Exemplaric Expressivity of Modal Logics

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joint work with

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It is about…

- Coalgebras

$$c : X \to TX$$

Behaviour functor !

Generalized transition systems

with coalgebra homomorphisms

$$h : X \to Y$$

$$\begin{tikzcd}
TX \arrow[r, "Th"] & TY \\
X \arrow[u, "c"] \arrow[r, "h"'] & Y \arrow[u, "d"']
\end{tikzcd}$$

- Modal logics

In a studied setting of dual adjunctions

Outline

- Expressivity:

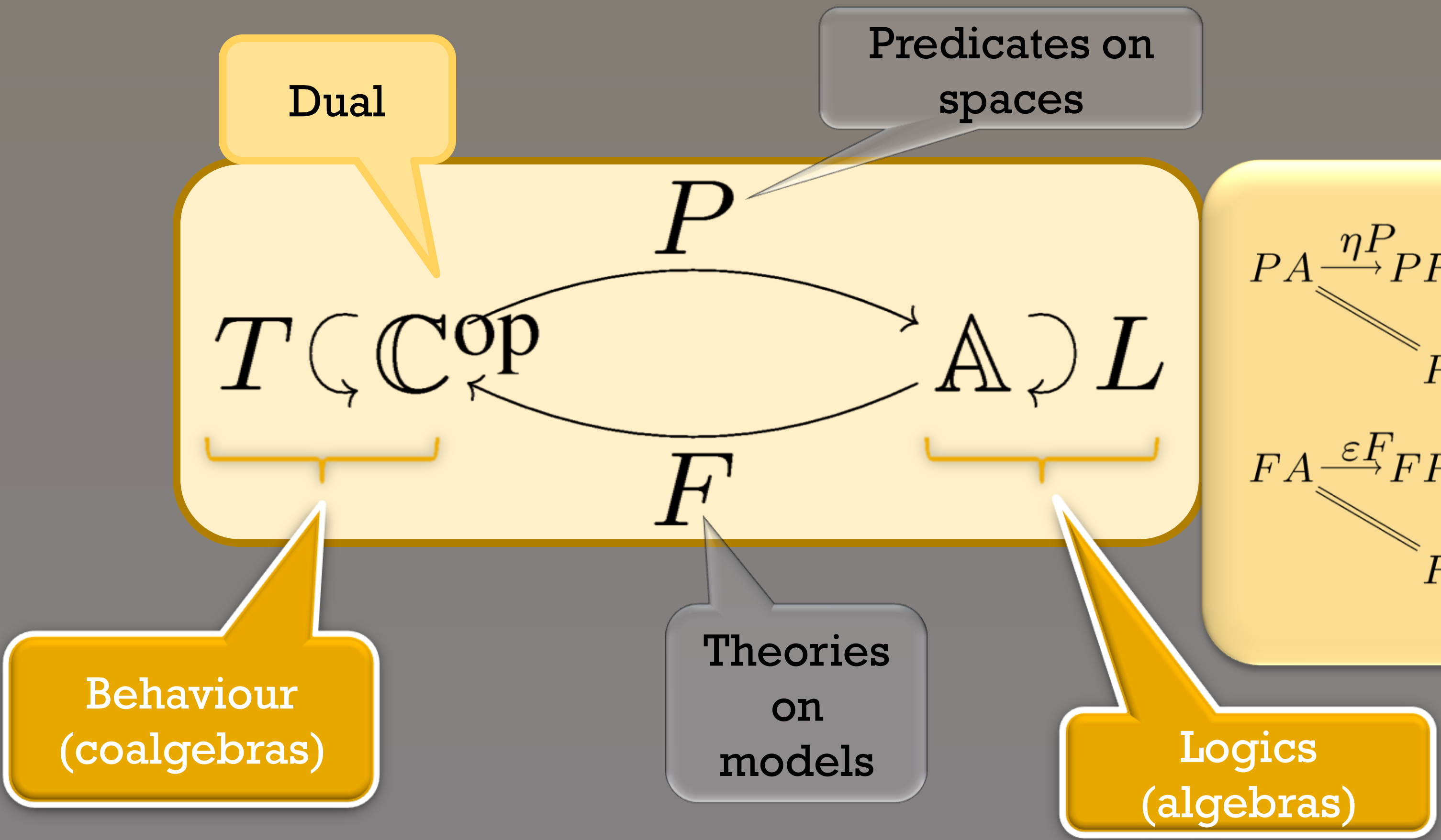
  logical equivalence = behavioral equivalence

