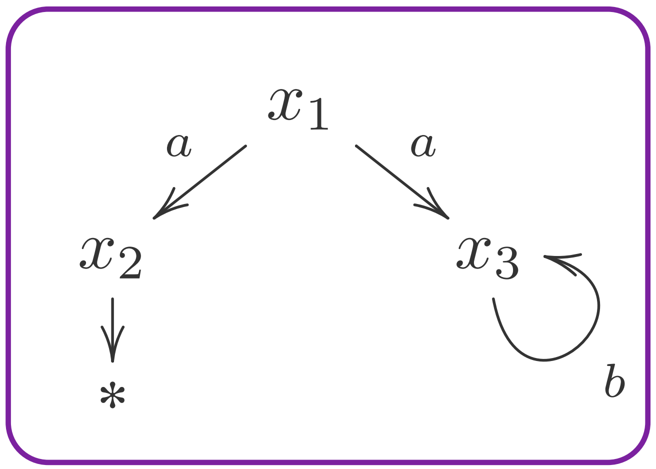


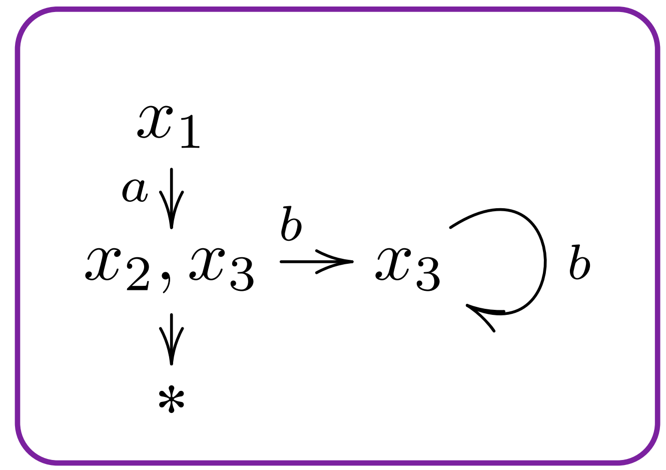


[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations

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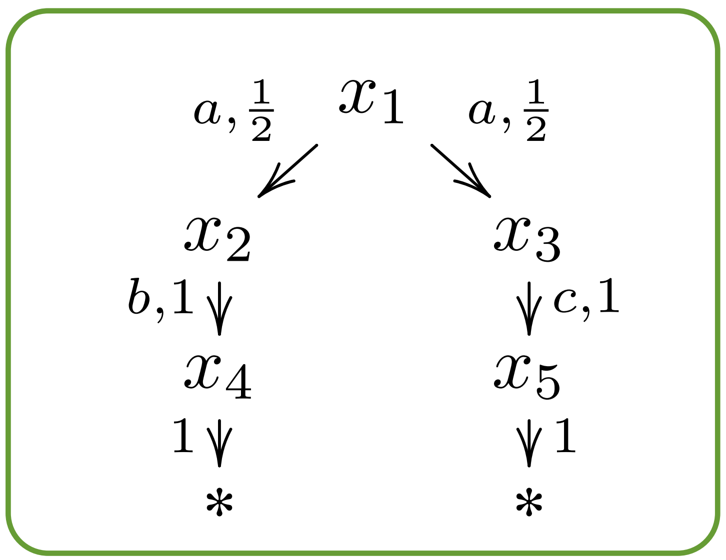
## Generative PTS

$X \to \mathcal{D}(1 + A \times X)$



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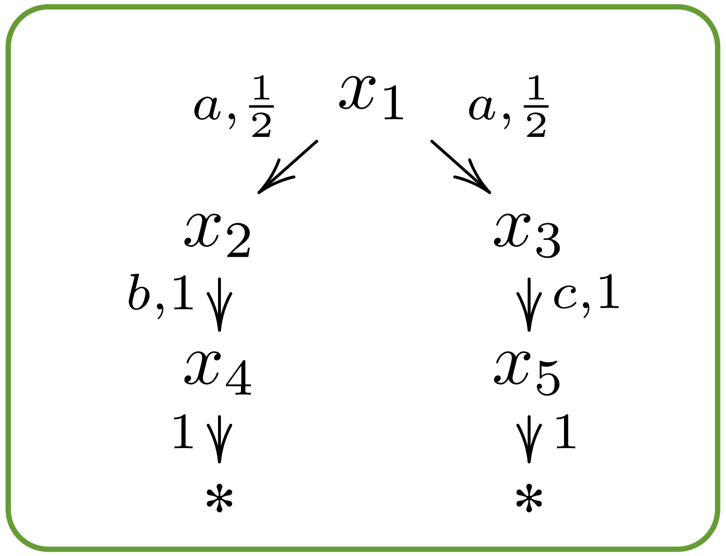

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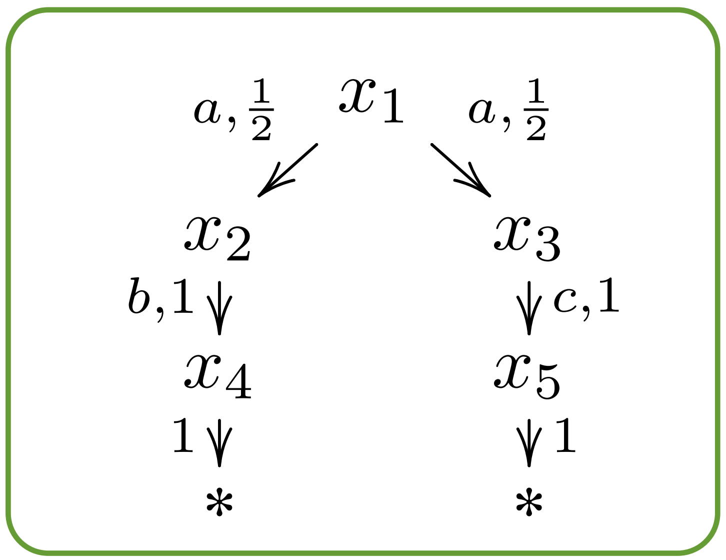


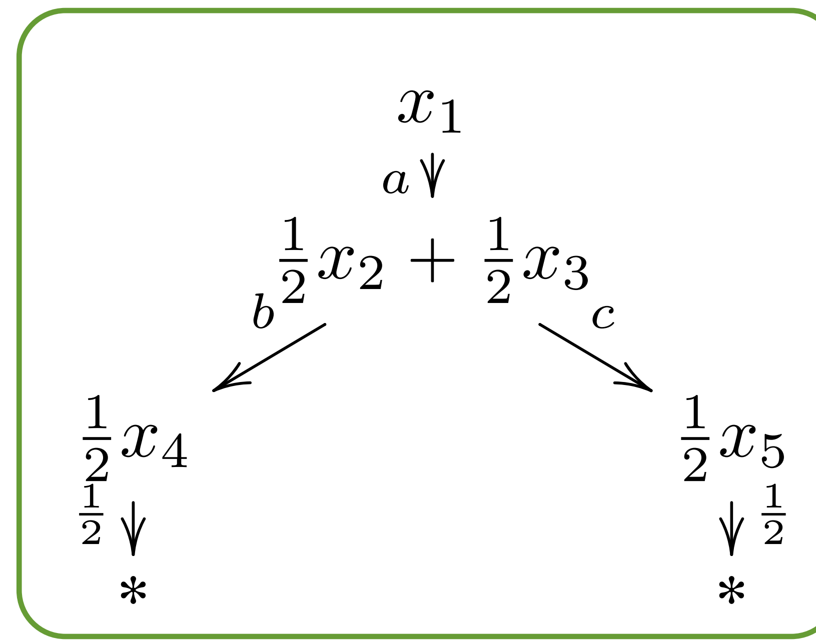

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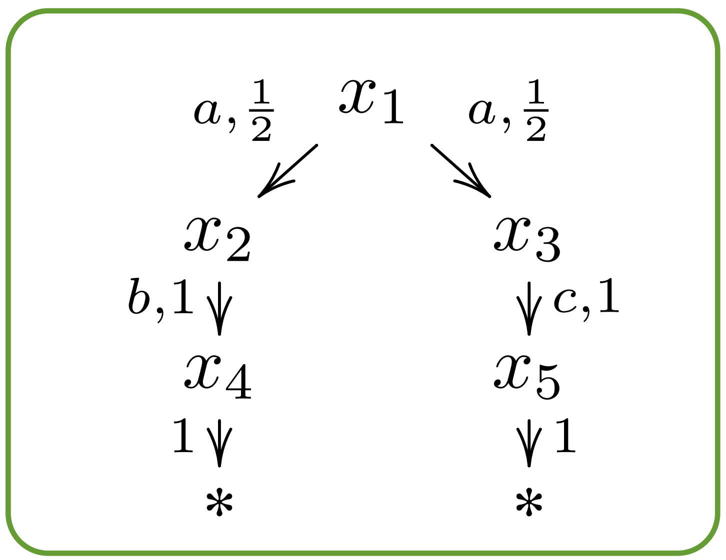
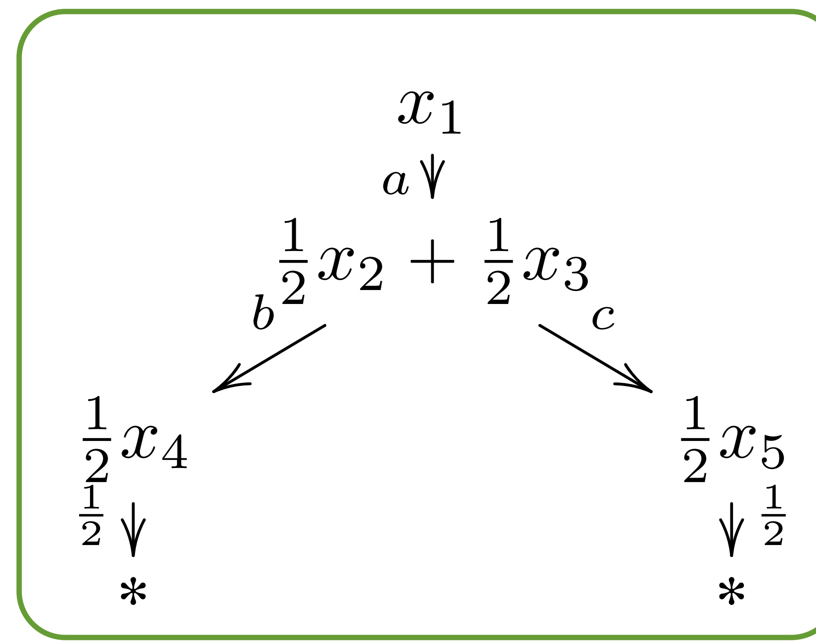
## Generative PTS

$X \to \mathcal{D}(1 + A \times X)$




[Silva, S. MFPS'11]

[Jacobs, Silva, S. JCSS'15]

# Determinisations

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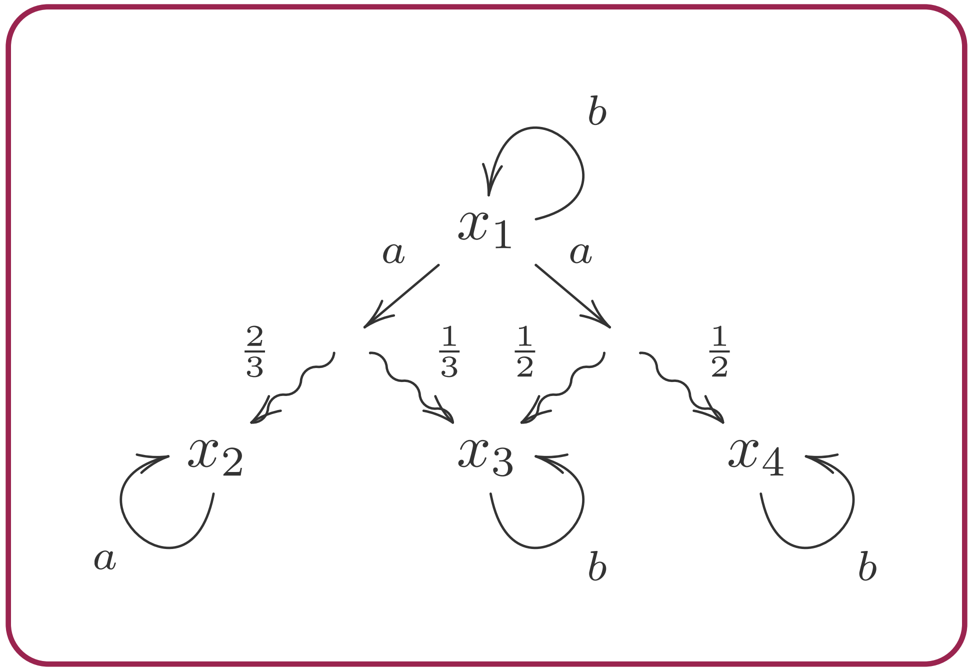
PA

$X \to (\mathcal{PD}(X))^A$



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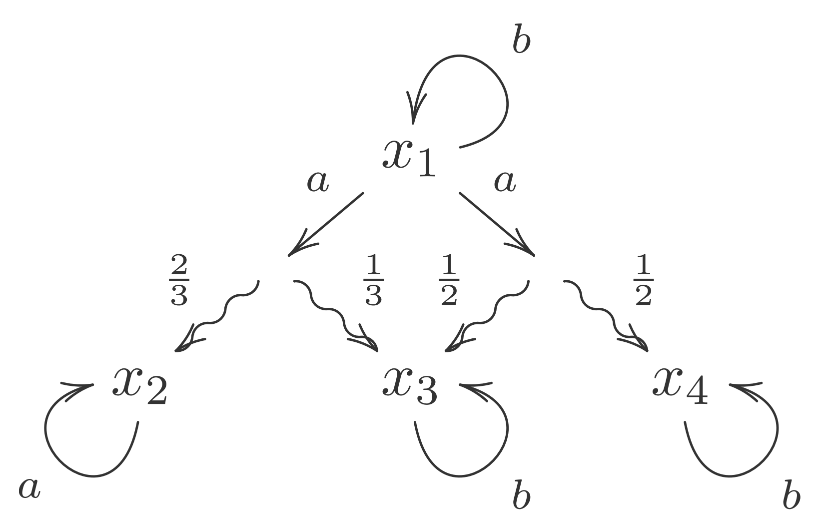


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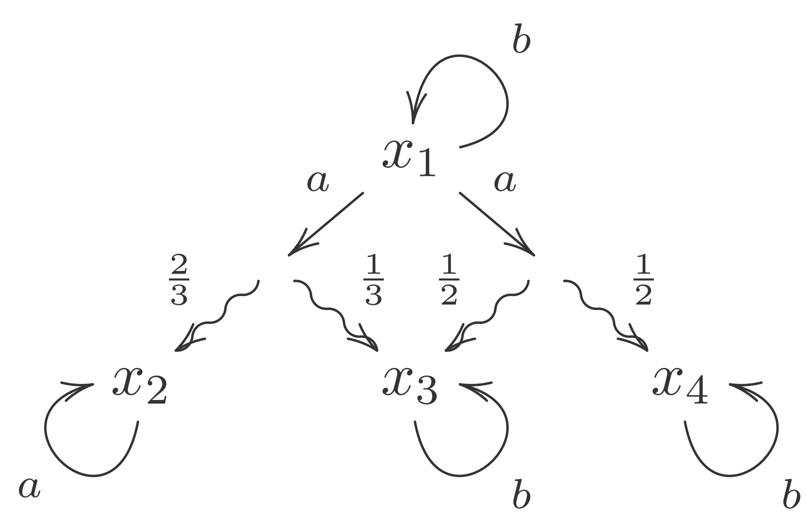


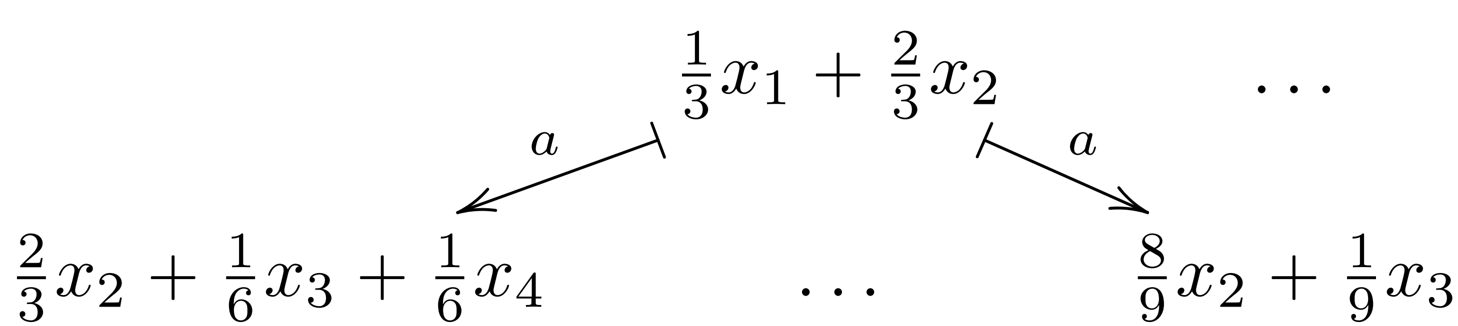

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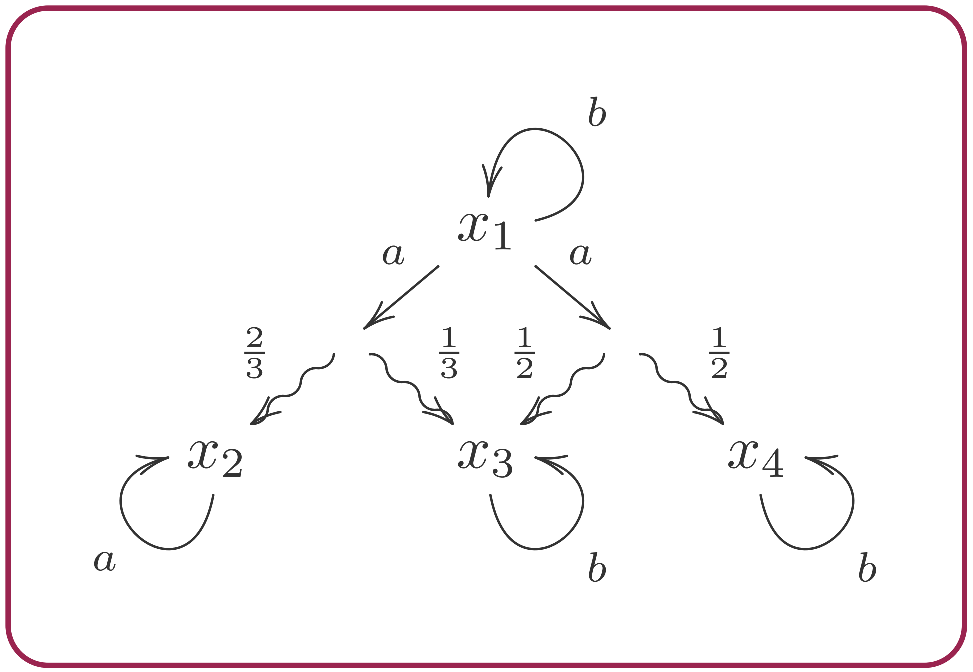
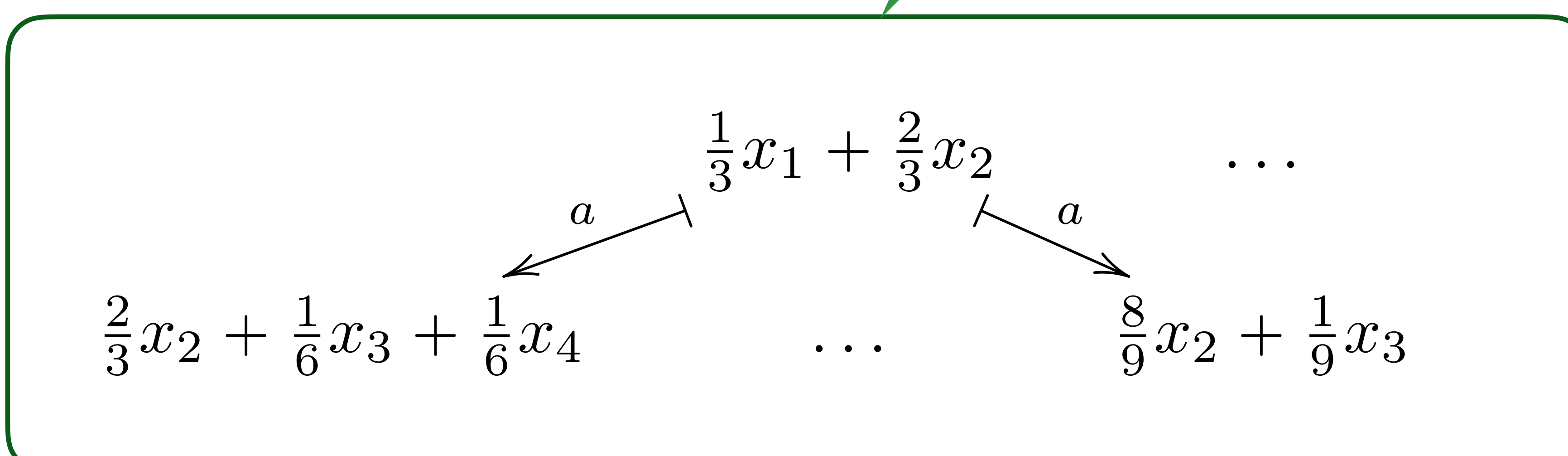



belief-state transformer



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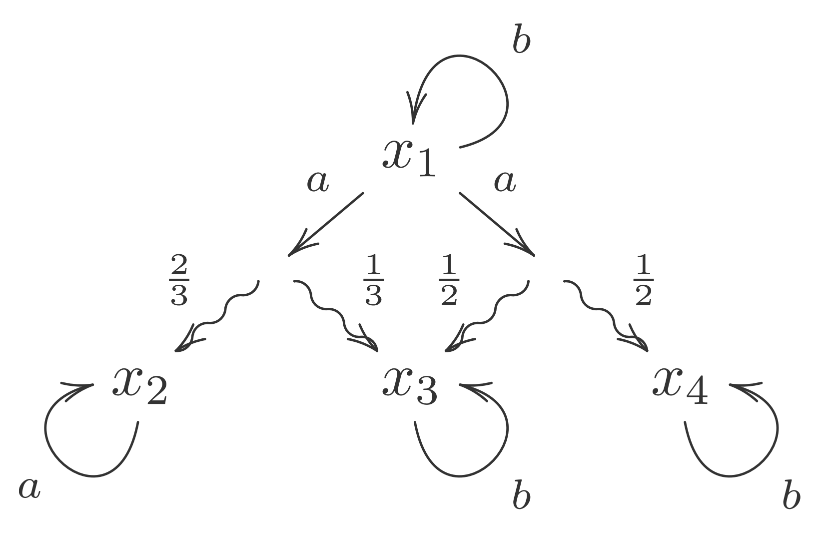
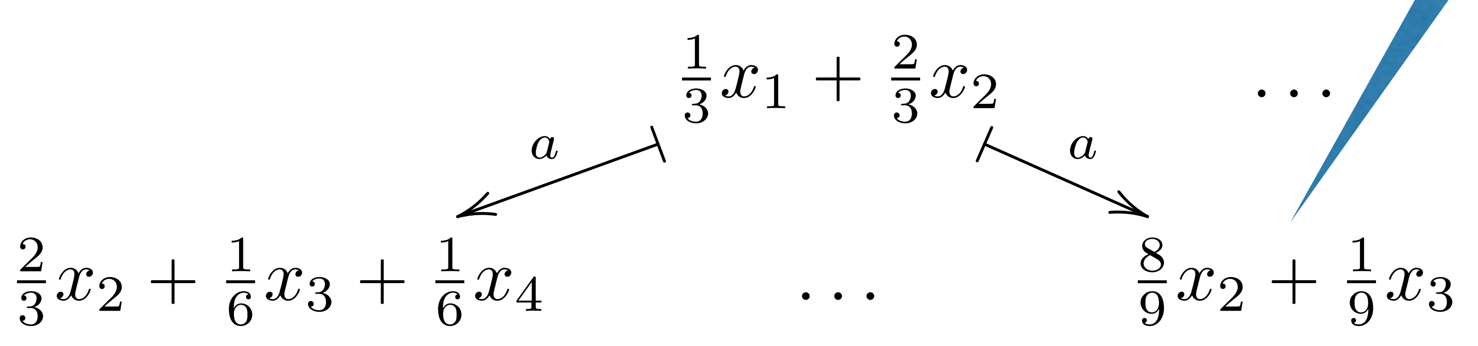



