Determinizations and non-determinizations for semantics

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Shonan NII Meeting on Coinduction, 9.10.2013

Two parts

1. Categorical treatment of determinizations
   joint work with Bart Jacobs and Alexandra Silva
   CMCS 2012 / JSS in preparation

2. Non-determinization of probabilistic automata for verification
   very early-stage work with Filippo Bonchi and Alexandra Silva

Determinization of NFA

NFA

2 x ($\mathcal{P}$(-))^A ≅ $\mathcal{P}$ (1 + A x (-))

$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
x_2 \arrow[d] & & x_3 \arrow[loop right, "b"] \\
* & &
\end{tikzcd}$$

DFA

2 x (-)^A states $\mathcal{P}$(-)

$$\begin{tikzcd}
x_1 \arrow[d, "a"'] & \\
\{x_2, x_3\} \arrow[r, "b"] \arrow[d] & x_3 \arrow[loop right, "b"] \\
* &
\end{tikzcd}$$

Determinization of PTS

PTS

$\mathcal{D}(1 + A \times (-))$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{4}}"] & \\
x_2 \arrow[d, "{b, \frac{1}{3}}"'] & & x_3 \arrow[d, "{c, \frac{1}{2}}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

DFA

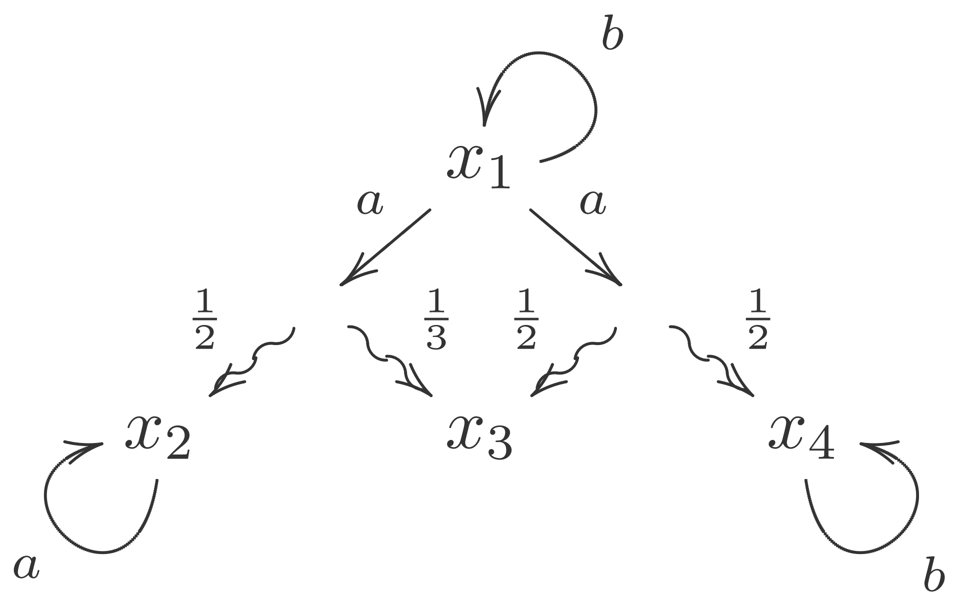
$[0,1] \times (-)^A$ states $\mathcal{D}(-)$

$$\begin{tikzcd}
 & x_1 \arrow[d, "a"] & \\
 & \frac{1}{2}x_2 + \frac{1}{4}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\frac{1}{6}x_4 \arrow[d, "\frac{1}{6}"'] & & \frac{1}{8}x_5 \arrow[d, "\frac{1}{8}"] \\
* & & *
\end{tikzcd}$$