- For four examples:
  1. Transition systems
  2. Markov chains
  3. Multitransition systems
  4. Markov processes

Boolean modal logic

Finite conjunctions „valuation“ modal logic

Via dual adjunctions



Dual

Predicates on spaces

$$\begin{tikzcd}
\mathbb{C}^{op} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[loop right, "L"] \arrow[l, bend left, "F"]
\end{tikzcd}$$

Behaviour (coalgebras)

Theories on models

Logics (algebras)

$$\begin{tikzcd}
PA \arrow[r, "\eta P"] \arrow[dr, equal] & PFPA \arrow[d, "P\varepsilon"] \\
& PA
\end{tikzcd}$$

$$\begin{tikzcd}
FA \arrow[r, "\varepsilon F"] \arrow[dr, equal] & FPFA \arrow[d, "F\eta"] \\
& FA
\end{tikzcd}$$

$F \dashv P, \quad \eta_A : A \to PFA$ in $\mathbb{A}$, $\quad \varepsilon_X : X \to FPX$ in $\mathbb{C}$

Logical set-up

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[l, bend left, "F"] \arrow[loop right, "L"] \end{tikzcd}$$

- If L has an initial algebra of formulas

$$L : \mathsf{Form} \xrightarrow{\cong} \mathsf{Form}$$

- A natural transformation

$$\sigma : LP \Rightarrow PT$$

gives interpretations

for arbitrary coalgebra
$$\begin{tikzcd} TX \\ X \arrow[u, "c"'] \end{tikzcd}$$

$$\begin{tikzcd} L(\mathsf{Form}) \arrow[r, dashed, "L[\![-]\!]"] \arrow[dd, "\cong"'] & LPX \arrow[d, "\sigma_X"] \\ & PTX \arrow[d, "Pc"] \\ \mathsf{Form} \arrow[r, dashed, "[\![-]\!]"'] & PX \end{tikzcd}$$

Logical equivalence behavioural equivalence

- The interpretation map yields a theory map

$$\begin{array}{c} [\![-]\!]\colon \mathit{Form} \to PX \\ \hline\hline \mathit{th}\colon X \to F(\mathit{Form}) \end{array}$$

Aim: expressivity

- which defines logical equivalence

$$x \equiv y \quad \Leftrightarrow \quad \mathit{th}(x) = \mathit{th}(y)$$

- behavioural equivalence is given by

$$x \sim y \quad \Leftrightarrow \quad h_1(x) = h_2(y)$$

for some coalgebra homomorphisms h_1 and h_2

$$\begin{tikzcd} \mathbb{C}^{\mathrm{op}} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[l, bend left, "F"] \arrow[loop right, "L"] \end{tikzcd}$$

Expressivity

- Bijective correspondence between

$$\begin{tikzcd}[column sep=huge] \mathbb{C}^{\mathrm{op}} \arrow[r, bend left, "LP", ""{name=U, below}] \arrow[r, bend right, "PT"', ""{name=D}] & \mathbb{A} \arrow[Rightarrow, from=U, to=D, "\sigma"] \end{tikzcd}$$

and

$$\begin{tikzcd}[column sep=huge] \mathbb{A} \arrow[r, bend left, "FL", ""{name=U, below}] \arrow[r, bend right, "TF"', ""{name=D}] & \mathbb{C}^{\mathrm{op}} \arrow[Rightarrow, from=U, to=D, "\tau"] \end{tikzcd}$$

If ☀ and the transpose of the interpretation is componentwise abstract mono, then expressivity.