belief-state transformer

belief state

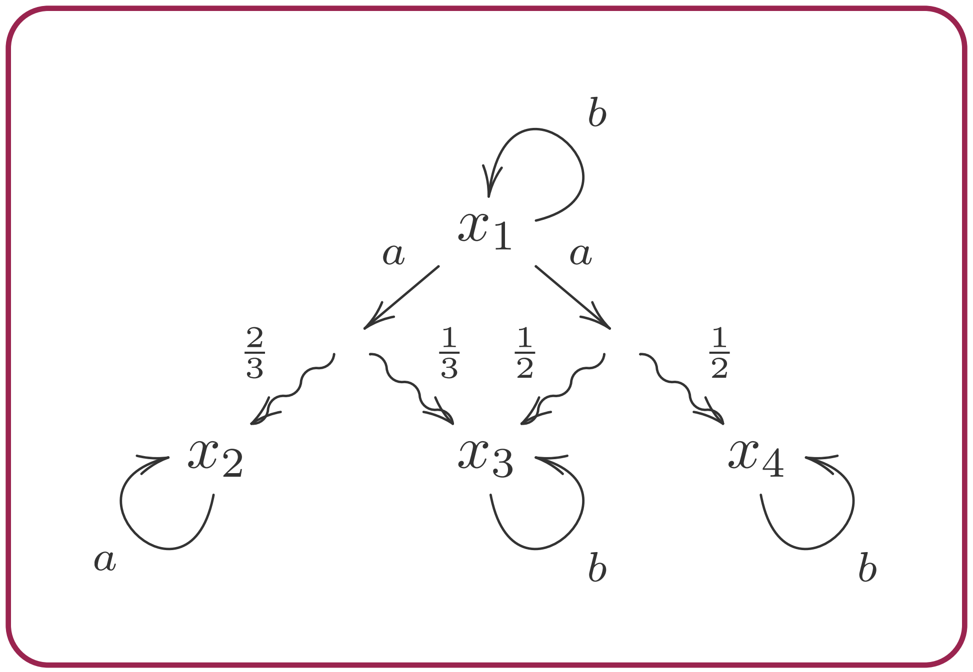


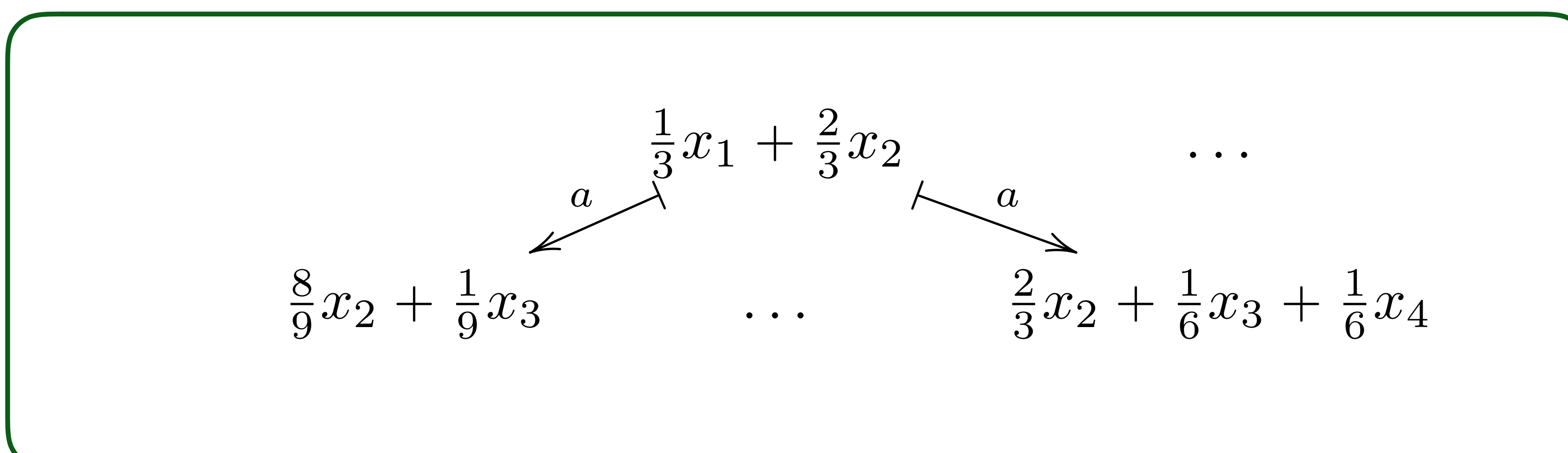
# Belief-state transformer

foundation ?

PA

X → $(\mathcal{PD}(X))^A$



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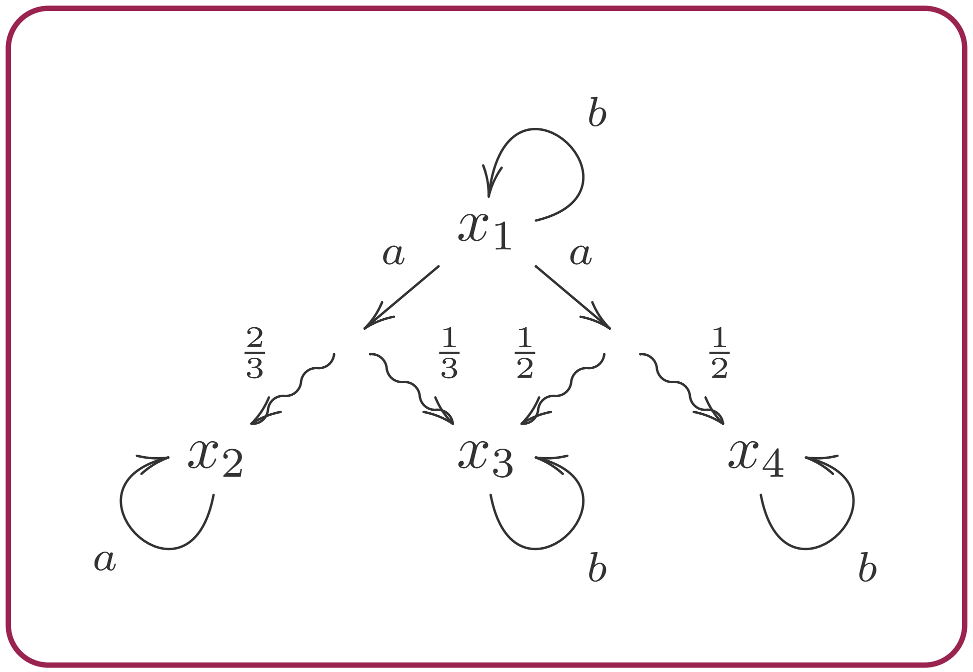
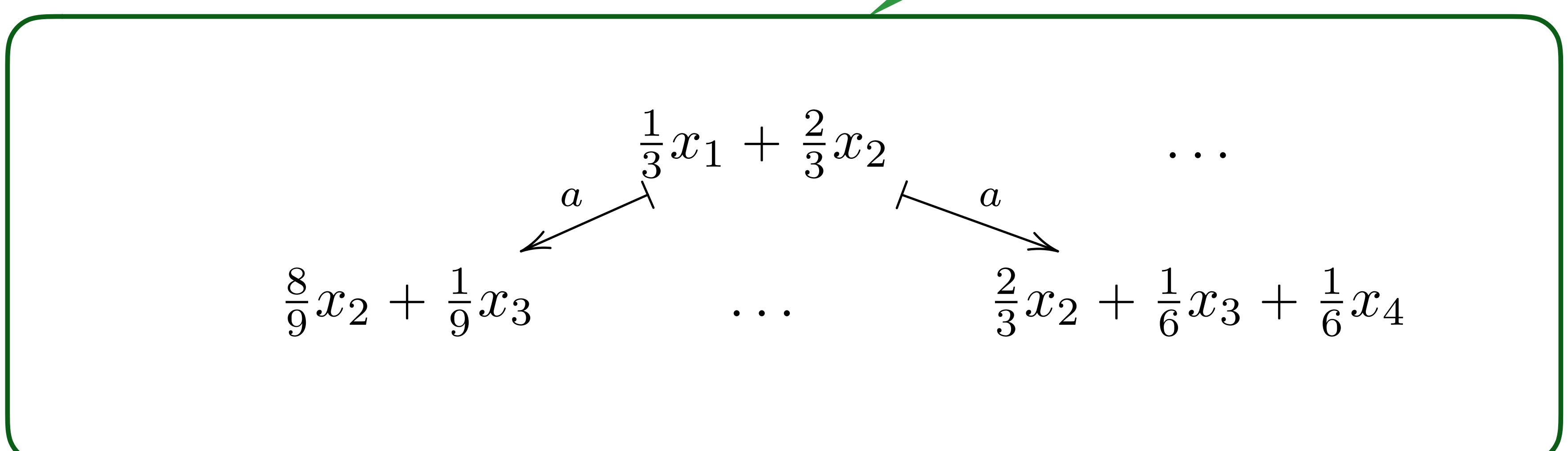



what is it?



# Belief-state transformer

foundation ?

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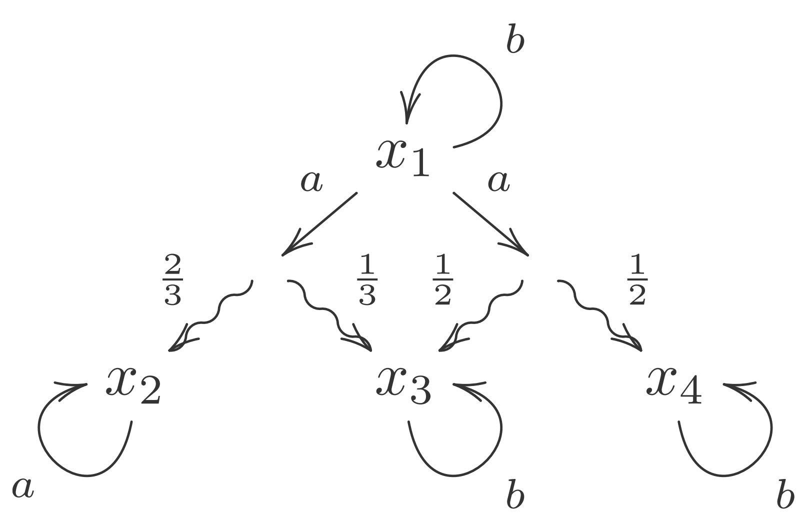
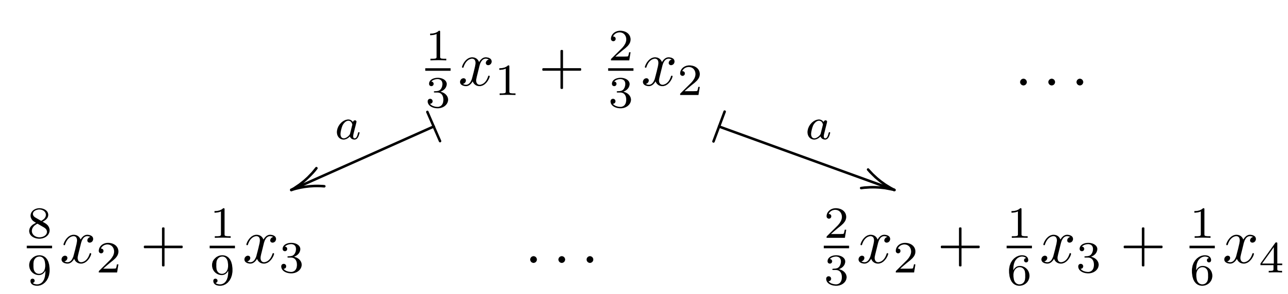
$X \to (\mathcal{PD}(X))^A$

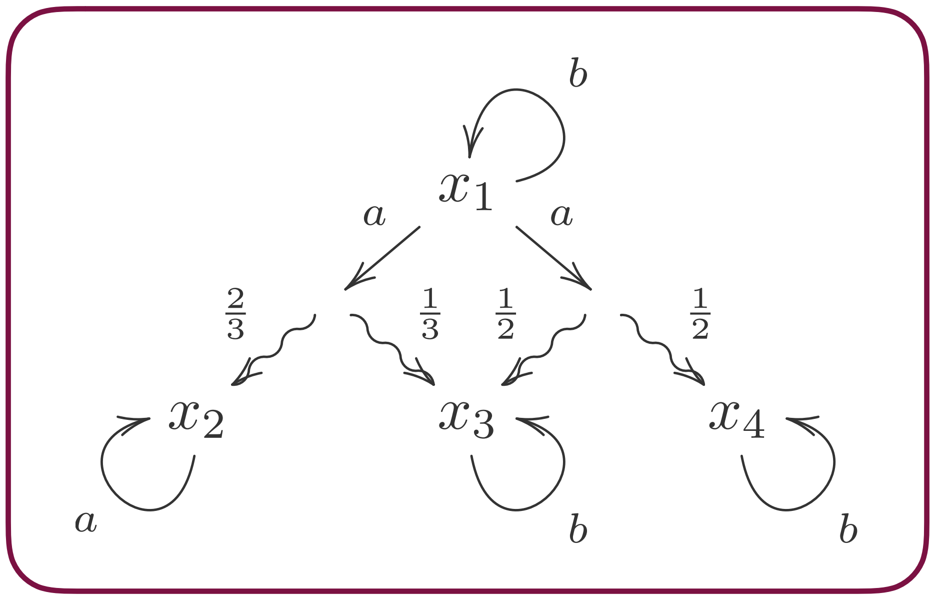


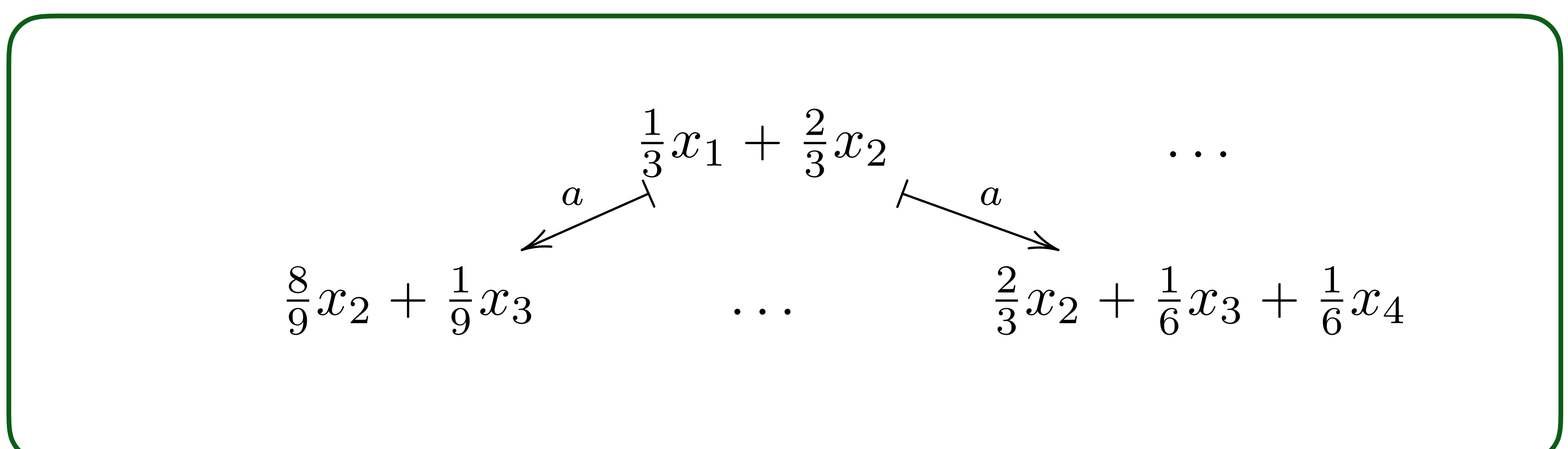

how does it emerge?

what is it?

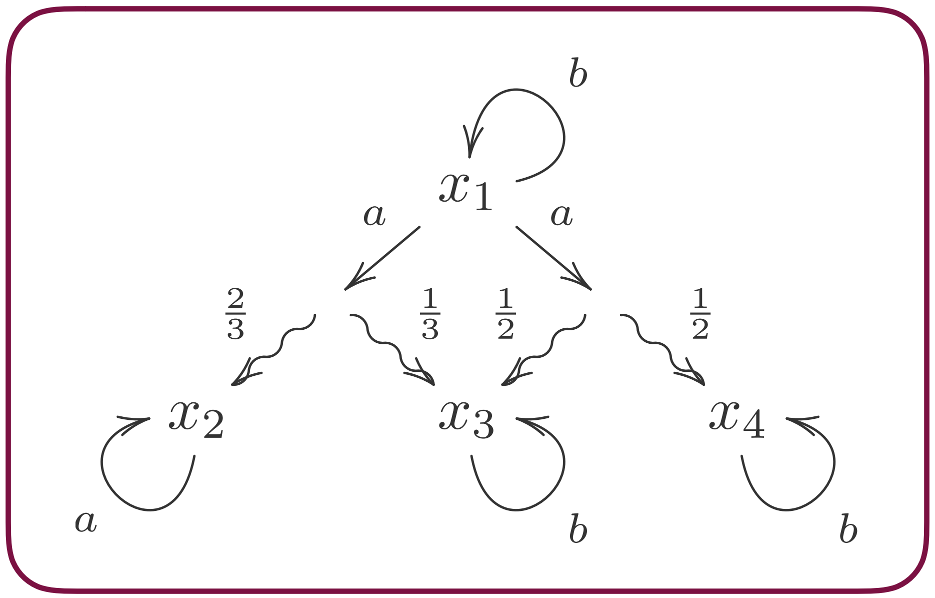


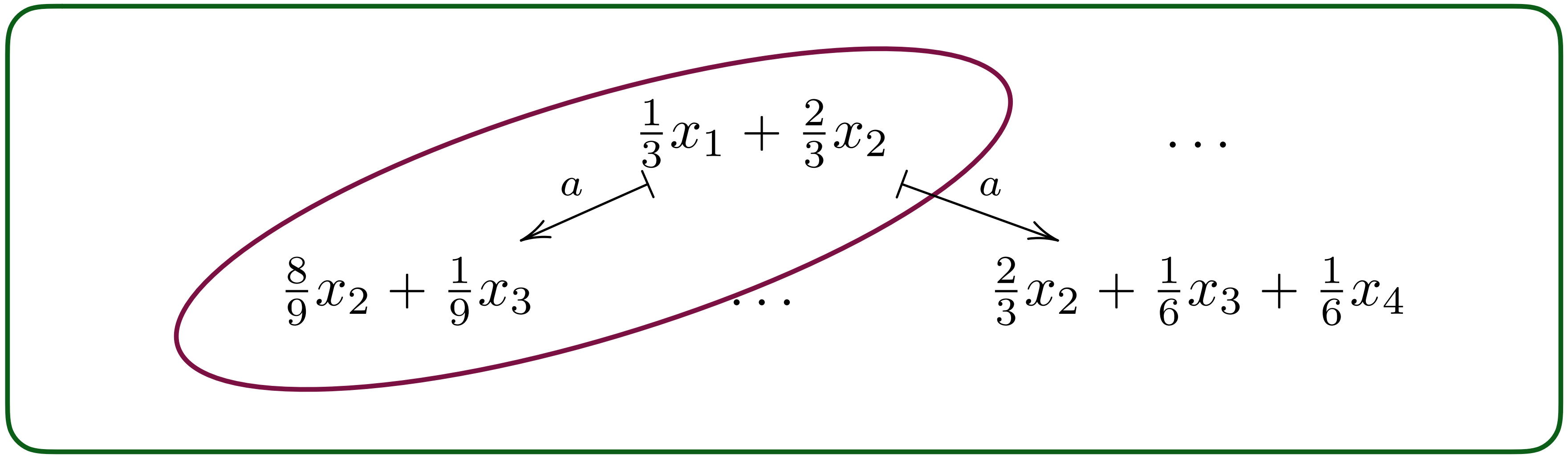
# Belief-state transformer



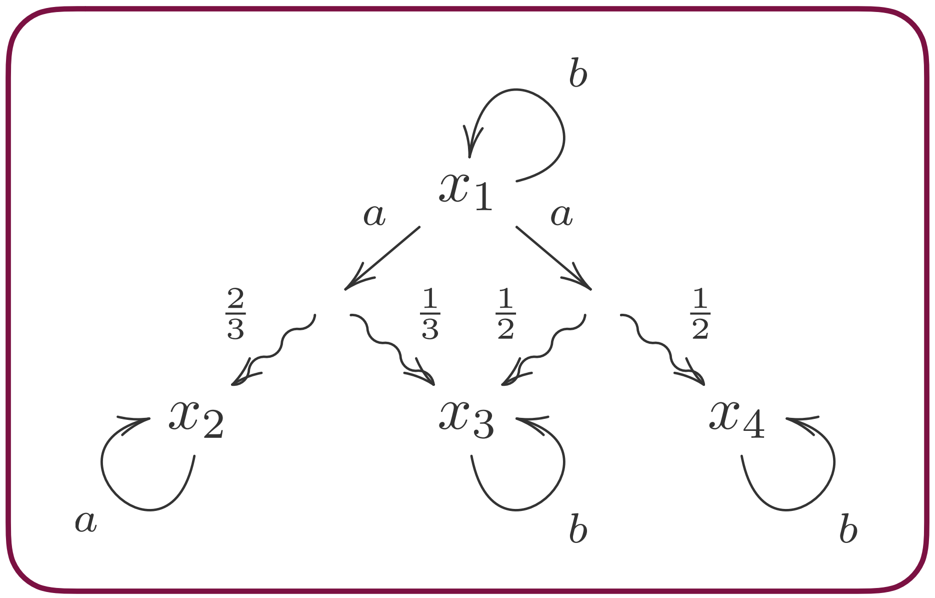



# Belief-state transformer

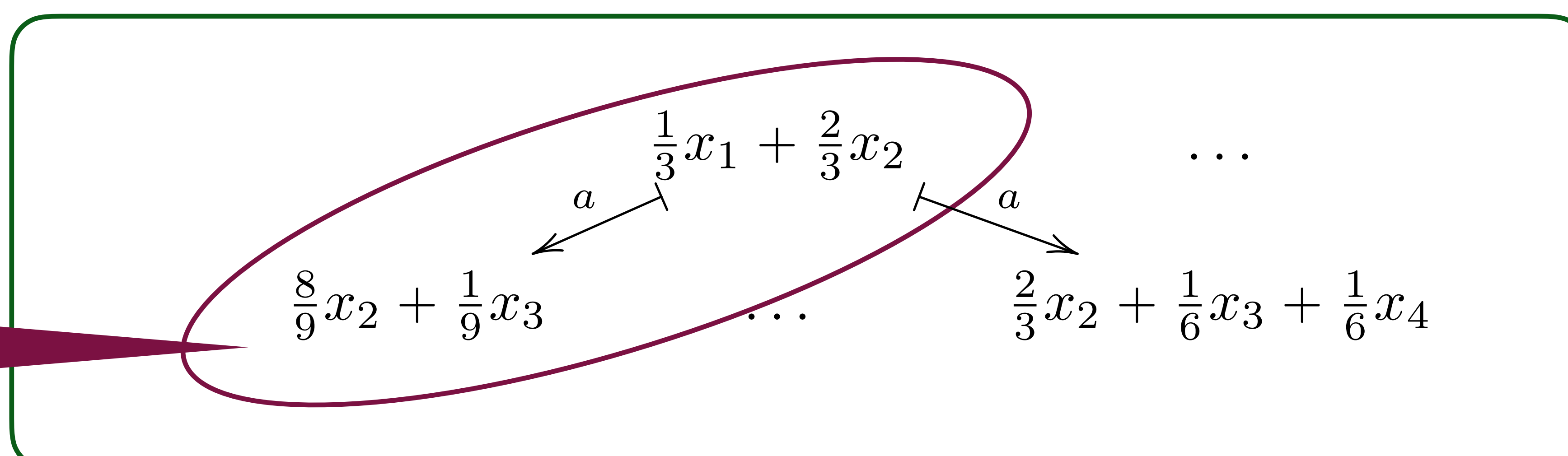


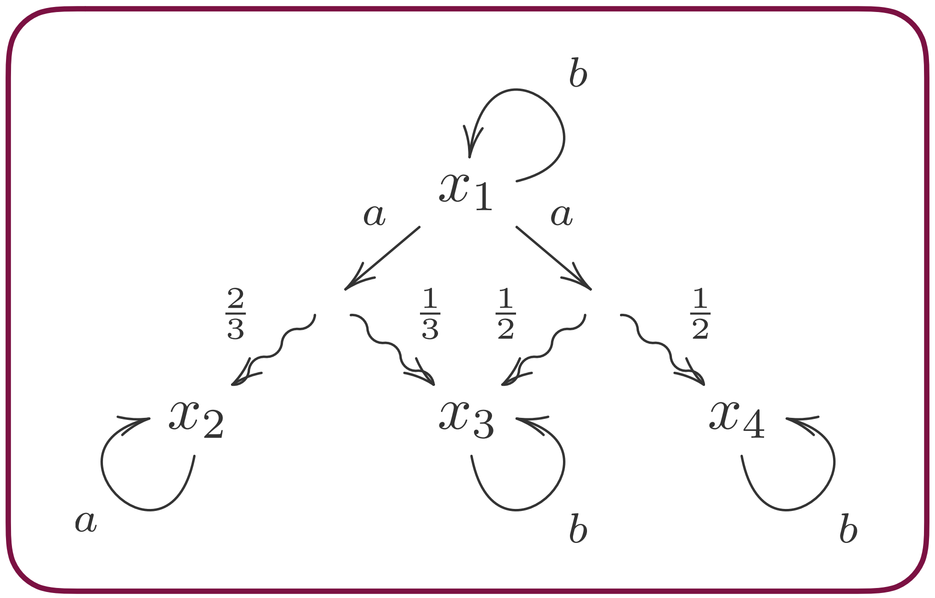
# Belief-state transformer

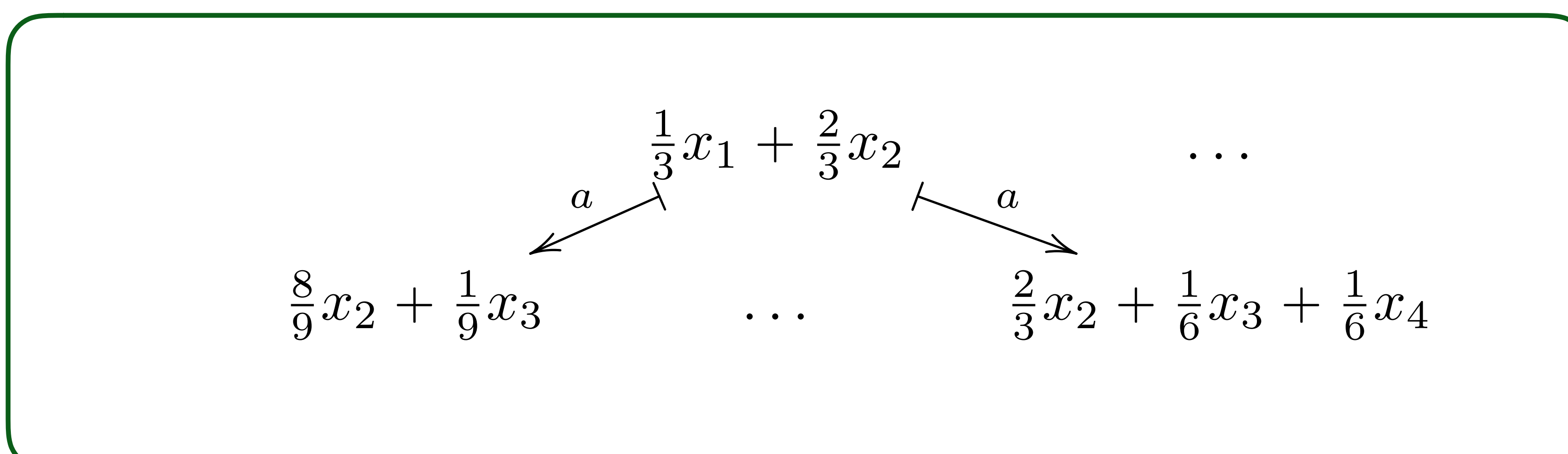



$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

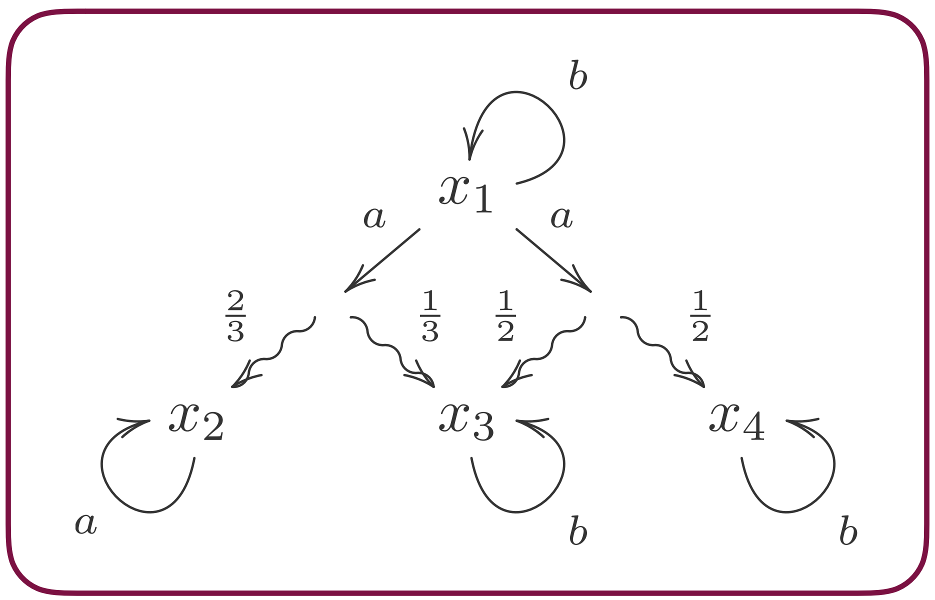


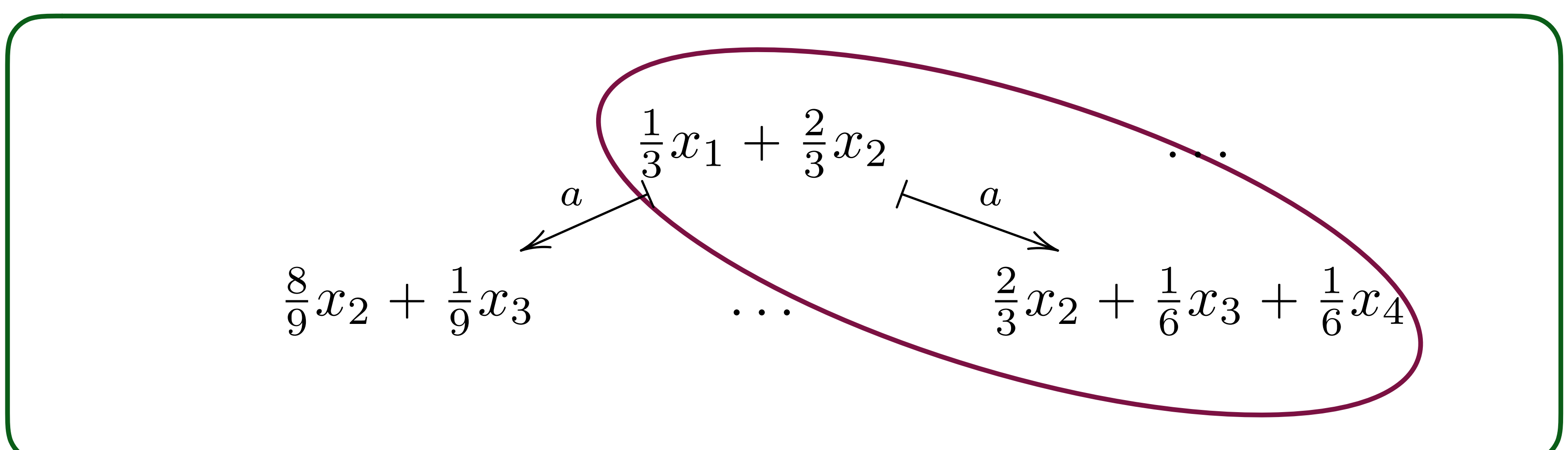
# Belief-state transformer



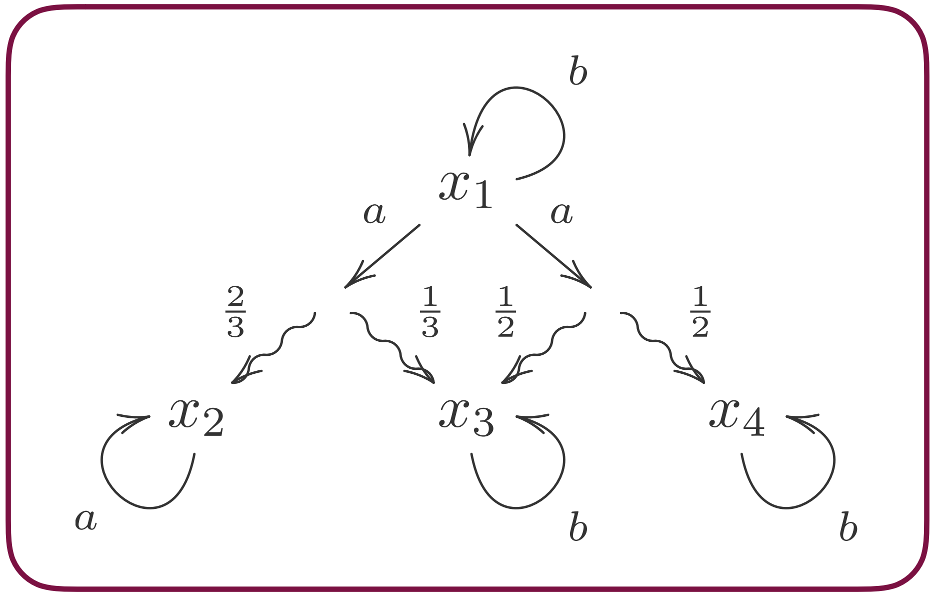



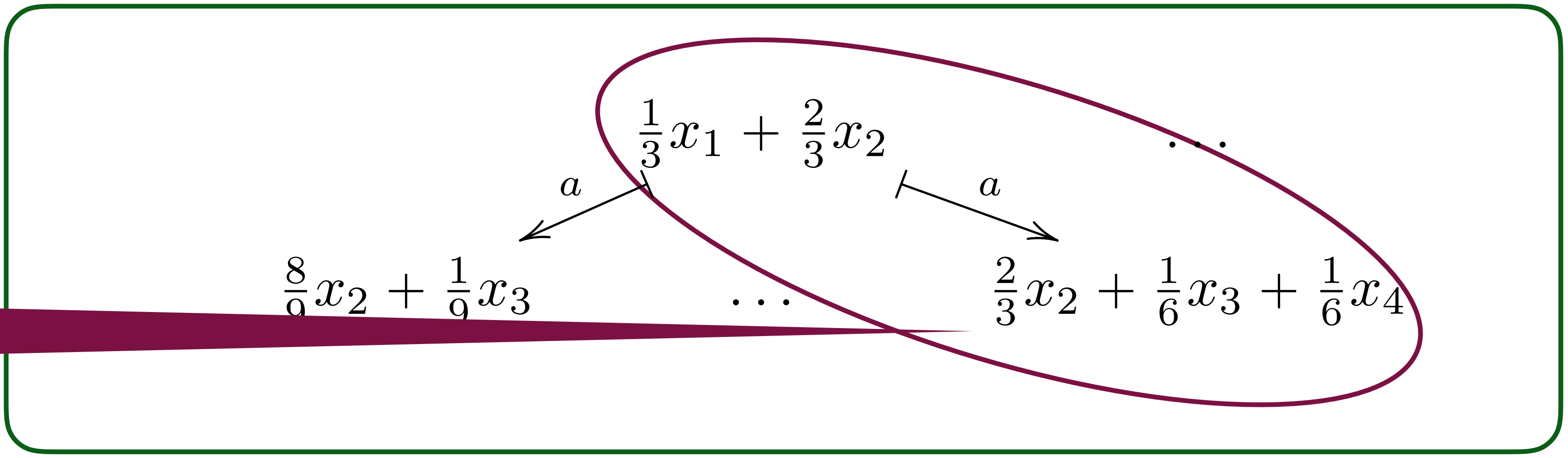
# Belief-state transformer



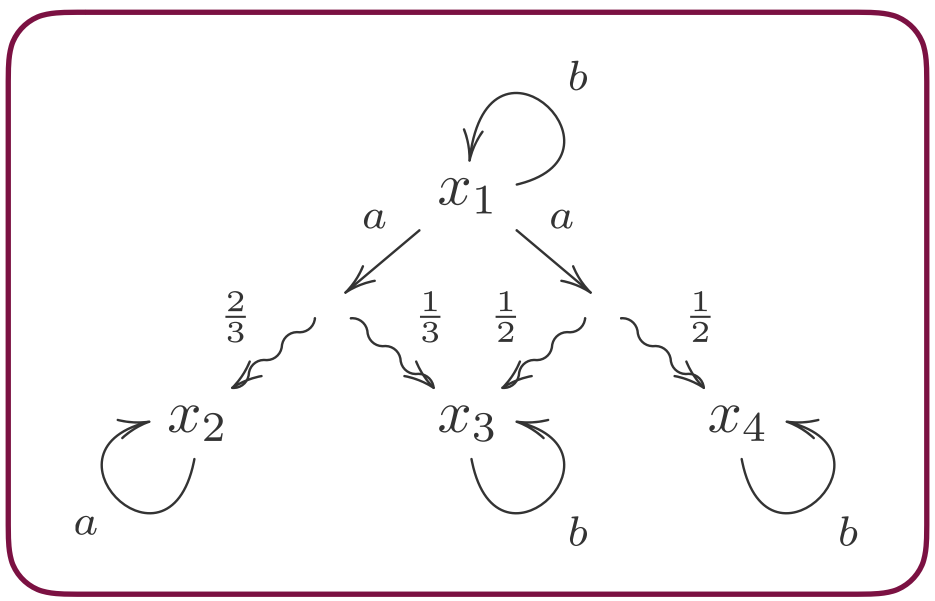
# Belief-state transformer

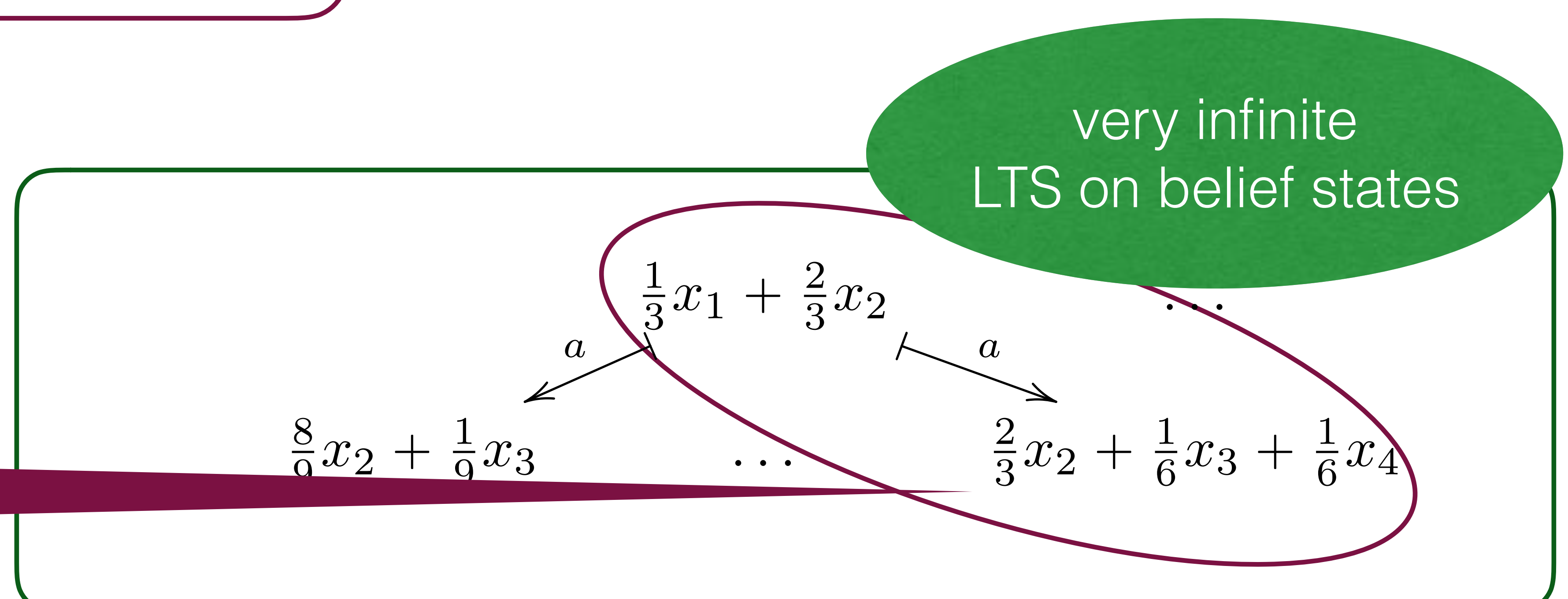



$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer




$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Probabilistic Automata

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Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity

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strong bisimilarity

probabilistic / combined bisimilarity

# Probabilistic Automata

Can be given different semantics:

1. Bisimilarity — strong bisimilarity
2. Convex bisimilarity — probabilistic / combined bisimilarity
3. Distribution bisimilarity — belief-state bisimilarity

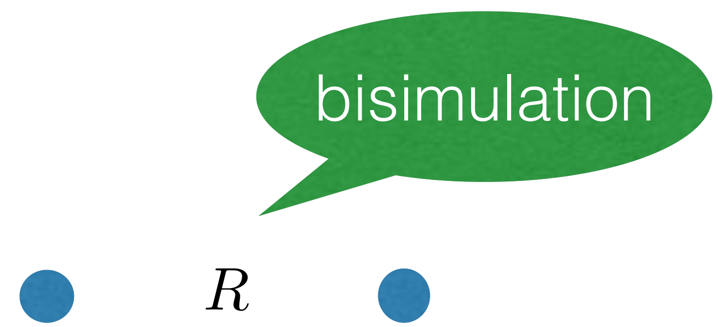
# Bisimilarity

# Bisimilarity

bisimulation

$R$

# Bisimilarity



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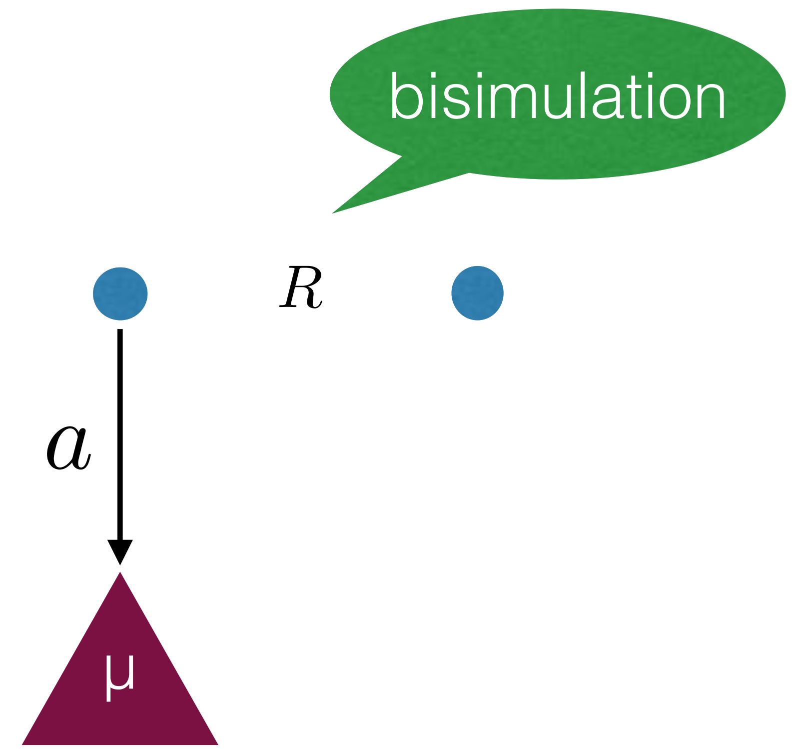
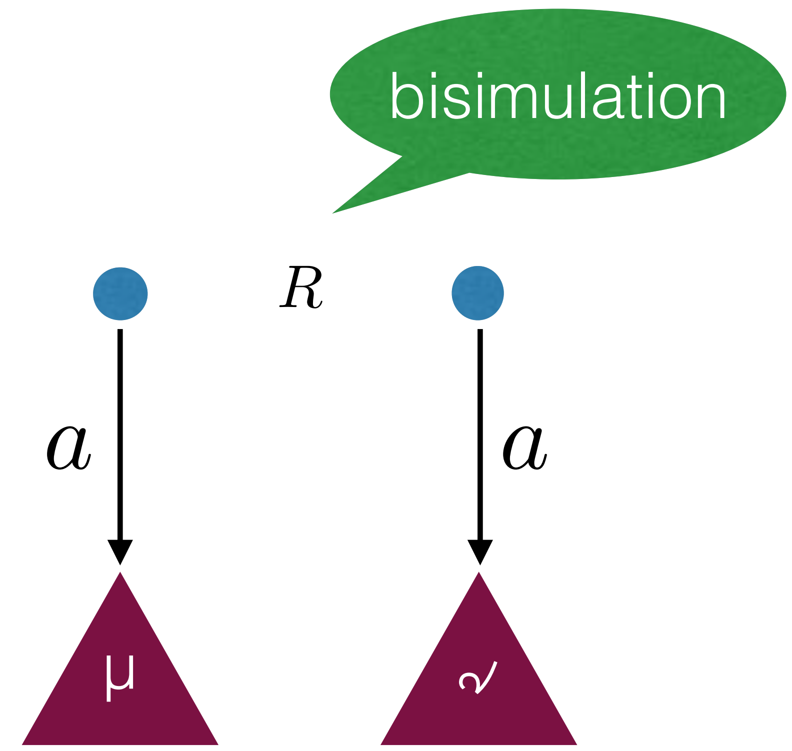
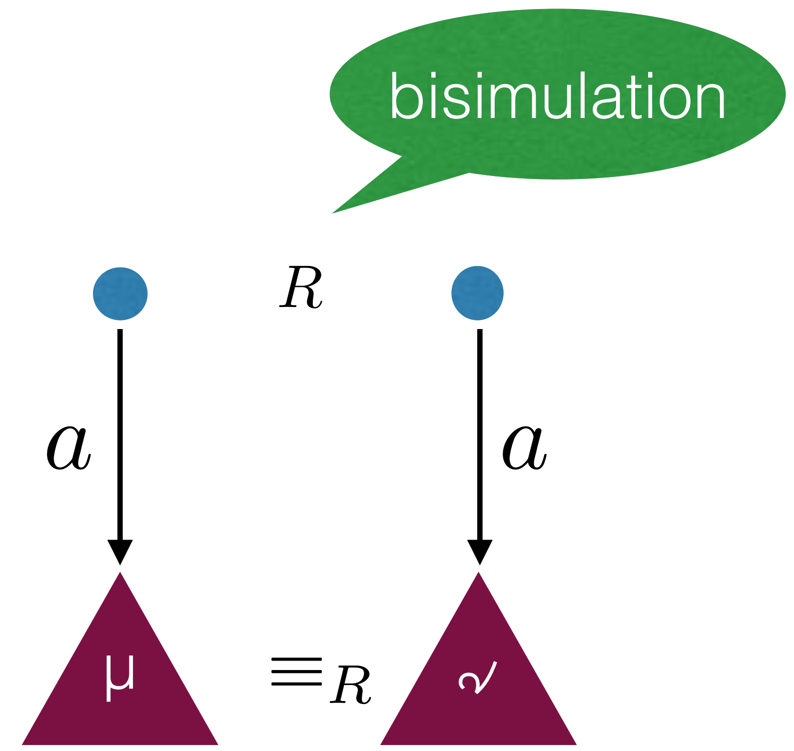