Factorisation system on $\mathbb{C}$

$(\mathcal{M}, \mathcal{E}), \ \mathcal{M} \subseteq \text{Monos}, \ \mathcal{E} \subseteq \text{Epis}$

with diagonal fill-in

T preserves $\mathcal{M}$

Sets vs. Boolean algebras

unit $\eta : A \to \mathcal{P}\mathcal{F}_{\mathcal{U}}(A)$

$\eta(a) = \{\alpha \in \mathcal{F}_{\mathcal{U}}(A) \mid a \in \alpha\}$

contravariant powerset

$$\begin{tikzcd}
\mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{BA} \arrow[l, bend left, "\mathcal{F}_{\mathcal{U}}"]
\end{tikzcd}$$

Boolean algebras

standard correspondence

$$\frac{f : X \longrightarrow \mathcal{F}_{\mathcal{U}}(A) \quad \text{in } \mathbf{Sets}}{g : A \longrightarrow \mathcal{P}(X) \quad \text{in } \mathbf{BA}}$$

via

$$\frac{a \in f(x)}{x \in g(a)}$$

ultrafilters

upsets $\alpha \subseteq A$, $\top \in \alpha$

$a, b \in \alpha \Rightarrow a \wedge b \in \alpha$

$\forall a \in A. a \in \alpha$ xor $\neg a \in \alpha$

Sets vs. meet semilattices

unit $\eta : A \to \mathcal{P}\mathcal{F}(A)$

$\eta(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$

$$\begin{tikzcd} \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \textsf{\textbf{MSL}} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

contravariant powerset

meet semilattices

filters

upsets $\alpha \subseteq A$, $\top \in \alpha$

$a, b \in \alpha \Rightarrow a \wedge b \in \alpha$

“the same” correspondence

$$\begin{array}{c} f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \mathbf{Sets} \\ \hline\hline g : A \longrightarrow \mathcal{P}(X) \quad \text{in } \textbf{MSL} \end{array}$$

via

$$\begin{array}{c} a \in f(x) \\ \hline\hline x \in g(a) \end{array}$$

Measure spaces vs. meet semilattices

unit $\eta : A \to \mathcal{SF}(A)$
$\eta(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$

maps a measure space to its σ-algebra

$$\begin{tikzcd}
\textbf{Meas}^{\mathrm{op}} \arrow[rr, bend left, "\mathcal{S}"] & & \textbf{MSL} \arrow[ll, bend left, "\mathcal{F}"]
\end{tikzcd}$$

σ-algebra: "measurable" subsets closed under empty, complement, countable union

"the same" correspondence

$$\begin{array}{c}
f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \textbf{Meas} \\
\hline\hline
g : A \longrightarrow \mathcal{S}(X) \quad \text{in } \textbf{MSL}
\end{array}$$

via

$$\begin{array}{c}
a \in f(x) \\
\hline\hline
x \in g(a)
\end{array}$$

by

measure spaces

objects: pairs $(X, \mathcal{S}(X))$
arrows: measurable functions

$\to Y$ with $f^{-1}(\mathcal{S}(Y)) \subseteq \mathcal{S}(X)$

Behaviour via coalgebras

$$\begin{tikzcd} \mathbb{C}^{op} \arrow[loop left, "T"] \arrow[r, bend left, "P"] & \mathbb{A} \arrow[loop right, "L"] \arrow[l, bend left, "F"] \end{tikzcd}$$

- Transition systems

$\mathcal{P}_f$-coalgebras in **Sets**

$$\begin{tikzcd} \mathbf{Sets}^{op} \arrow[loop left, "\mathcal{P}_f"] \arrow[r, bend left, "\mathcal{P}"] & \textbf{BA} \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd}$$

- Markov chains/Multitransition systems

$\mathcal{M}_f$-coalgebras in **Sets**

$$\begin{tikzcd} \mathbf{Sets}^{op} \arrow[loop left, "\mathcal{M}_f"] \arrow[r, bend left, "\mathcal{P}"] & \textbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

$$\mathcal{M}_f(X) = \{\varphi : X \to \mathbb{N} \mid \text{supp}(\varphi) \text{ finite}\}$$

Giry monad

- Markov processes

$\mathcal{G}$-coalgebras in **Meas**

$$\begin{tikzcd} \mathbf{Meas}^{op} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "\mathcal{P}"] & \textbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \end{tikzcd}$$

What do they have in common?