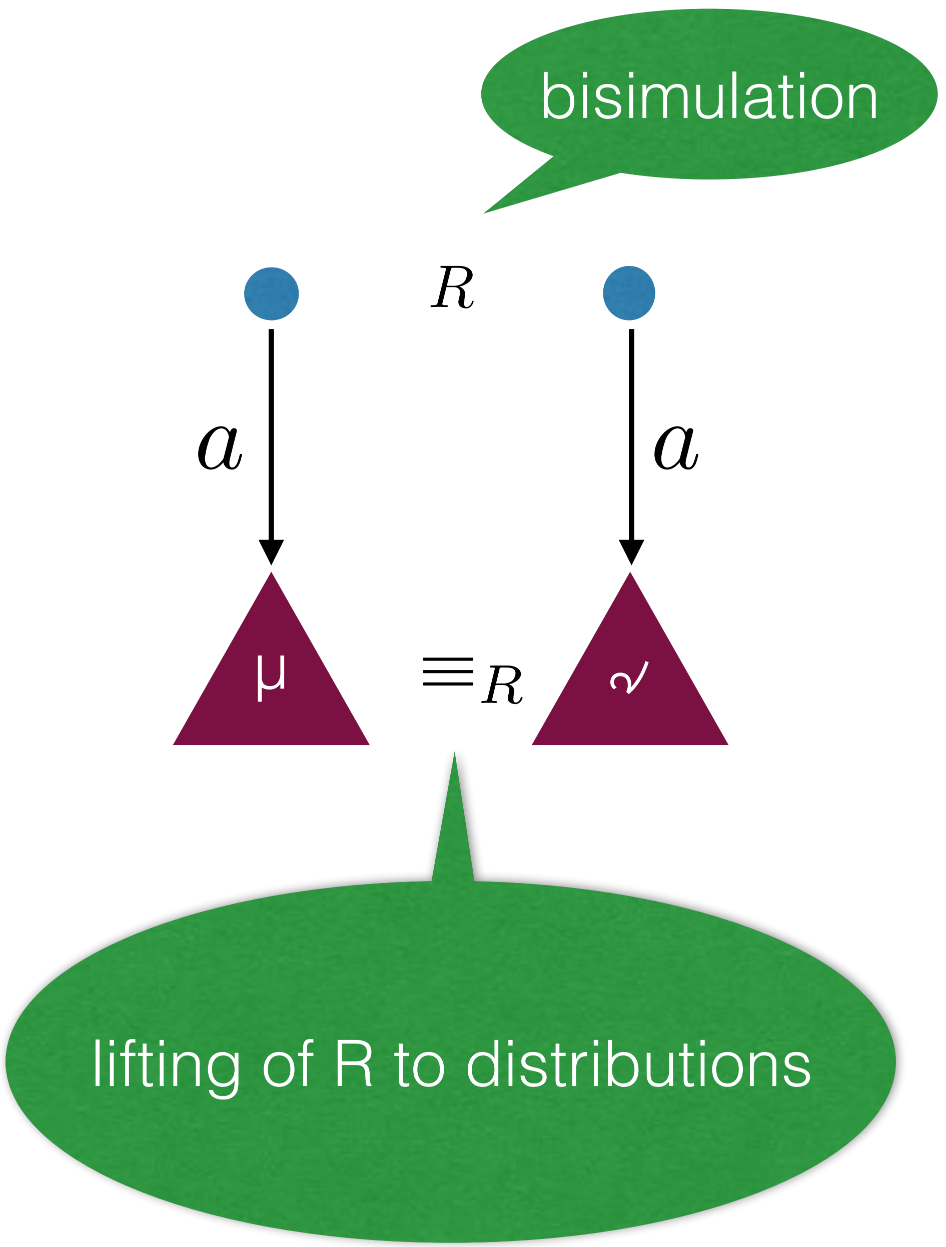
# Bisimilarity



# Bisimilarity

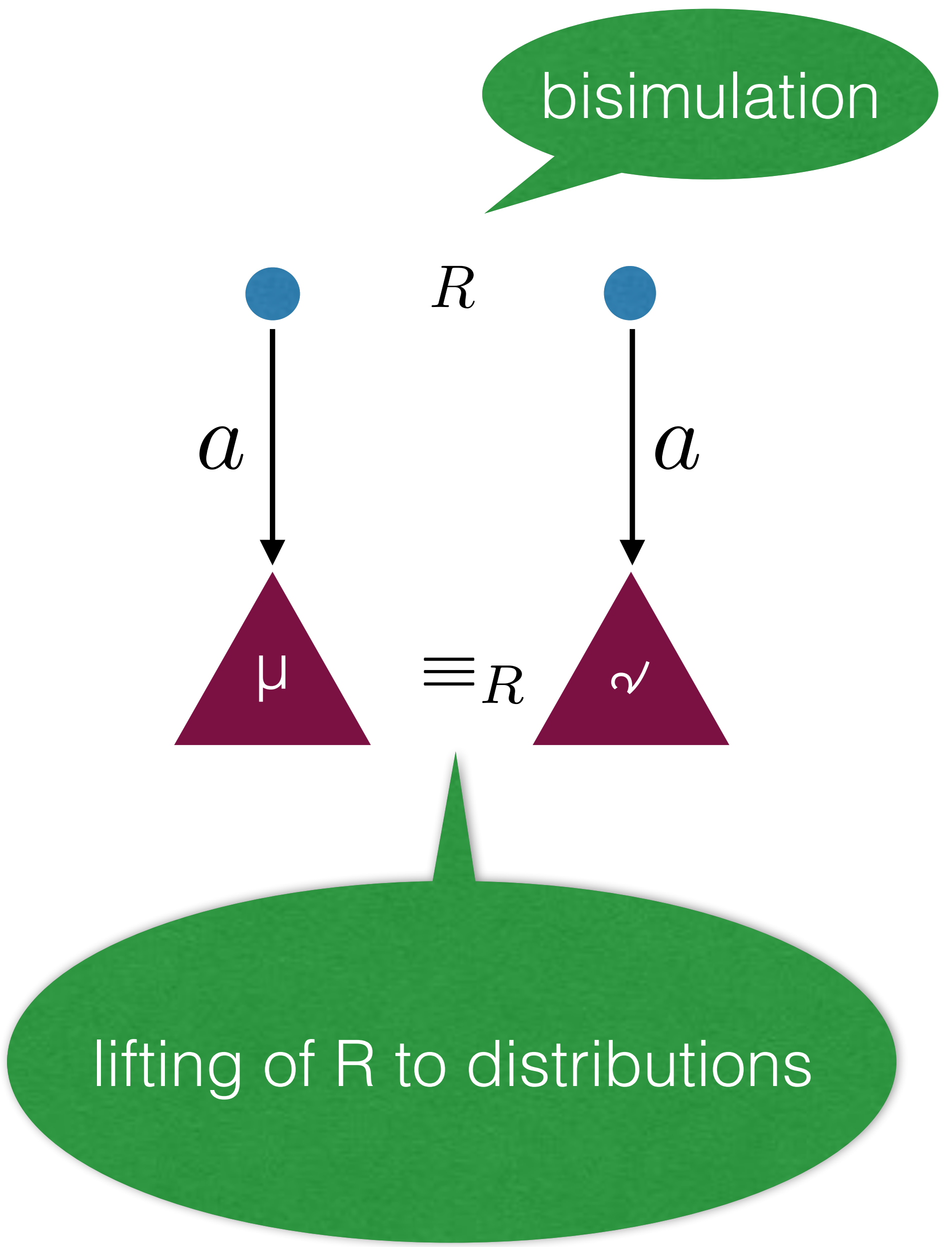


# Bisimilarity



# Bisimilarity

$\sim$ largest bisimulation



# Convex bisimilarity

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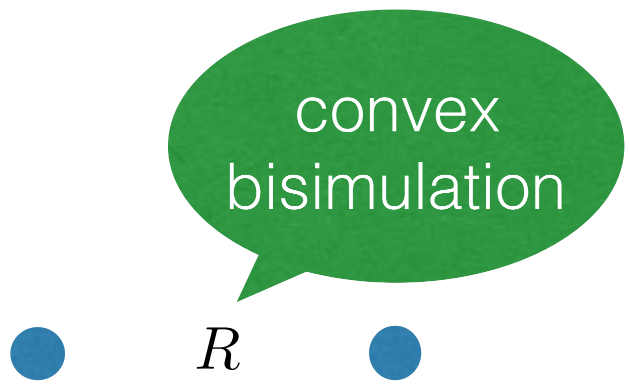
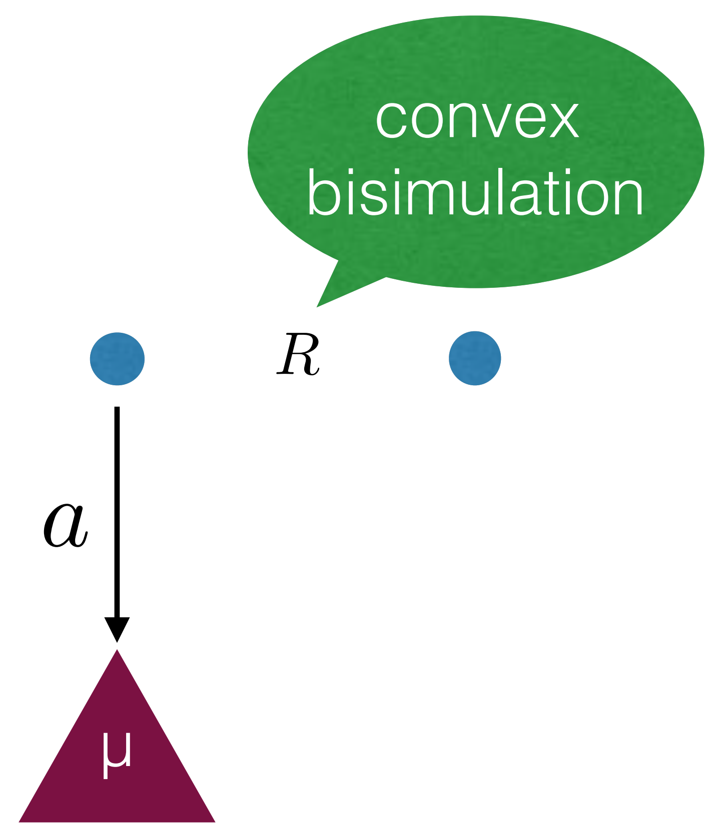
$R$

# Convex bisimilarity



# Convex bisimilarity



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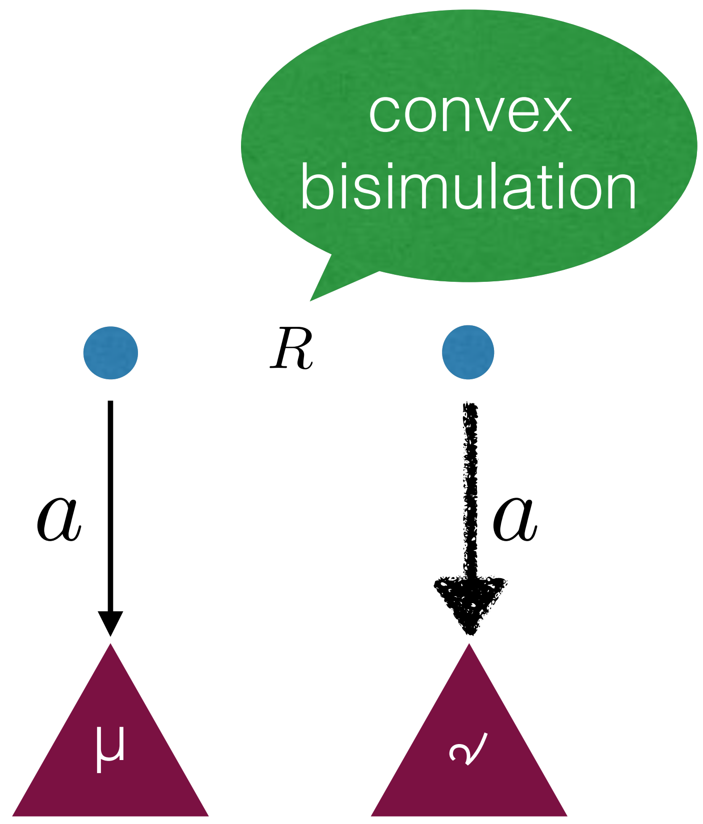

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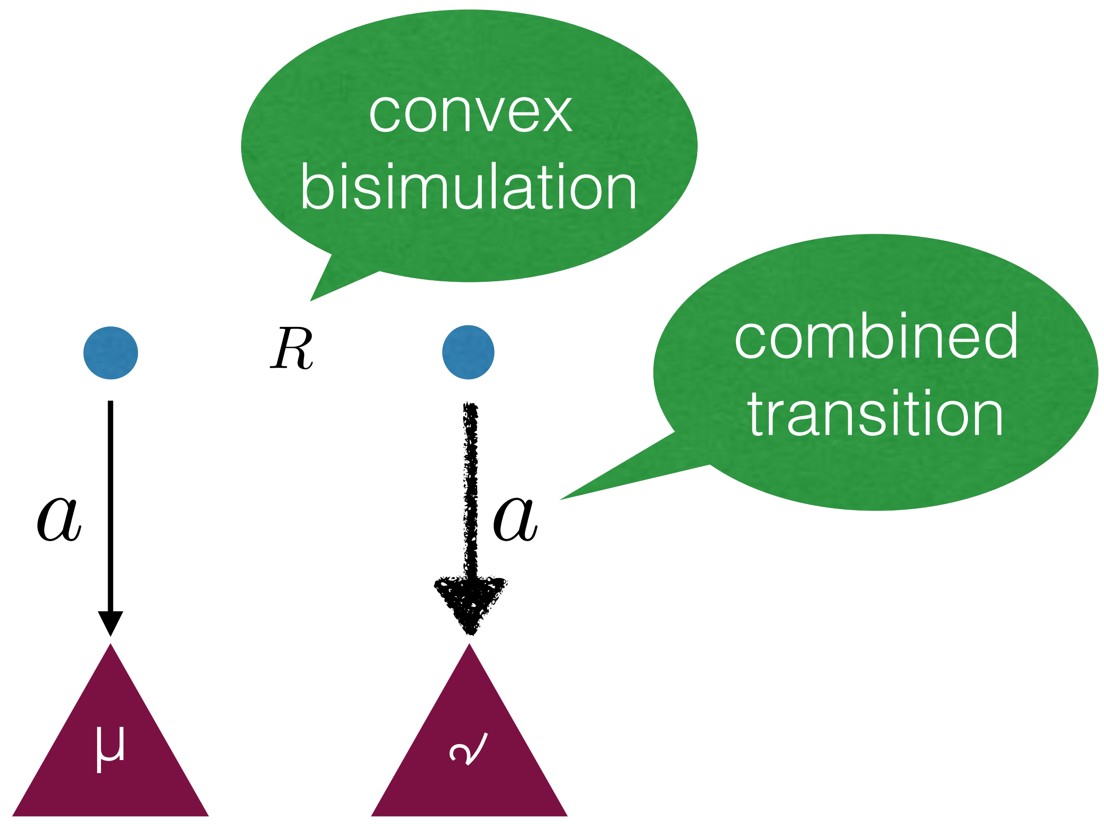

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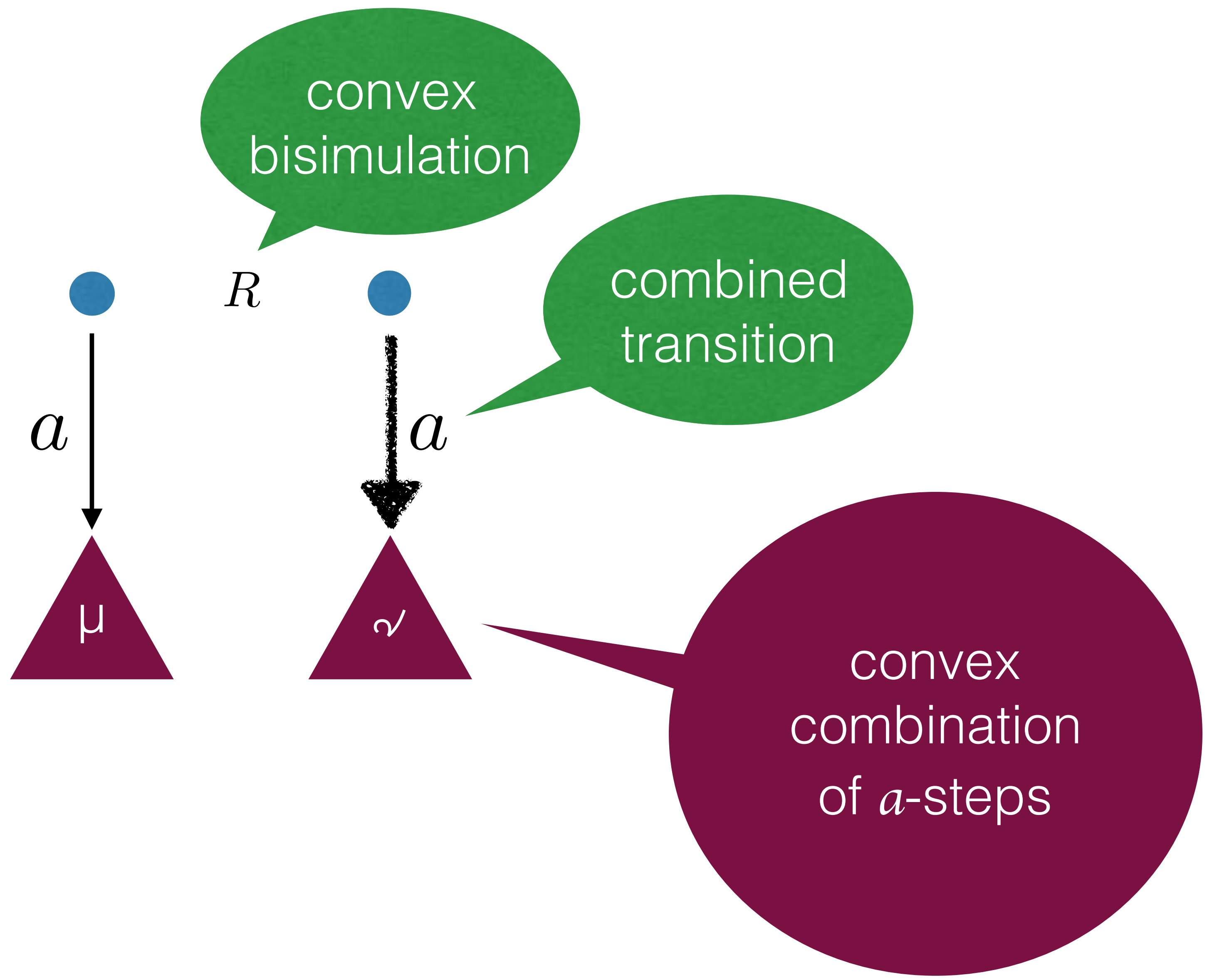

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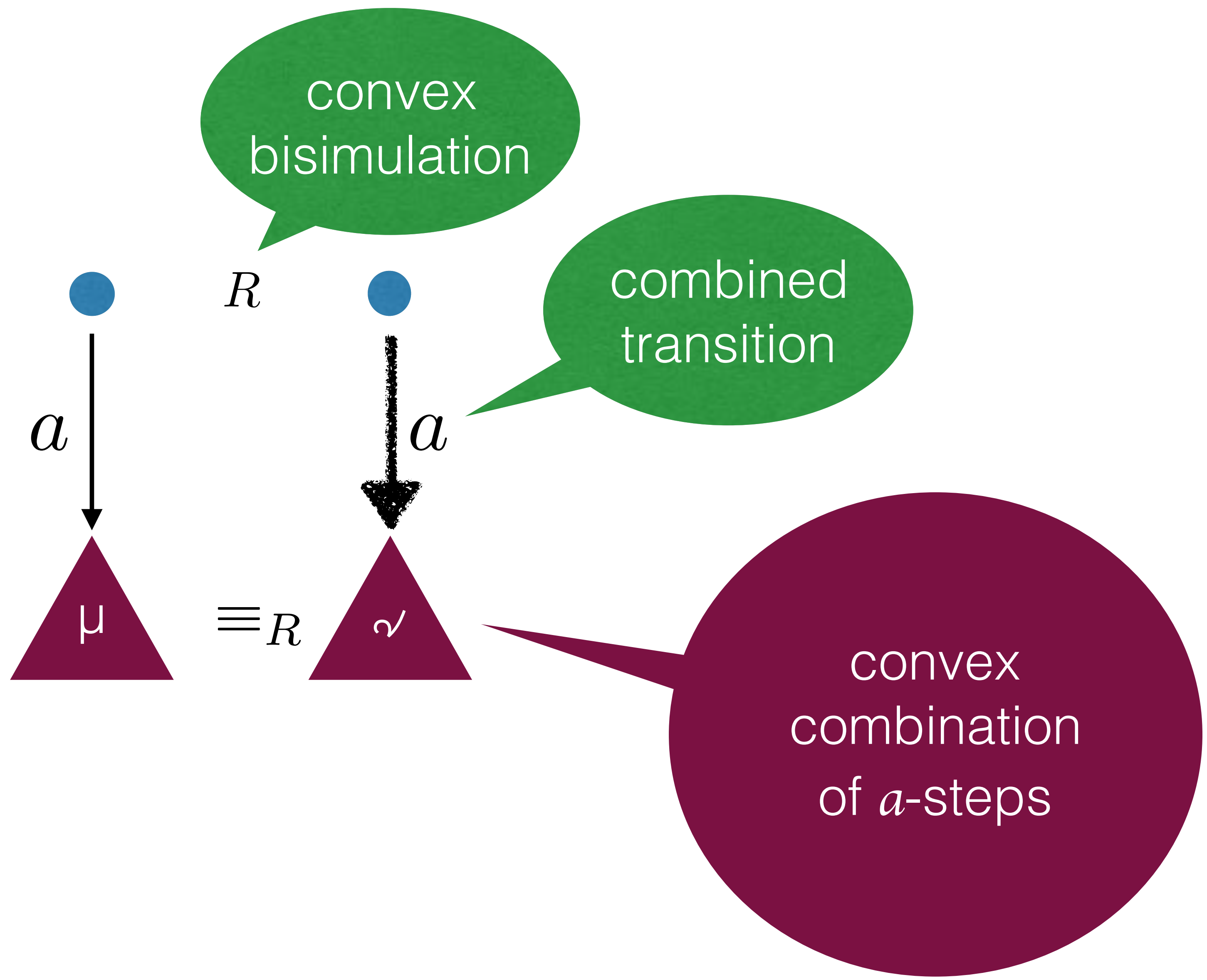