They are instances of the same functor

Given $(M, +, 0, \leq)$ with $x \leq x + y$, $O \subseteq M$ - downward closed

$$V_O(X) = \{\varphi : X \to O \mid \text{supp}(\varphi) \text{ is finite}\} \qquad V_O(f)(\varphi)(y) = (\varphi \circ f^{-1})(\{y\})$$

$$\mathcal{P}_f = V_O \quad \text{for} \quad M = (\{0,1\}, \vee, 0, \leq) \ O = M$$

$$\mathcal{M}_f = V_O \quad \text{for} \quad M = (\mathbb{N}, +, 0, \leq) \quad O = M$$

$$\mathcal{D}_f = V_O \quad \text{for} \quad M = (\mathbb{R}^{\geq 0}, +, 0, \leq) \quad O = [0,1]$$

Not cancellative

The Giry monad

$$(X, \mathcal{S}X) \mapsto (\mathcal{G}X, \mathcal{S}\mathcal{G}X)$$

countable
union of
pairwise
disjoint

$$\mathcal{G}X = \{\varphi : \mathcal{S}X \to [0,1] \mid \varphi(\quad = 0, \varphi(\cup_i M_i) = \textstyle\sum_i \varphi(M_i)\}$$

the smallest making

$$\begin{aligned} ev_M : \mathcal{G}X &\to [0,1] \\ \varphi &\mapsto \varphi(M) \end{aligned}$$

measurable

generated by

$$\{\Box_r(M) \mid r \in \mathbb{Q} \cap [0,1]\}$$

$$\Box_r(M) = \{\varphi \in \mathcal{G}X \mid \varphi(M) \geq r\}$$

subprobability
measures

Logic for transition systems

- Modal operator

$$\Box(S) = \{u \in \mathcal{P}_f(X) \mid u \subseteq S\}$$

☐ induces ☒: $L\mathcal{P} \Rightarrow \mathcal{P}\mathcal{P}_f$

models of boolean logic with fin.meet preserving modal operators

$$\begin{tikzcd} \mathcal{P}_f \circlearrowright \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{BA} \circlearrowleft L \arrow[l, bend left, "\mathcal{F}_u"] \end{tikzcd}$$

componentwise mono tr

: $\mathcal{P}_f\mathcal{F}_u \Rightarrow \mathcal{F}_u L$

expressivity

GV

rgetful

Logic for Markov chains

- Probabilistic modalities

$$\Box_r(S) = \{\varphi \in \mathcal{D}_f(X) \mid \sum_{x\in S}\varphi(x) \geq r\}$$

$\Box$ induces $\boxtimes$: $K\mathcal{P} \Rightarrow \mathcal{P}\mathcal{D}_f$

models of
logic with
fin.conj. and
monotone
modal
operators

$$\begin{tikzcd}\mathcal{D}_f \circlearrowright \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{MSL} \circlearrowleft K \arrow[l, bend left, "\mathcal{F}"]\end{tikzcd}$$

$\rfloor_r HV$
getful

componentwise mono tr $: \mathcal{D}_f\mathcal{F} \Rightarrow \mathcal{F}K$

expressivity

Logic for multitransition ...