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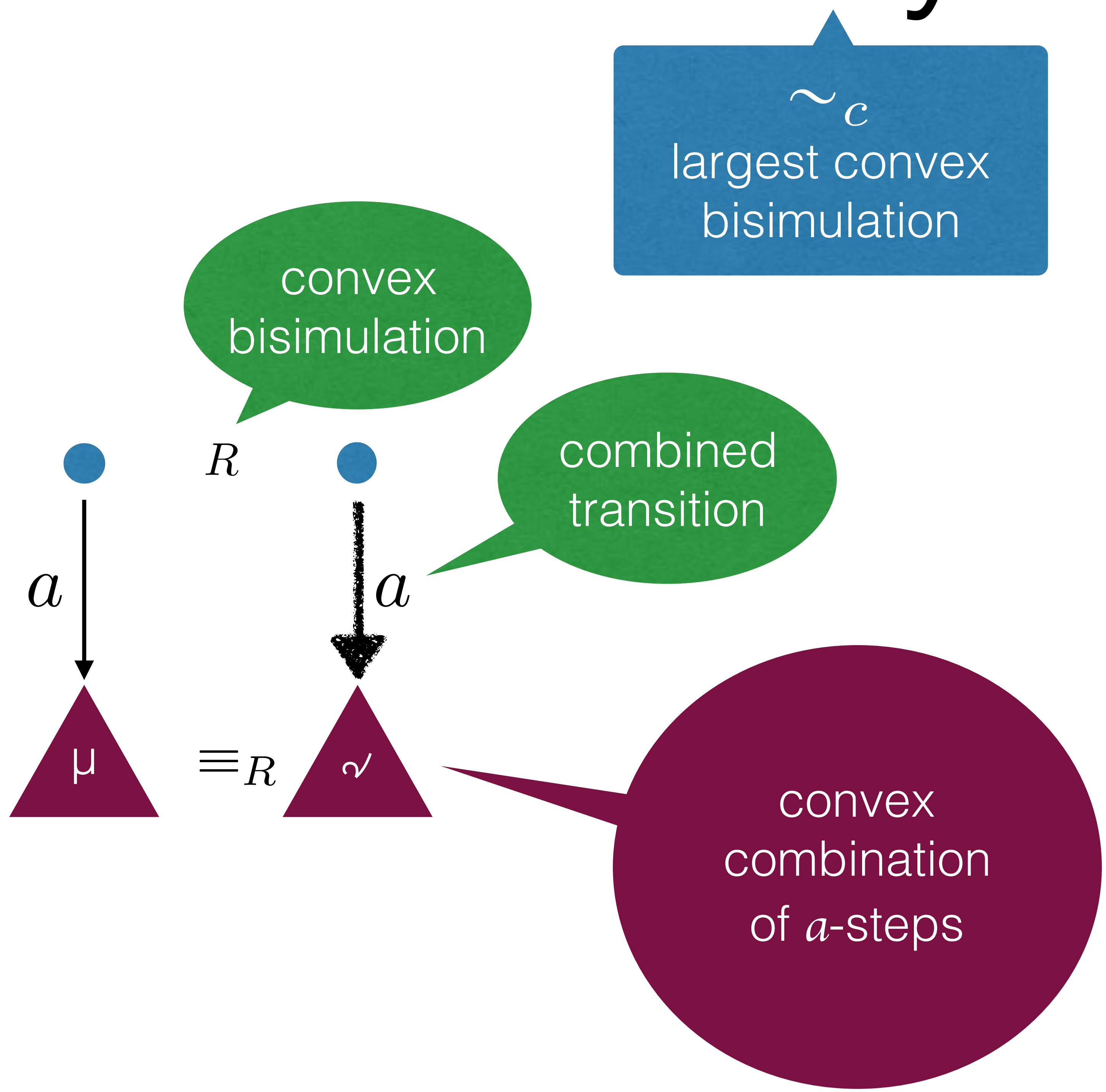

# Distribution bisimilarity

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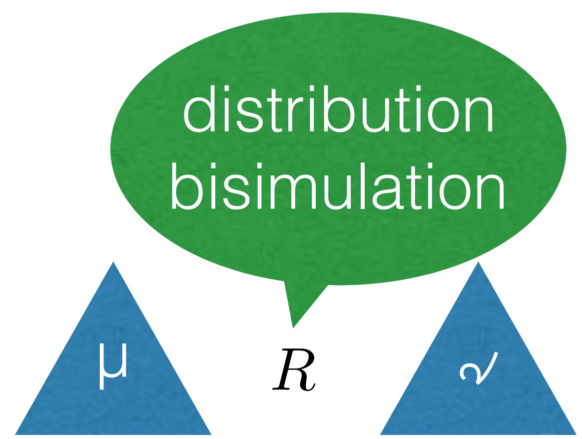
distribution bisimulation

$R$

# Distribution bisimilarity



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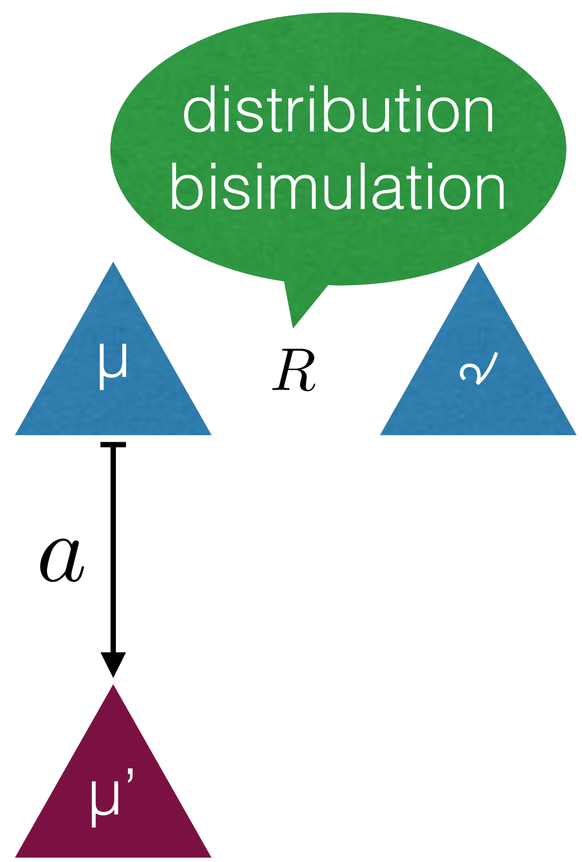

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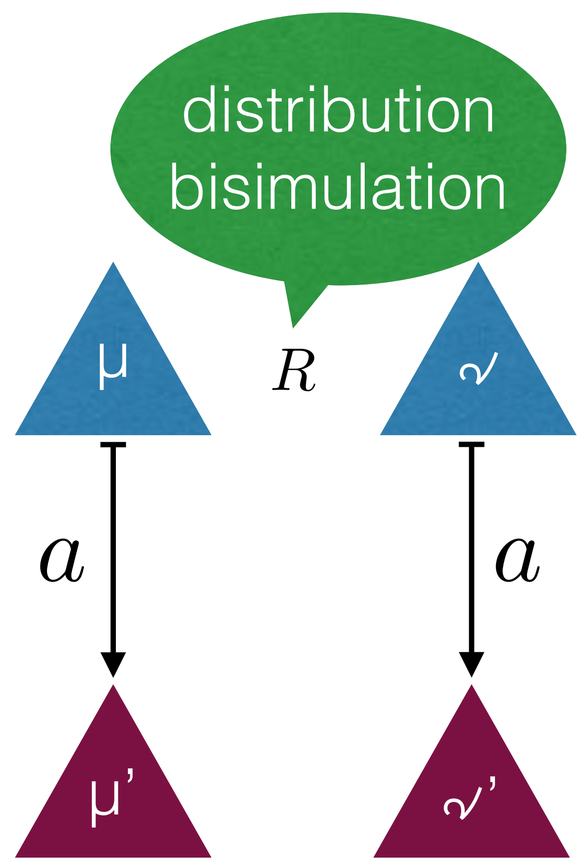

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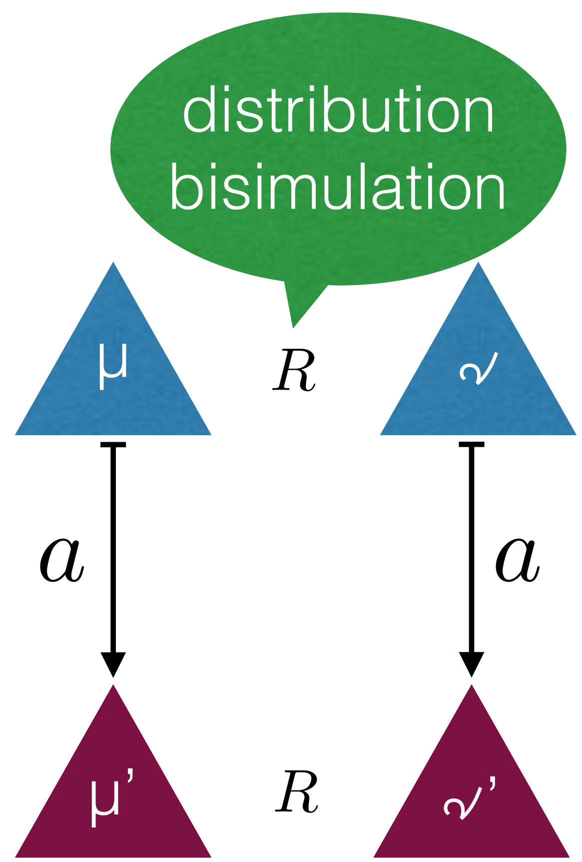

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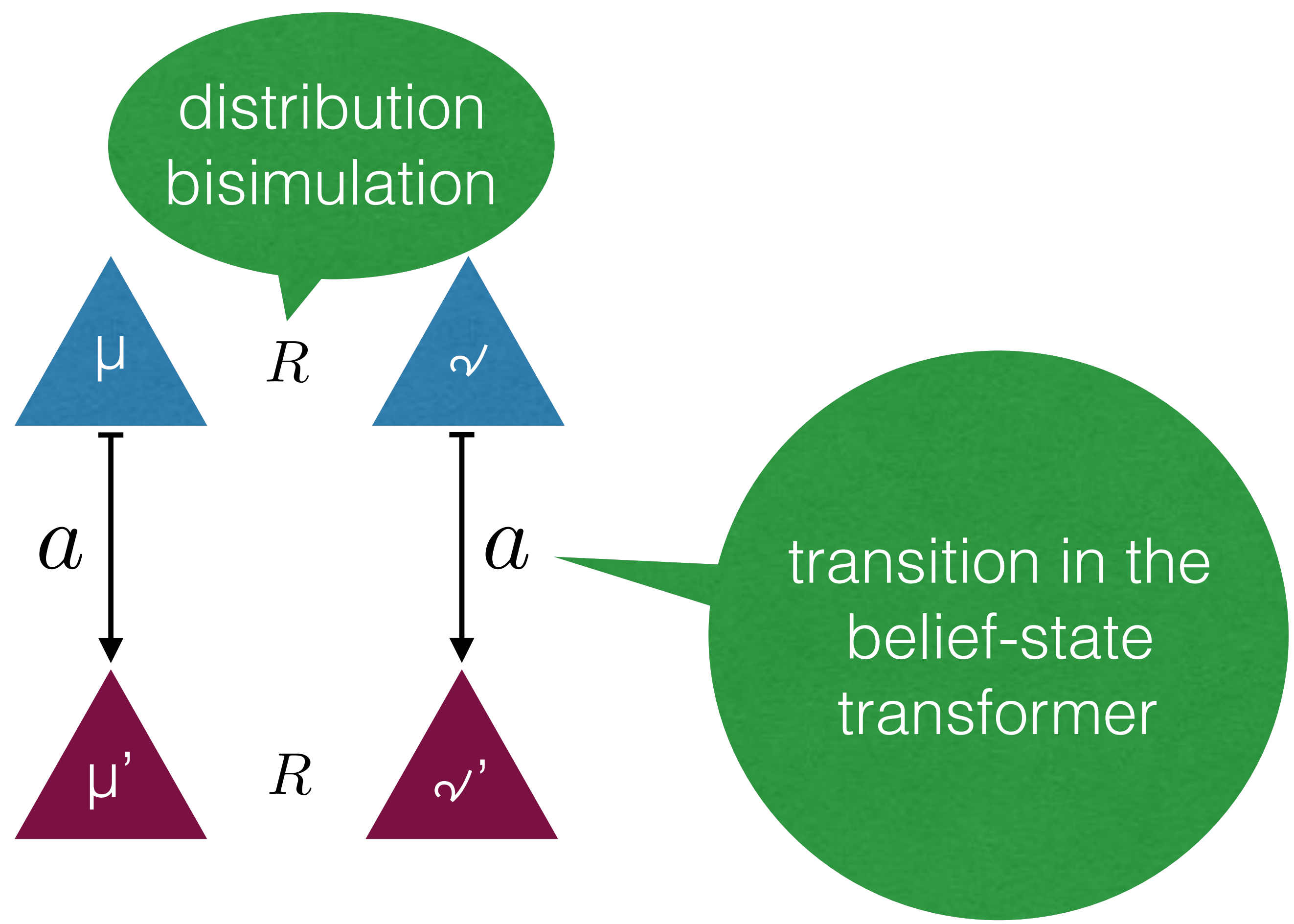

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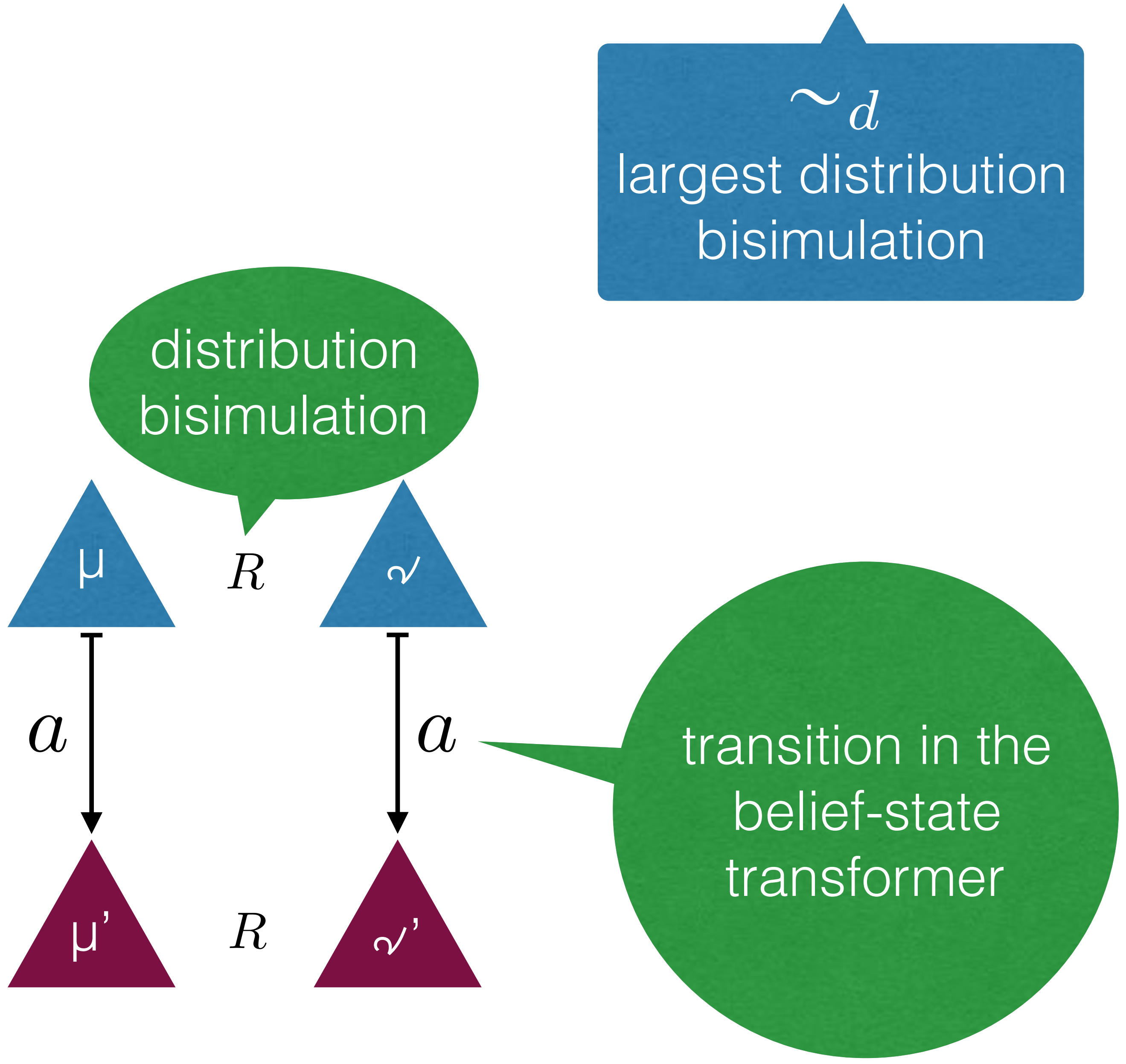
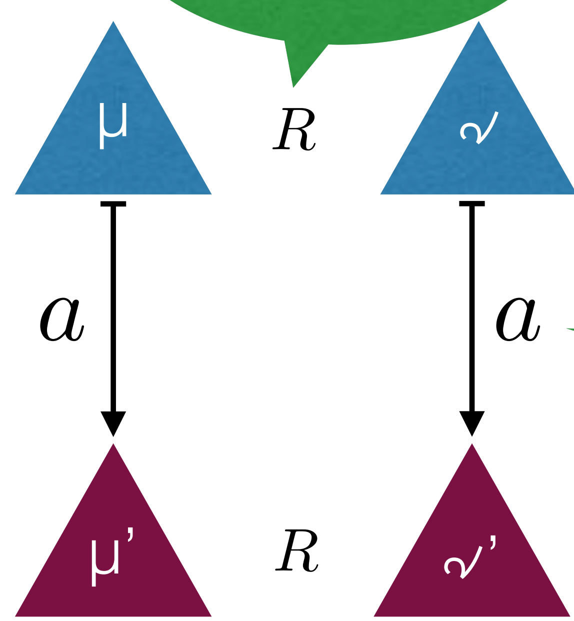


# Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation




transition in the belief-state transformer

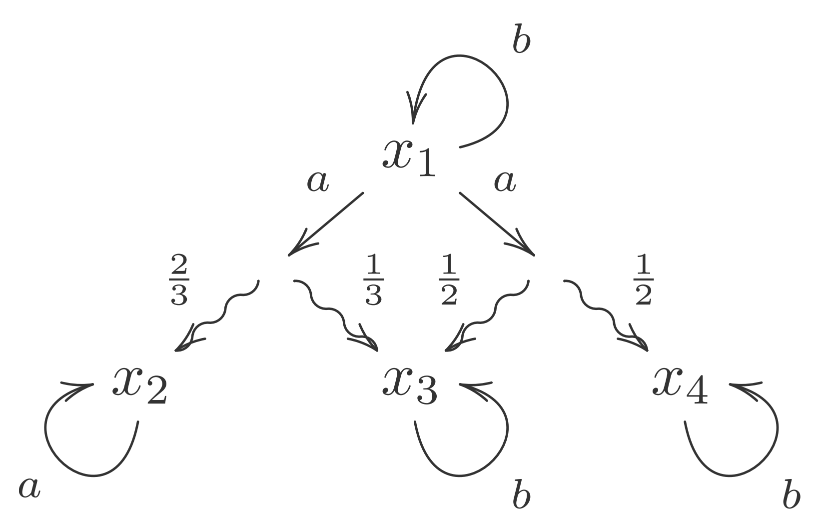
$\sim_d$
is LTS bisimilarity on the belief-state transformer

# Belief-state transformer

foundation ?

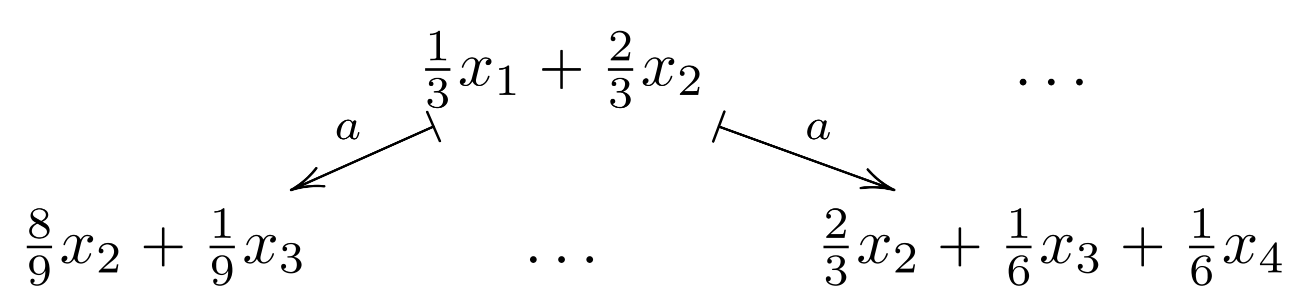
PA

$X \to (\mathcal{PD}(X))^A$




how does it emerge?

what is it?



# PA coalgebraically

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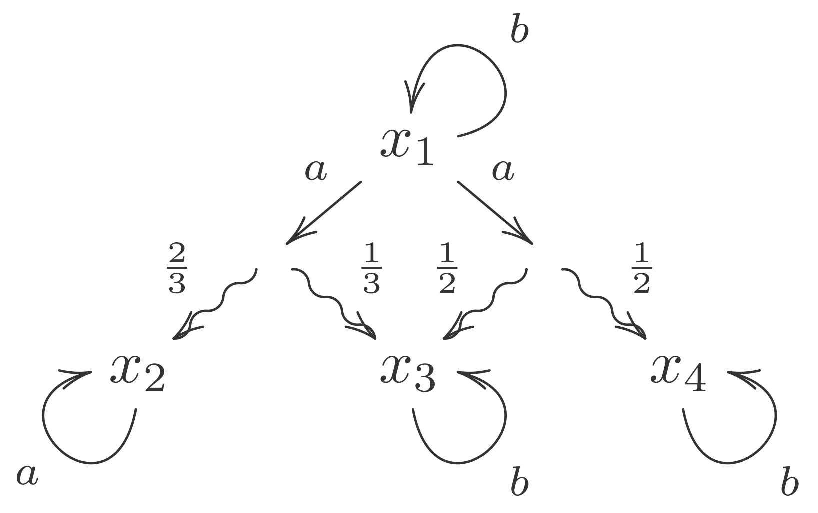

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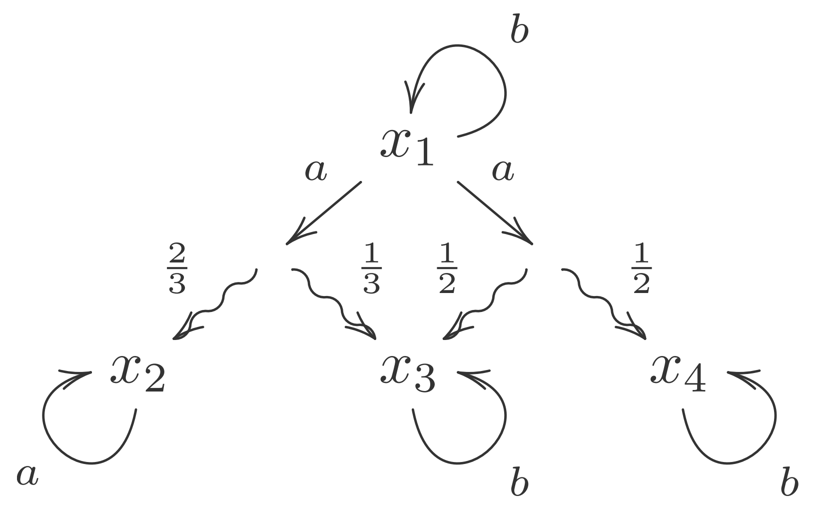
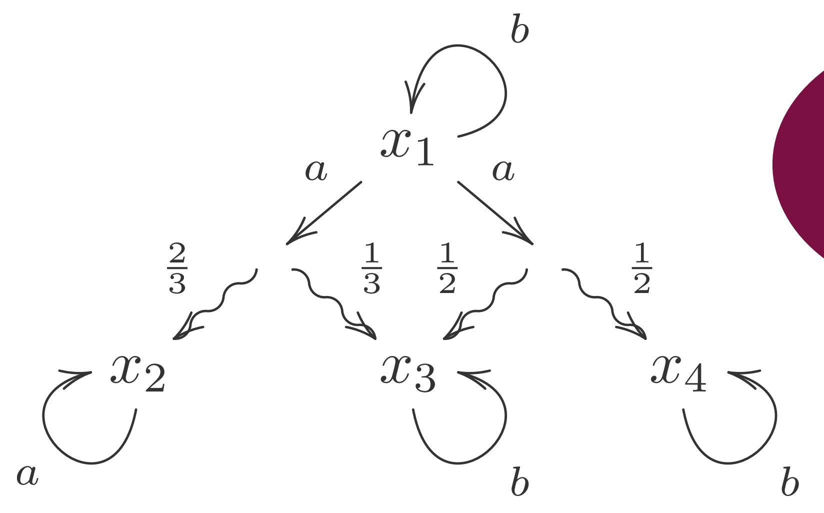



on **Sets**

# PA coalgebraically

$X \to (\mathcal{PD}(X))^A$

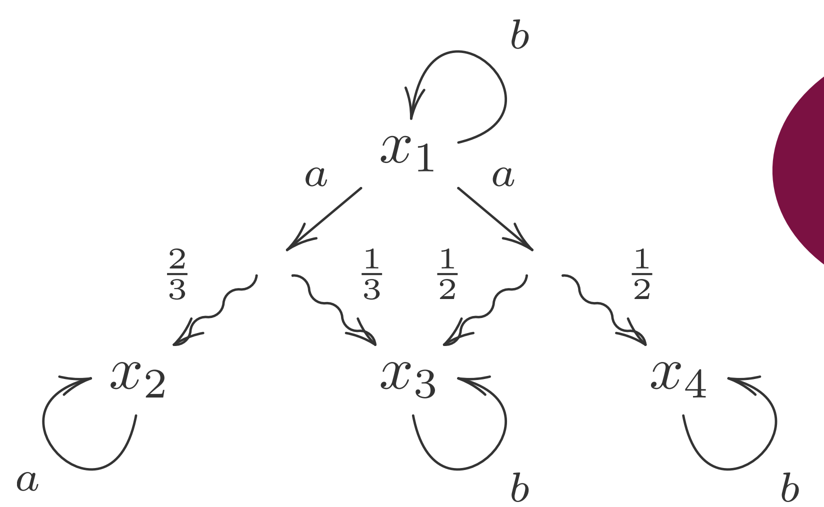



on **Sets**

$\sim \; = \; \approx$

# PA coalgebraically

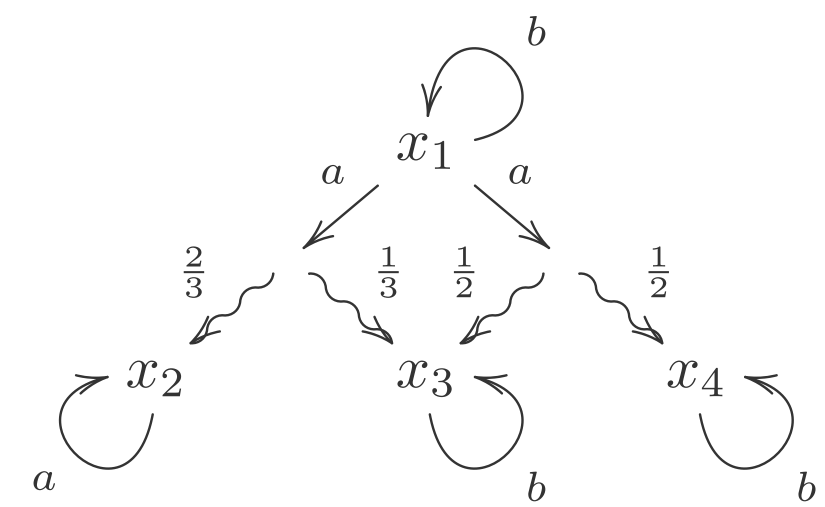
$X \to (\mathcal{PD}(X))^A$




on **Sets**

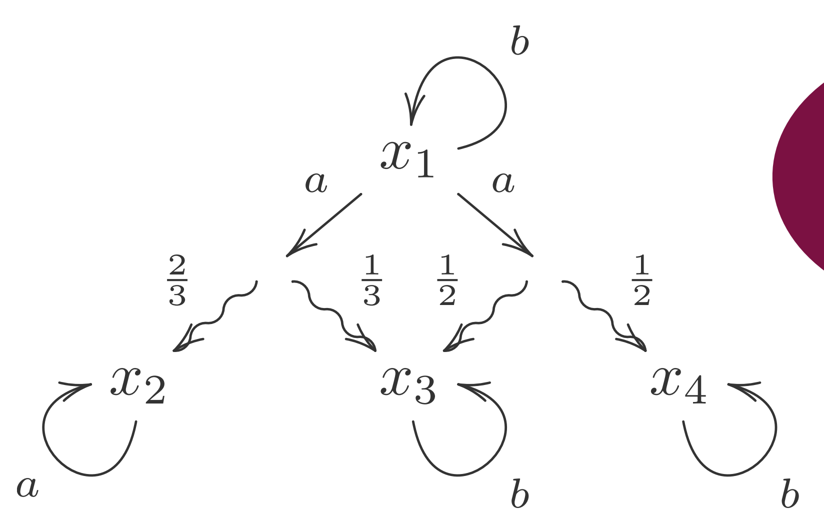
$\sim\ =\ \approx$

$X \to (\mathcal{C}(X))^A$



# PA coalgebraically

$X \to (\mathcal{PD}(X))^A$

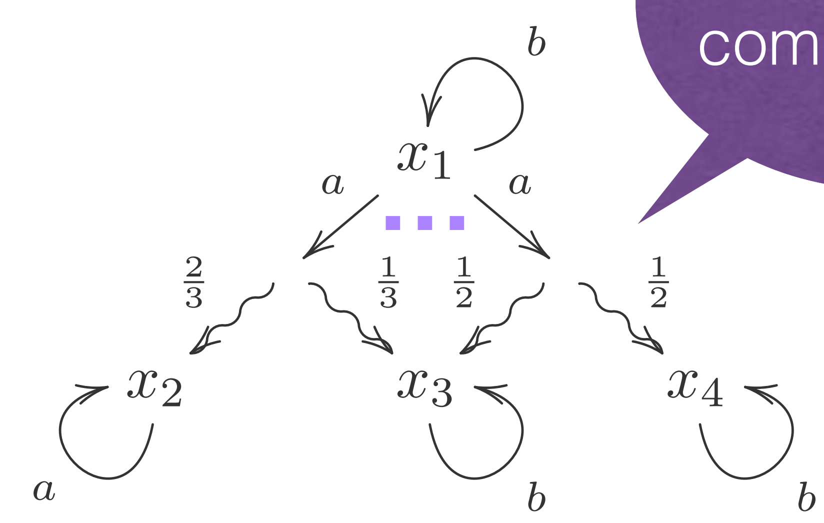



on **Sets**

$\sim \; = \; \approx$

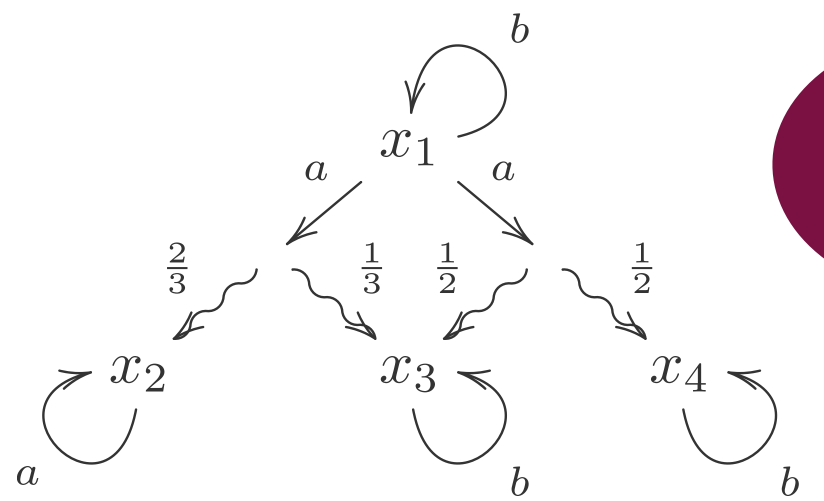
$X \to (\mathcal{C}(X))^A$

and all convex combinations



# PA coalgebraically

$X \to (\mathcal{PD}(X))^A$



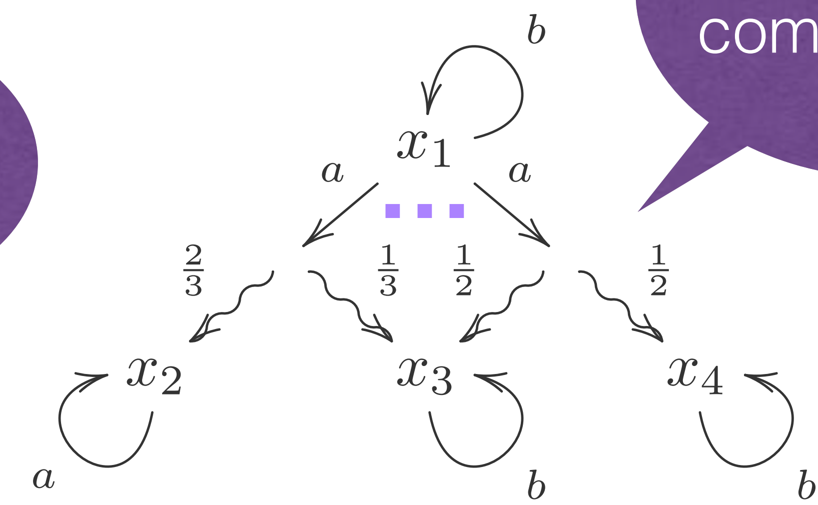

on **Sets**

$\sim \; = \; \approx$

$\sim_c \; = \; \approx$

$X \to (\mathcal{C}(X))^A$

and all convex combinations



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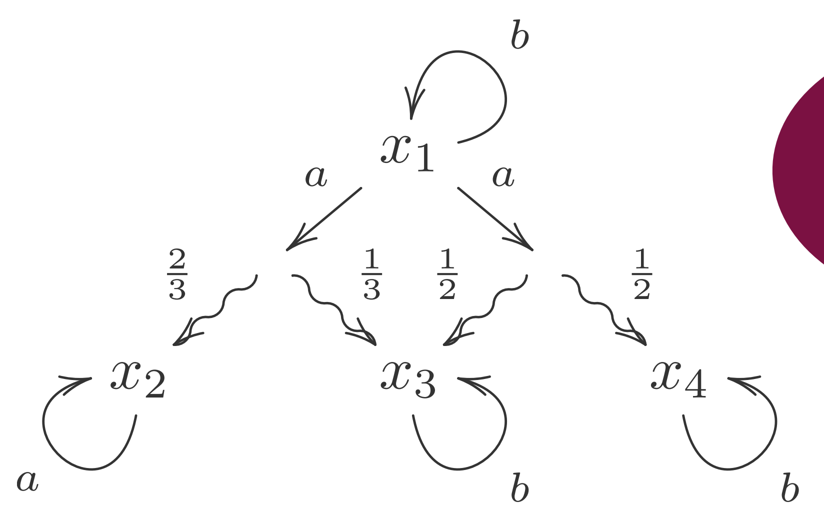
$X \to (\mathcal{PD}(X))^A$




on **Sets**

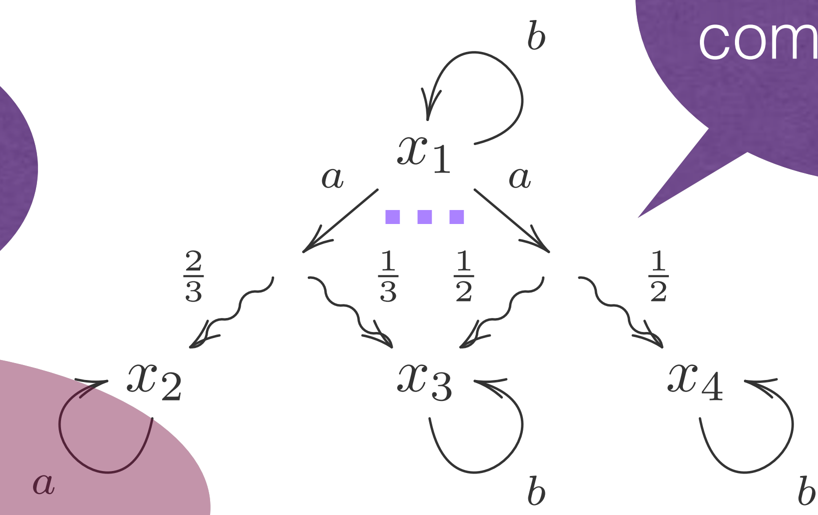
$\sim \,=\, \approx$

$\sim_c \,=\, \approx$

Mio FoSSaCS '14

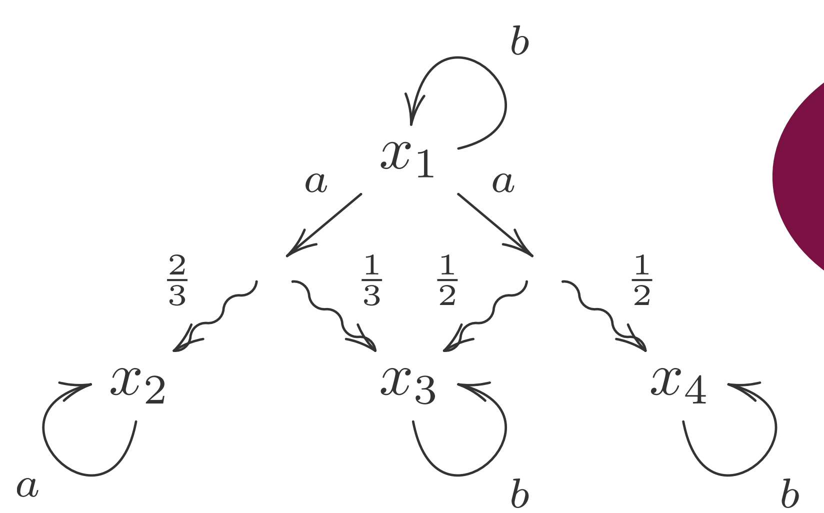
$X \to (\mathcal{C}(X))^A$

and all convex combinations



# PA coalgebraically

$X \to (\mathcal{PD}(X))^A$



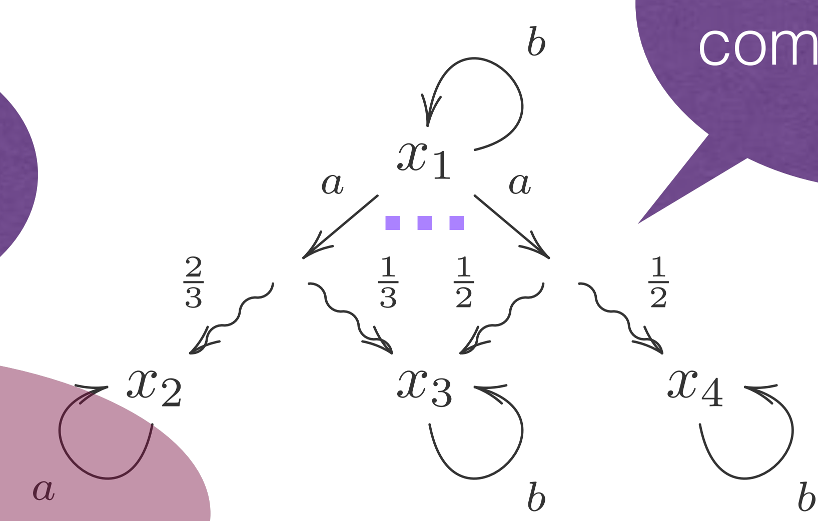

on **Sets**

$\sim \, = \, \approx$

$\sim_c \, = \, \approx$

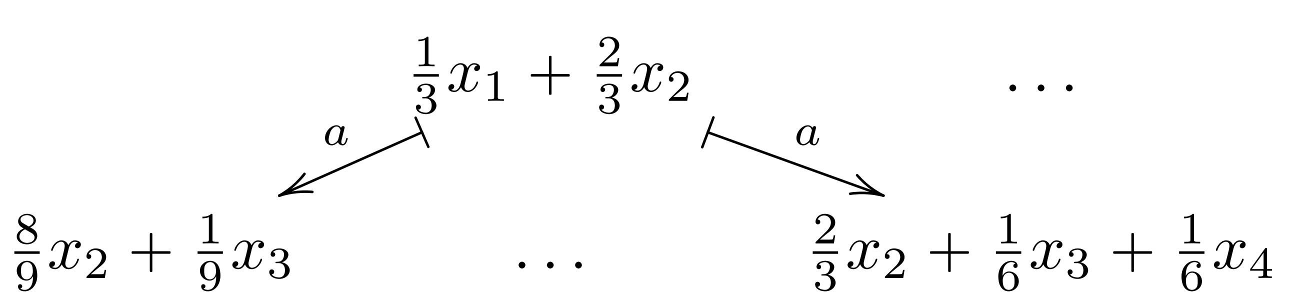
Mio FoSSaCS '14

$X \to (\mathcal{C}(X))^A$



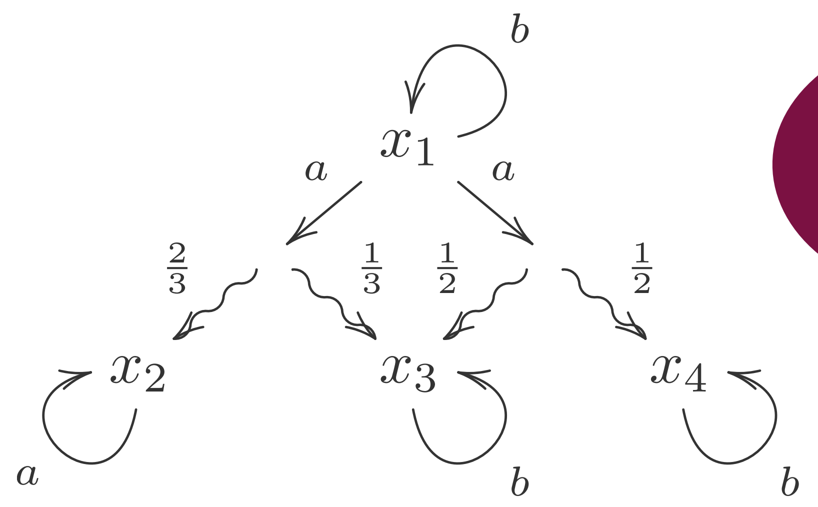

and all convex combinations

$X \to (\mathcal{P}_c(X)+1)^A$



# PA coalgebraically

$X \to (\mathcal{PD}(X))^A$




on **Sets**

$\sim \;=\; \approx$

$\sim_c \;=\; \approx$

Mio FoSSaCS '14

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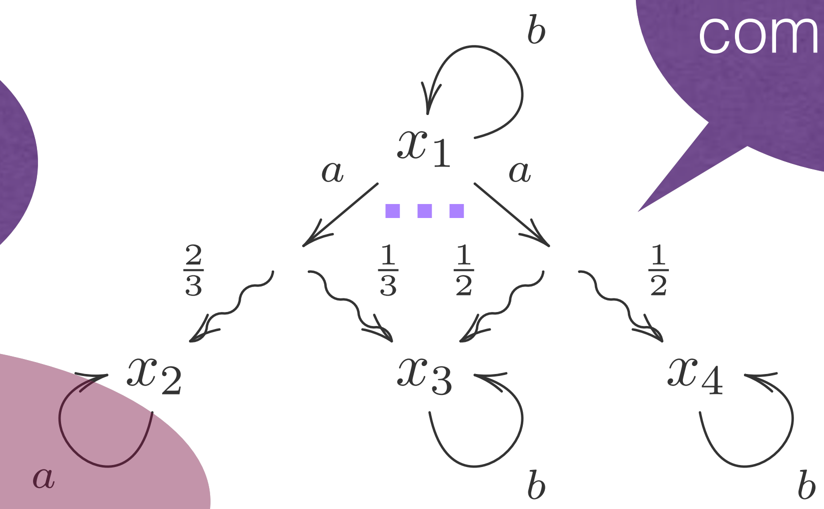


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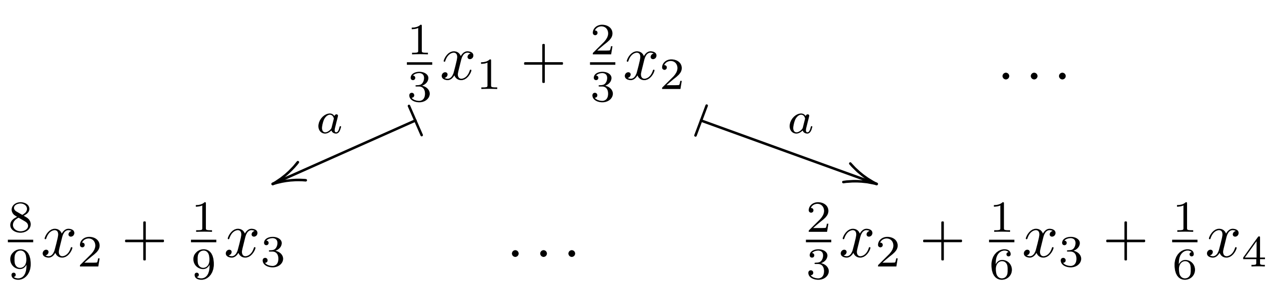


on convex algebras

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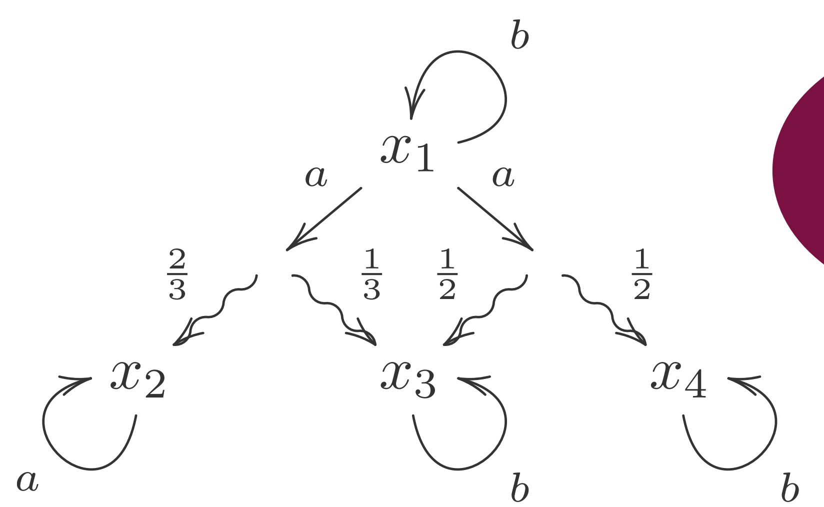


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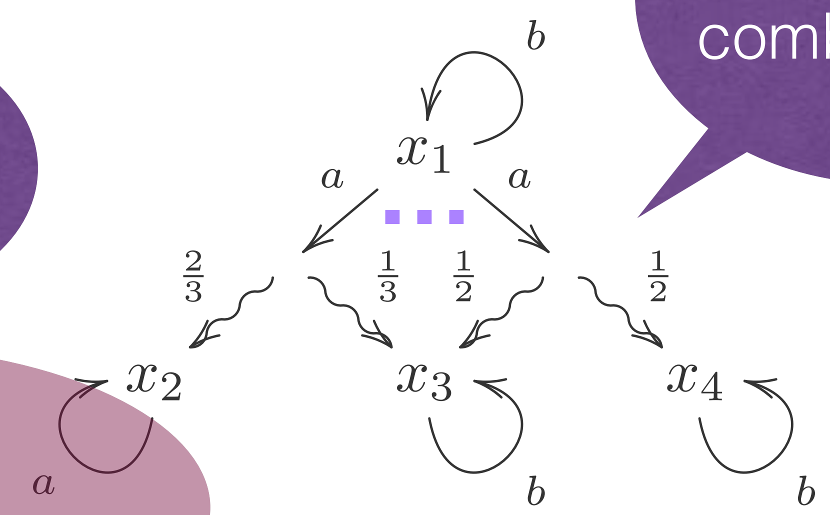


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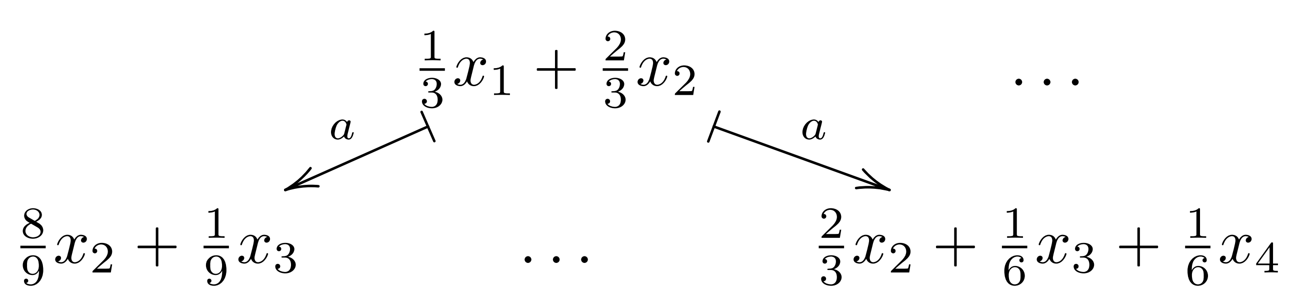


on convex algebras

$\mathcal{EM}(\mathcal{D})$

# PA coalgebraically

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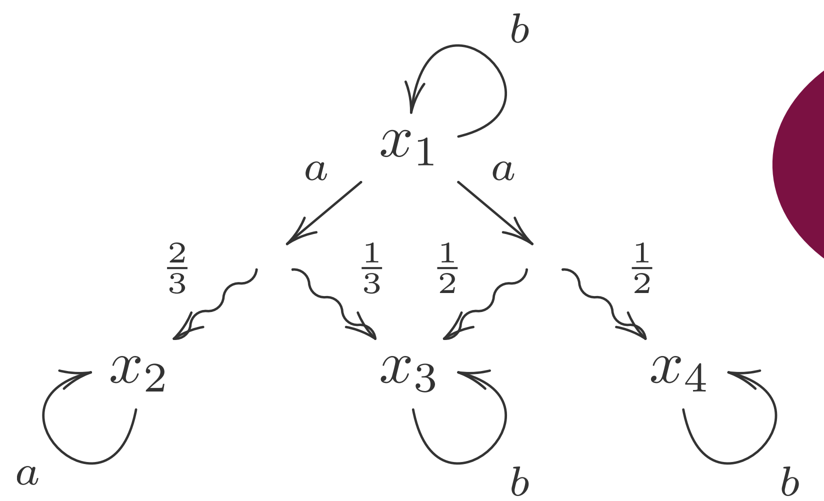


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$X \to (\mathcal{C}(X))^A$

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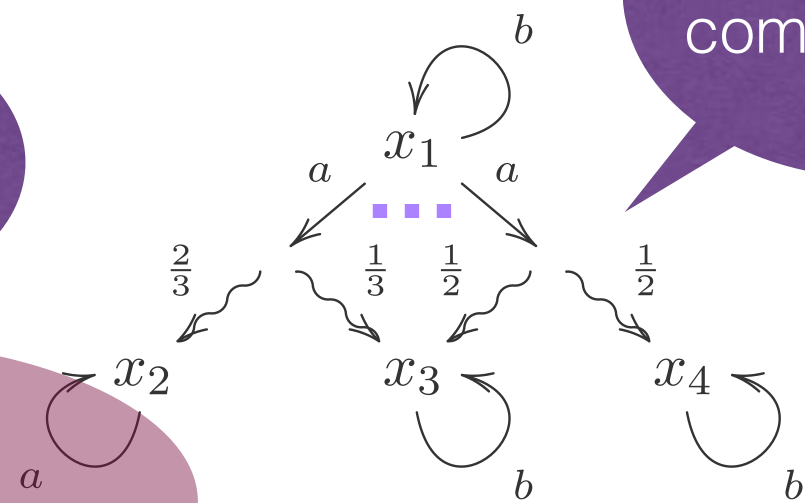


$X \to (\mathcal{P}_c(X)+1)^A$

$\frac{1}{3}x_1 + \frac{2}{3}x_2$ $\xrightarrow{a}$ $\frac{8}{9}x_2 + \frac{1}{9}x_3$ ... $\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4$ ...

on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \, = \, \approx$

# Belief-state transformers

coalgebras on
free convex algebras

# Belief-state transformers

$$\mathbb{D}_X = \begin{array}{c}\mathcal{D}\mathcal{D}X\\ \big\downarrow\mu\\ \mathcal{D}X\end{array}$$

free convex algebra

convex combinations

coalgebras on free convex algebras

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$$\mathbb{D}_X \to (\mathcal{P}_c(\mathbb{D}_X) + 1)^A$$

constant exponent

nonempty convex powerset

termination

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$$pA_1 + (1-p)A_2 = \{pa_1 + (1-p)a_2 \mid a_1 \in A_1, a_2 \in A_2\}$$

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Minkowski sum

# Belief-state transformer

foundation ?