- Graded modal logic modalities

$$\diamond_k(S) = \{\varphi \in \mathcal{M}_f(X) \mid \textstyle\sum_{x \in S} \varphi(x) \geq k\}$$

$\diamond$ induces ☒: $K\mathcal{P} \Rightarrow \mathcal{P}\mathcal{M}_f$

models of logic with fin.conj. and monotone modal operators

$$\begin{tikzcd}\mathcal{M}_f \circlearrowright \mathbf{Sets}^{\mathrm{op}} \arrow[r, bend left, "\mathcal{P}"] & \mathbf{MSL} \circlearrowleft K \arrow[l, bend left, "\mathcal{F}"]\end{tikzcd}$$

$\rfloor_{r}HV$
getful

componentwise mono tra $\mathcal{M}_f\mathcal{F} \Rightarrow \mathcal{F}K$

expressivity

Logic for Markov processes

- General probabilistic modalities

$$\Box_r(M) = \{\varphi \in \mathcal{G}(X) \mid \varphi(M) \geq r\}$$

☐ induces ☒: $KS \Rightarrow S\mathcal{G}$

models of
logic with
fin.conj. and
monotone
modal
operators

$$\begin{tikzcd} \mathbf{Meas}^{\mathrm{op}} \arrow[loop left, "\mathcal{G}"] \arrow[r, bend left, "\mathcal{S}"] & \mathbf{MSL} \arrow[l, bend left, "\mathcal{F}"] \arrow[loop right, "K"] \end{tikzcd}$$

ne K

componentwise abs.mon

expressivity

$\overline{\boxtimes}: \mathcal{G}\mathcal{F} \Rightarrow \mathcal{F}K$

Discrete to indiscrete

- The adjunctions are related:

$$\begin{tikzcd}
\mathbf{Meas}^{\mathrm{op}} \arrow[drr, "\mathcal{S}"] \arrow[dd, bend left, "U"] & & \\
& & \mathbf{MSL} \arrow[ull, bend right, "\mathcal{F}"'] \arrow[dll, bend left, "\mathcal{F}"] \\
\mathbf{Sets}^{\mathrm{op}} \arrow[uu, "D"] \arrow[urr, "\mathcal{P}"'] & &
\end{tikzcd}$$

forgetful functor

discrete measure space

$D(X) = (X, \mathcal{P}(X))$

Hence $\mathcal{F} \dashv \mathcal{P}$ via

$U\mathcal{F} \dashv \mathcal{S}D$

by composition

Discrete to indiscrete

- Markov chains as Markov processes

$$\begin{tikzcd}
\mathcal{G} \circlearrowleft \mathbf{Meas}^{\mathrm{op}} \arrow[dr, "\mathcal{S}"] \arrow[dd, bend left, "U"] & \\
& \mathbf{MSL} \arrow[ul, bend left, "\mathcal{F}"'] \arrow[dl, bend left, "\mathcal{F}"] \\
\mathcal{D}_f \circlearrowleft \mathbf{Sets}^{\mathrm{op}} \arrow[uu, "D"] \arrow[ur, "\mathcal{P}"] &
\end{tikzcd}$$

via an embedding natural transformation $\rho : \mathcal{D}_f U \Rightarrow U\mathcal{G}$

$$\rho(\varphi) = \left[M \mapsto \sum_{x \in M} \varphi(x) \right]$$

Discrete to indiscrete

- So we can translate chains into processes

$$X \xrightarrow{c} \mathcal{D}_f(X) = \mathcal{D}_f U D(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)$$

$$\begin{tikzcd}
\mathsf{Coalg}_{\mathcal{D}_f} \arrow[rr, "\rho D \circ -"'] \arrow[rrr, bend left, "\mathcal{T}"] \arrow[dr] & & \mathsf{Coalg}_{\mathcal{G}D} \arrow[r, "(*)"'] \arrow[dl] & \mathsf{Coalg}_{\mathcal{G}} \arrow[d] \\
& \mathsf{Sets} \arrow[rr, "D"] & & \mathbf{Meas}
\end{tikzcd}$$

Behavioural equivalence coincides

Also the logic theories do

$$\begin{array}{c}
X \longrightarrow U\mathcal{G}D(X) \quad \text{in Sets} \\
\hline\hline
D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}
\end{array}$$