PA

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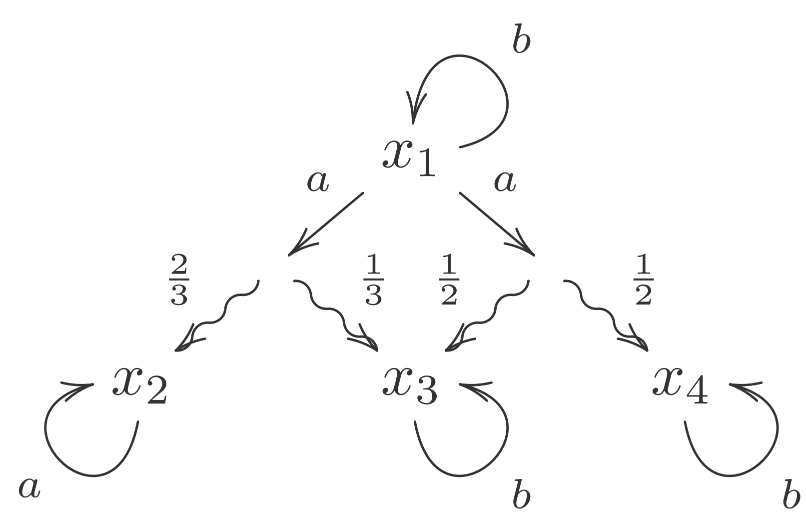
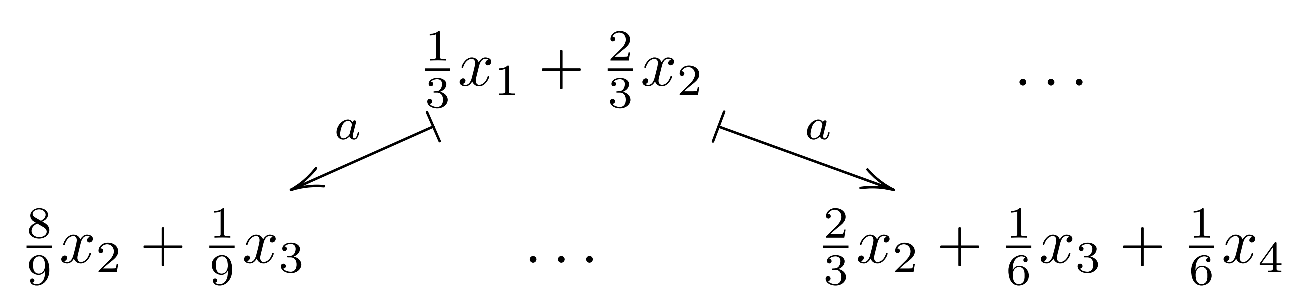



how does it emerge?

what is it?



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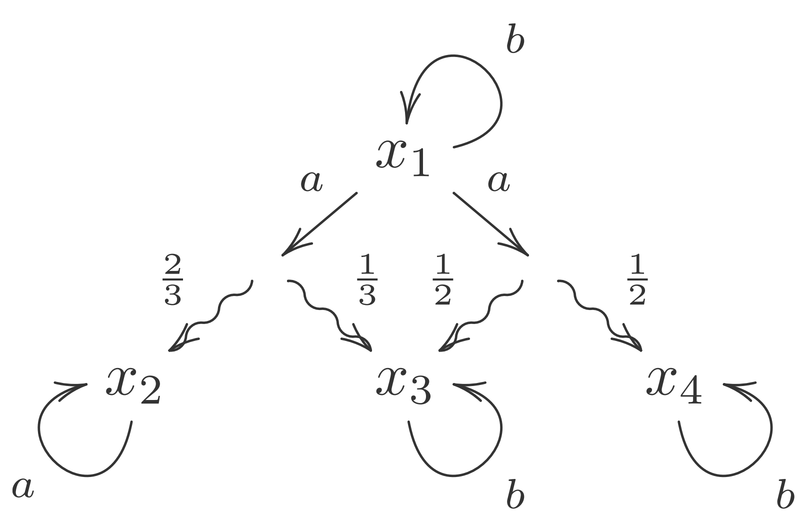
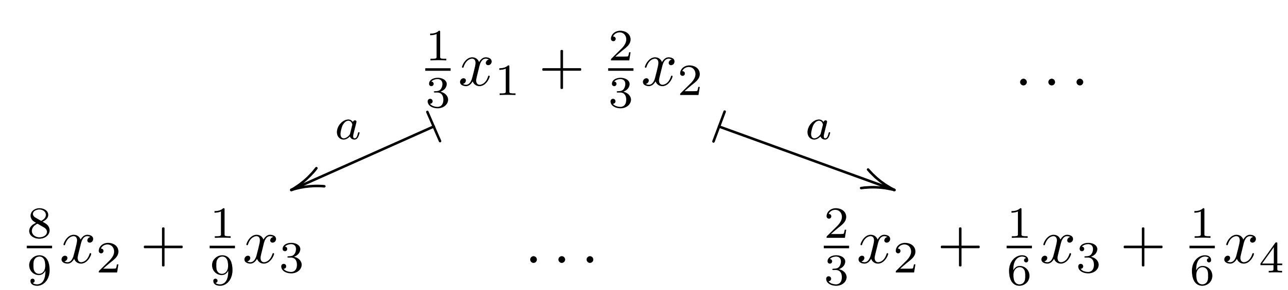



how does it emerge?

coalgebra over free convex algebra



# Determinisations I

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations I

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(F)$$

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

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determinise

forget

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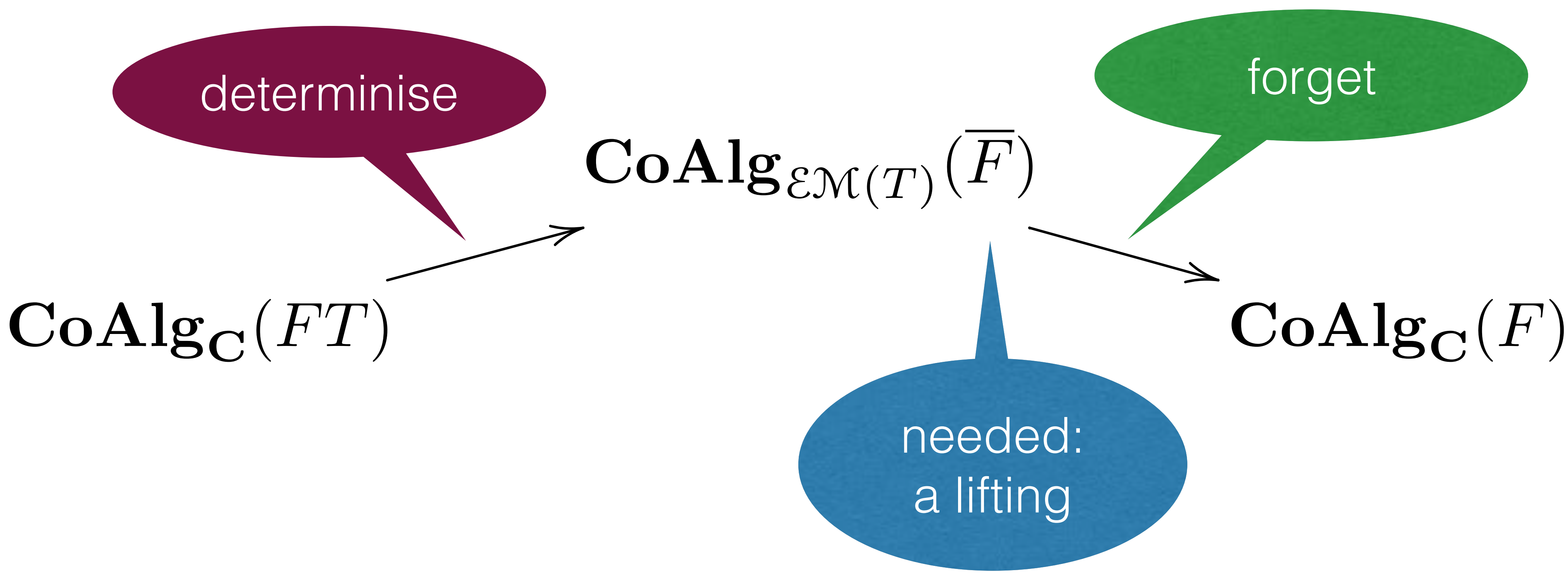


[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations I

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F})$

forget

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(F)$$

needed:
a lifting

works for NFA

not for generative PTS
not for PA / belief-state transformer

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations II

[Silva, S.  MFPS'11]

[Jacobs, Silva, S  JCSS'15]

# Determinisations II

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

[Silva, S. MFPS'11]

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determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

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determinise

$\mathbf{CoAlg}_{\mathbf{C}}(TF) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

forget

[Silva, S. MFPS’11]

[Jacobs, Silva, S JCSS’15]

# Determinisations II

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(TF)$ $\rightarrow$ $\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$ $\rightarrow$ $\mathbf{CoAlg}_{\mathbf{C}}(G)$

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[Silva, S. MFPS'11]

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determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$

forget

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[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

# Determinisations III

# Determinisations III

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$$

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \nearrow \qquad \searrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

# Determinisations III

determinise

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$$

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

# Determinisations III

determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

forget

# Determinisations III

determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

forget

needed:
a quasi lifting
and
a lax lifting

# Determinisations III

determinise

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$$

forget

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

needed:
a quasi lifting
and
a lax lifting

works for PA /belief-state transformer

# Determinisations III

determinise

forget

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$

$\mathbf{CoAlg}_{\mathbf{C}}(F)$ → $\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$ → $\mathbf{CoAlg}_{\mathbf{C}}(G)$

needed:
a quasi lifting
and
a lax lifting

works for PA /belief-state transformer

$(\mathcal{P}_c+1)^A$ on $\mathcal{EM}(\mathcal{D})$

is a quasi lifting and lax lifting of

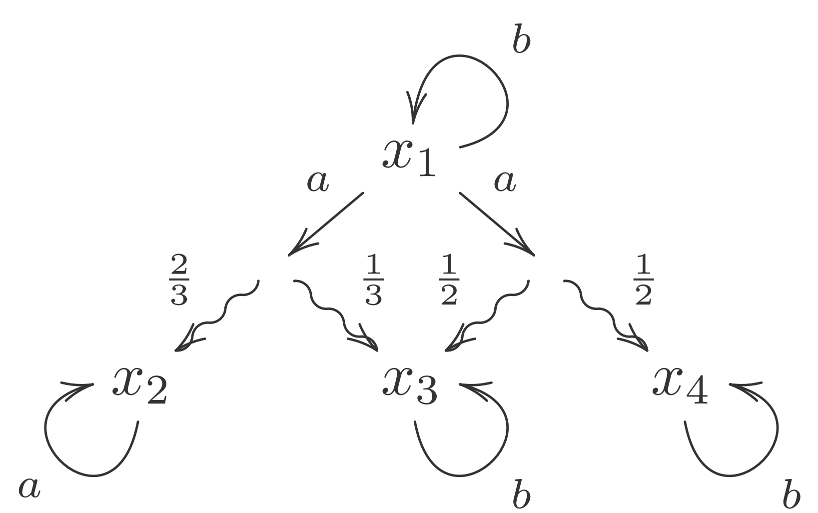
$\mathcal{C}^A$ and $\mathcal{P}^A$ on $\mathbf{Sets}$

# Belief-state transformer

foundation ?

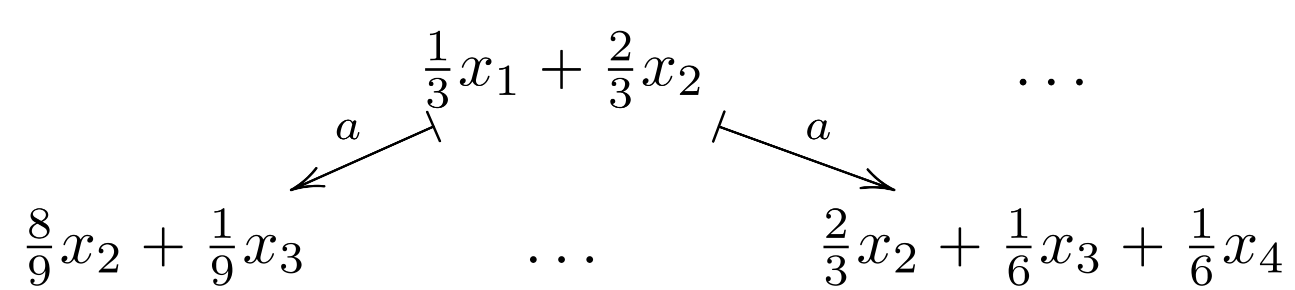
PA

$X \to (\mathcal{PD}(X))^A$




how does it emerge?

coalgebra over free convex algebra



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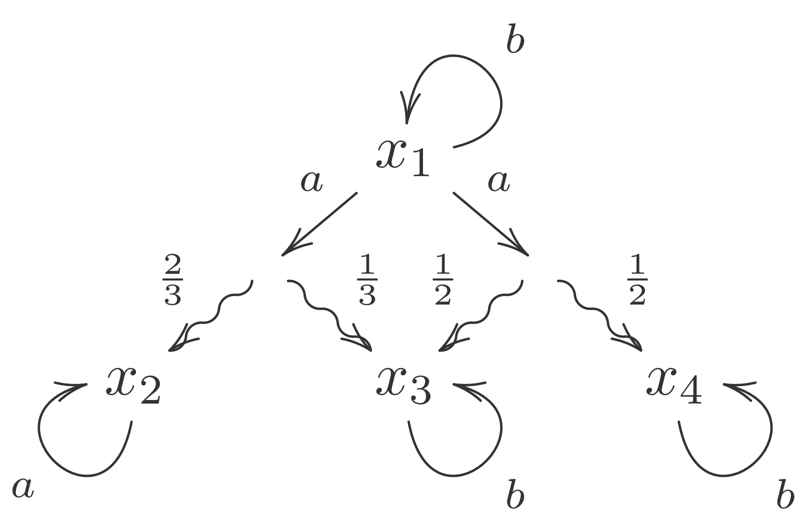
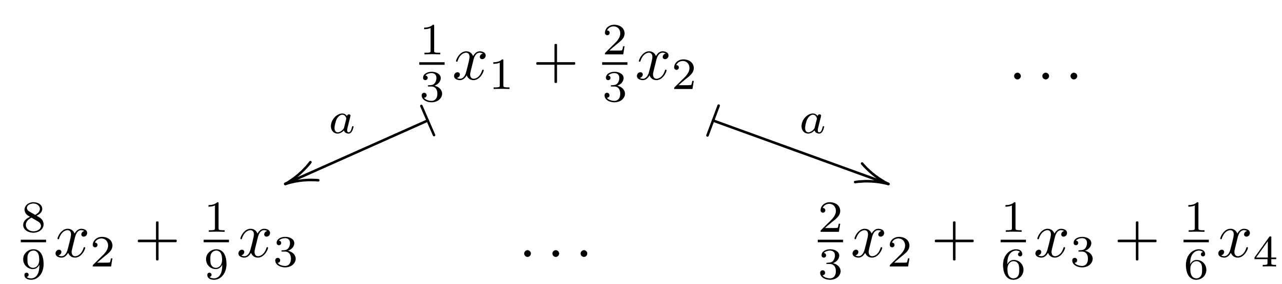



via a generalised[3] determinisation

coalgebra over free convex algebra

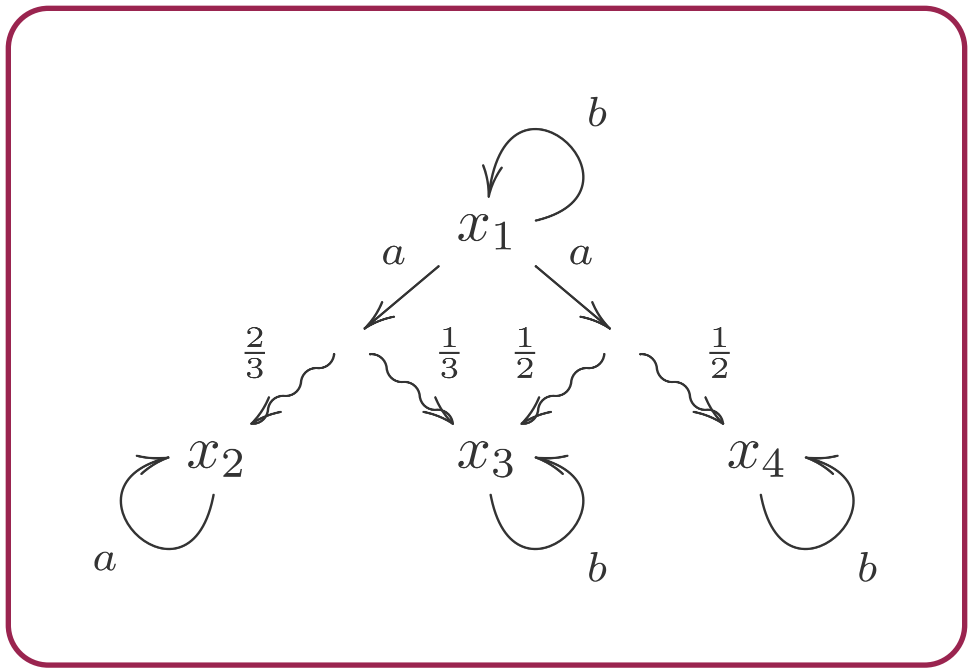


# Belief-state transformer

are natural indeed

PA

$X \to (\mathcal{PD}(X))^A$

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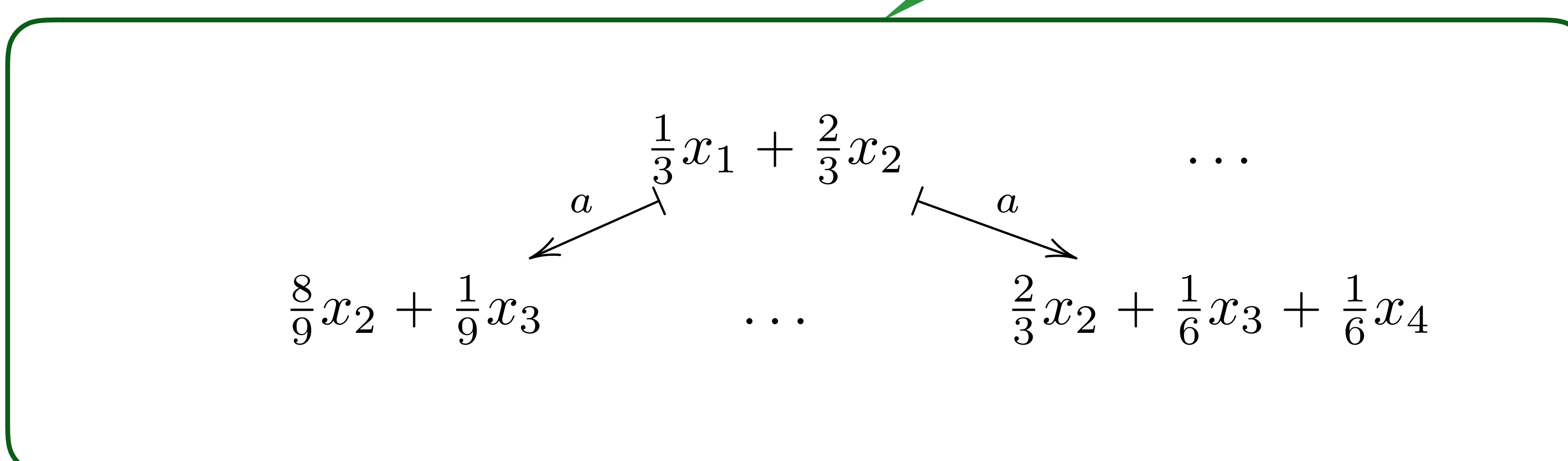

# Coinductive proof method for distribution bisimilarity

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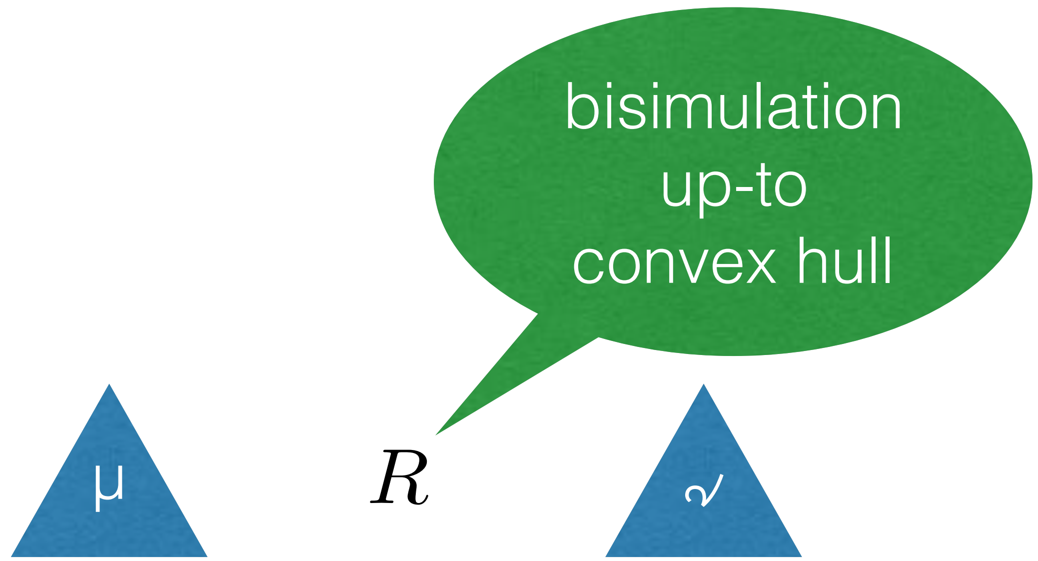
bisimulation up-to convex hull