Or directly …

$$\varphi \in \Box_r^{\mathcal{D}_f}(S) \Leftrightarrow \rho(\varphi) \in \Box_r^{\mathcal{G}}(S)$$

$\varphi \in \mathcal{D}_f(X)$

$S \in \mathcal{P}(X) = \mathcal{S}(DX)$

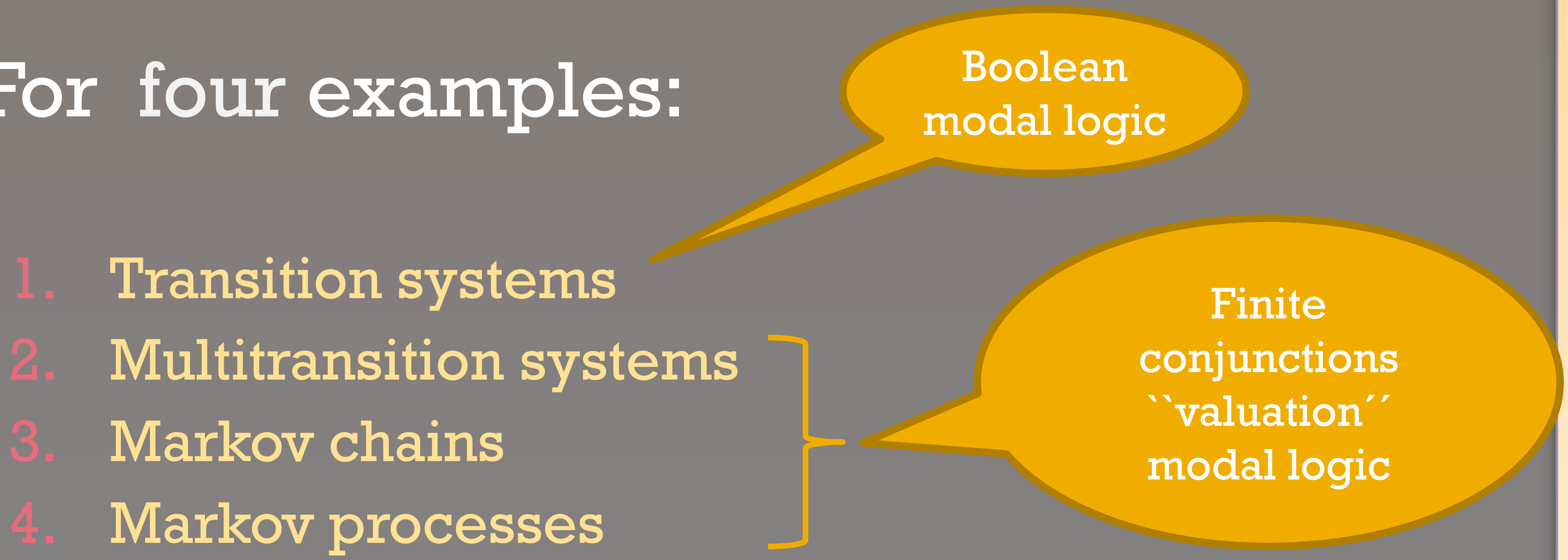
$\boxtimes^{\mathcal{D}_f} = \mathcal{P}(\rho) \circ \boxtimes$ and $\overline{\boxtimes}^{\mathcal{D}_f} = U(\overline{\boxtimes}^{\mathcal{G}}) \circ \rho\mathcal{F}$

$\rho\mathcal{F}\colon \mathcal{D}_f U\mathcal{F} \Rightarrow U\mathcal{G}\mathcal{F}$ is c ise mono

Expressivity for chains follows from expressivity for processes !

Conclusions

- Expressivity
- For four examples:



1. Transition systems
2. Multitransition systems
3. Markov chains
4. Markov processes

Boolean modal logic

Finite conjunctions ``valuation´´ modal logic

in the setting of dual adjunctions !