$R$

# Coinductive proof method for distribution bisimilarity



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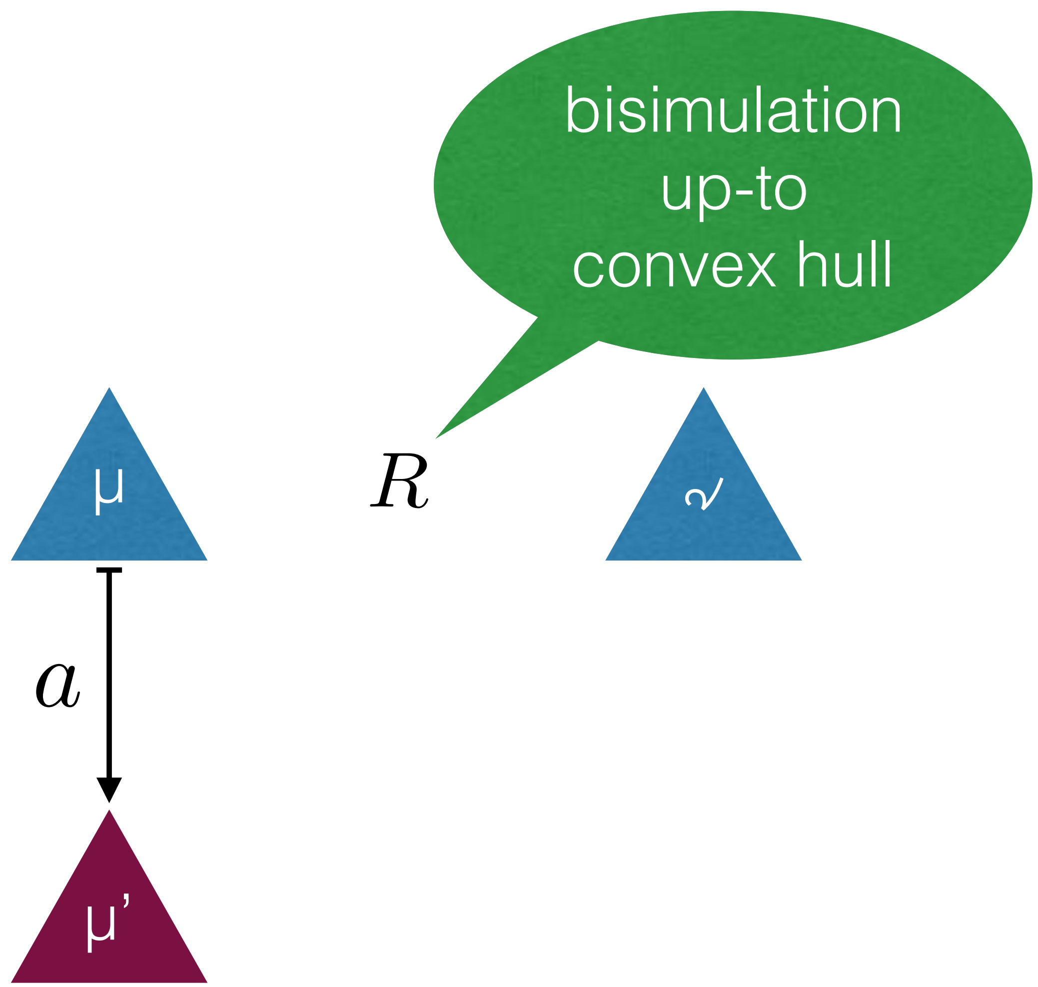

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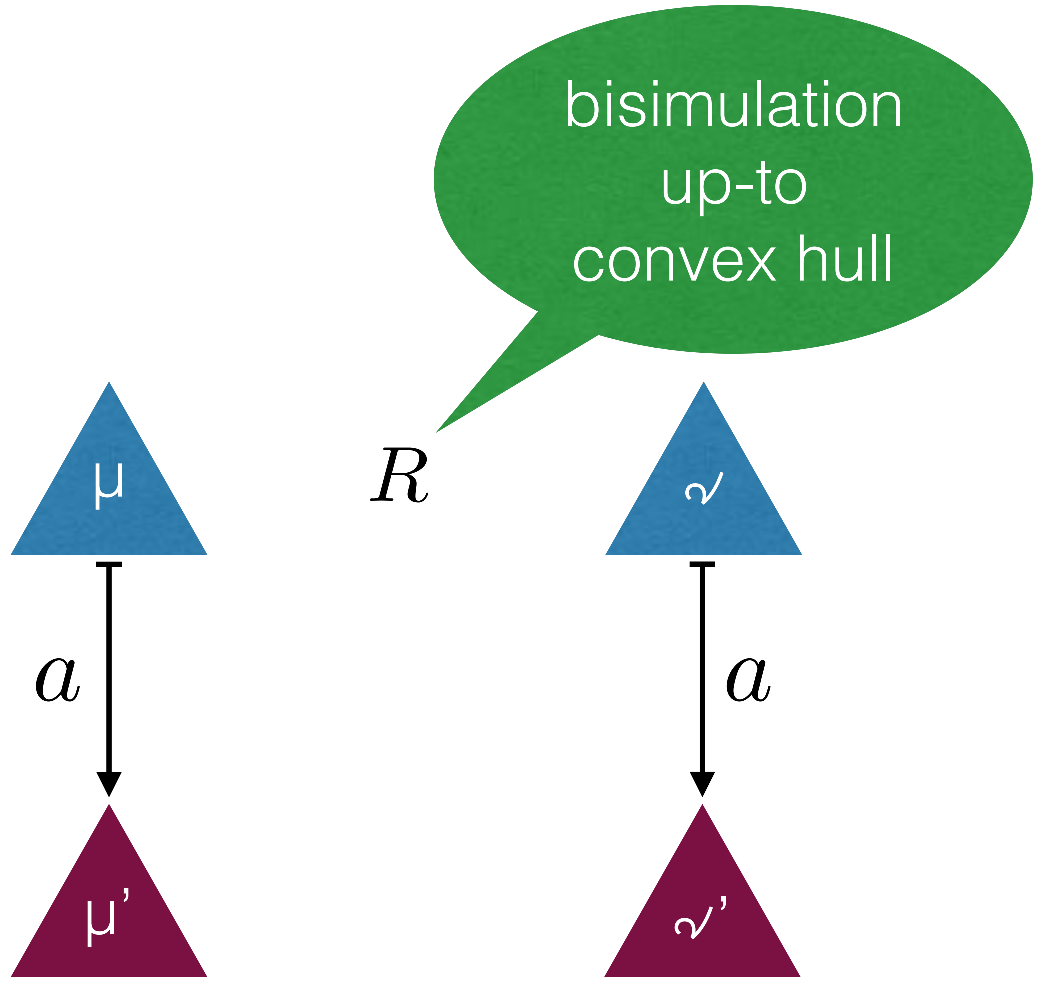

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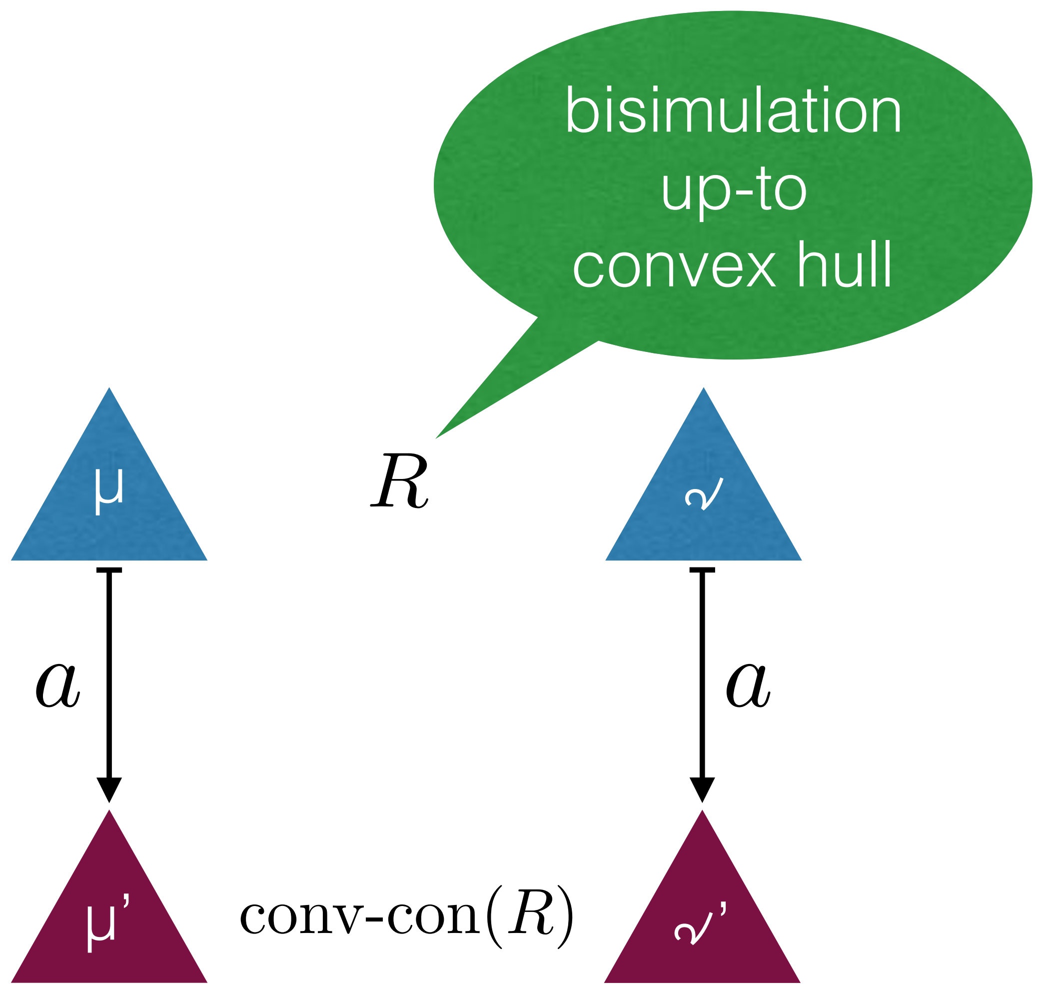

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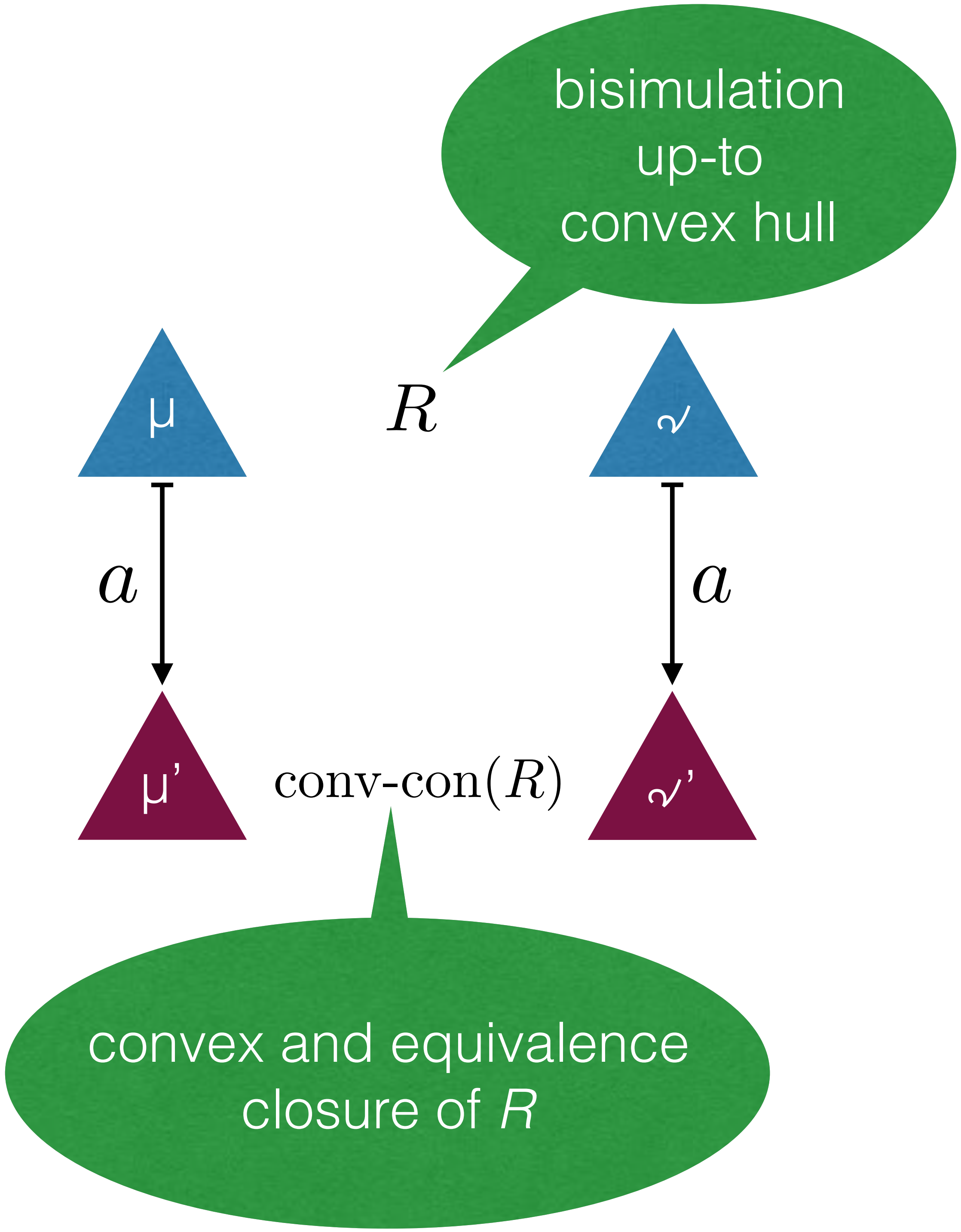

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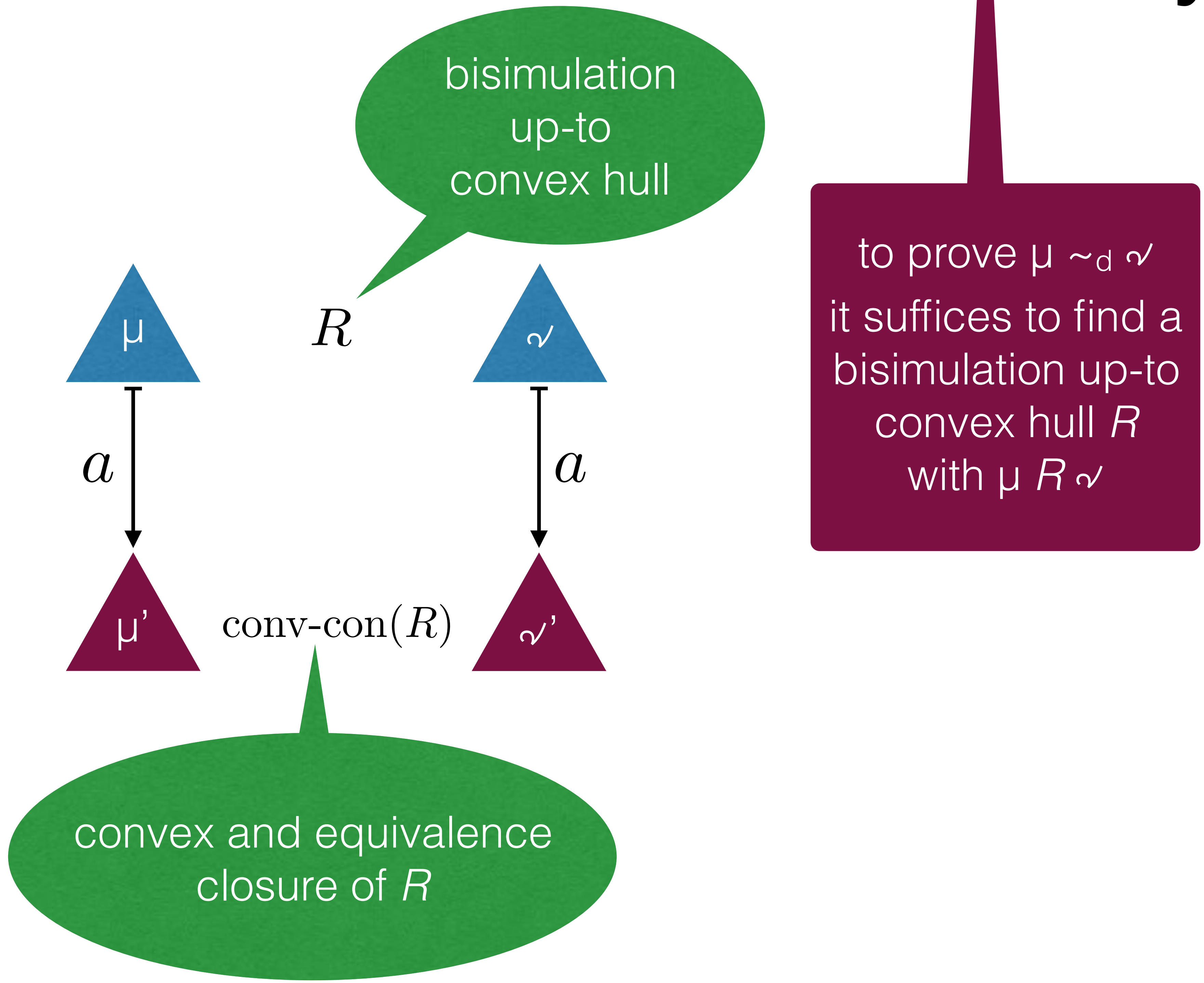

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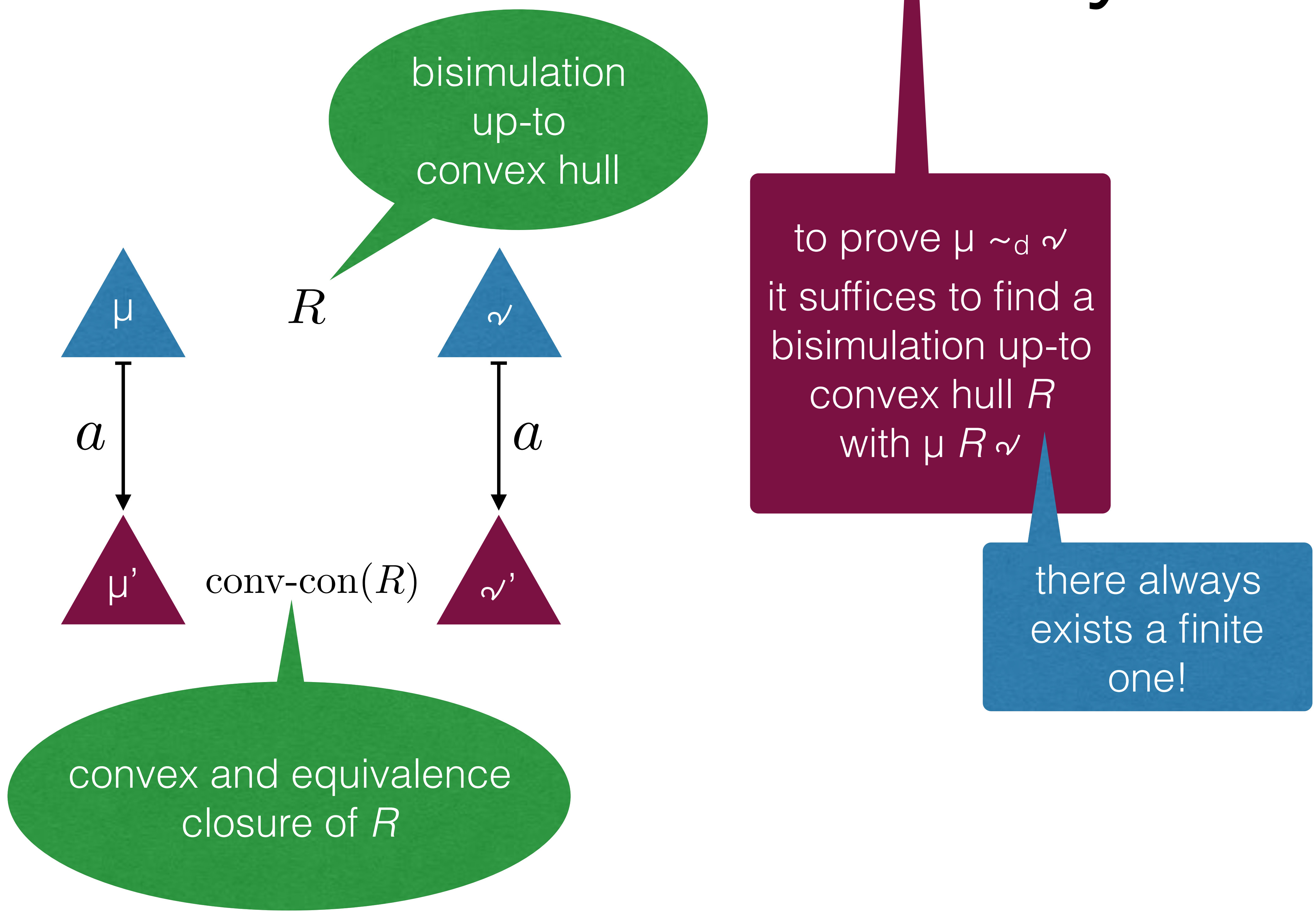
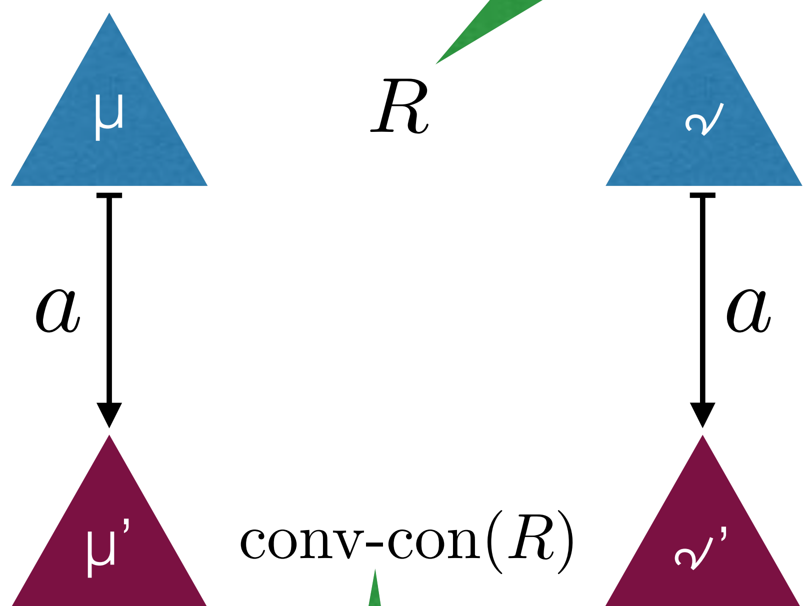


# Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull




convex and equivalence closure of *R*

to prove μ $\sim_d$ ν it suffices to find a bisimulation up-to convex hull *R* with μ *R* ν

there always exists a finite one!

[S., Woracek JPAA'15]

We looked at:

**Part 1.** Modelling probabilistic systems for branching-time semantics

bisimilarity

**Part 2.** Traces, linear-time semantics

trace equivalence

**Part 3.** Belief-state-transformer semantics via convexity

distribution bisimilarity

all with help of coalgebra

Thank You!

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