Non-determinization of PA

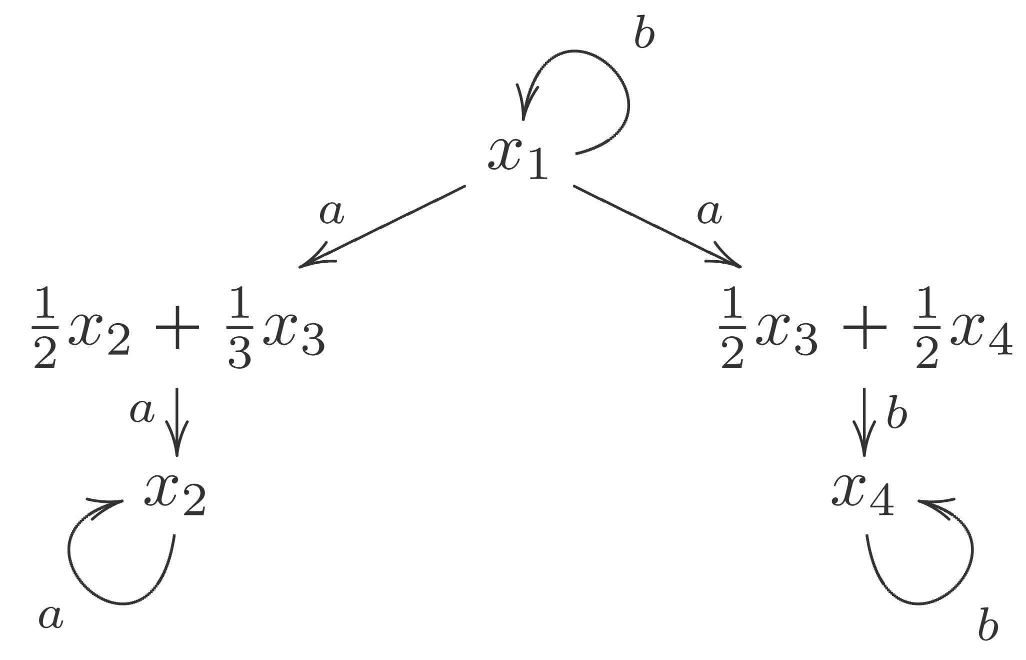
PA

$\mathcal{P}$ (A x $\mathcal{D}$(-))



LTS

$\mathcal{P}$ (A x (-)) states $\mathcal{D}$(-)



The functors / monads

Generalized powerset construction [SBBR'10]

system	functor	"determinization"	states
NFA (1)	GT	G	$\mathcal{P}(-)$
PA			$\mathcal{D}(-)$ T(-)
NFA (2)	TF	G	$\mathcal{P}(-)$
PTS			$\mathcal{D}(-)$

T - monad

Kleisli traces
[HJS'07]

semantics via coinduction

The functors / monads

Generalized powerset construction [SBBR'10]

system	functor	"determinization"	states
NFA (1)	GT	$\hat{G}$	$\mathcal{P}(-)$
PA			$\mathcal{D}(-)$
NFA (2)	TF	$\hat{G}$	$\mathcal{P}(-)$
PTS			$\mathcal{D}(-)$

$T(-)$

Kleisli traces [HJS'07]

T - monad

semantics via coinduction

The big picture

$$
\begin{tikzcd}
& \mathbf{CoAlg}(\widehat{F}) \arrow[rr, "\widehat{E}"] \arrow[dl] \arrow[drr, phantom, "6"] & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
\mathcal{K}\ell(T) \arrow[rr, "E"] \arrow[dd, bend right] \arrow[dd, phantom, "\vdash"] \arrow[dr, phantom, "3"] & & \mathcal{EM}(T) \arrow[dd, bend right] \arrow[dd, phantom, "\vdash"] \arrow[dr, phantom, "5"] \arrow[dl, phantom, "4"] & \\
& \mathbf{CoAlg}(TF) \arrow[uu, "\mathcal{F}_{\mathcal{K}\ell}"'] \arrow[rr, "\mathcal{E}"] \arrow[dl] \arrow[ur, phantom, "2"] \arrow[d, phantom, "1"] & & \mathbf{CoAlg}(GT) \arrow[uu, "\mathcal{F}_{\mathcal{EM}}"'] \arrow[dl] \\
C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[rr, equal, "\mathrm{Id}"] & & C \arrow[uu, bend right, "\mathcal{F}"'] &
\end{tikzcd}
$$

The big picture

$$
\begin{tikzcd}
 & \mathbf{CoAlg}(\widehat{F}) \arrow[rr, "\widehat{E}"] \arrow[dl] \arrow[dr, phantom, "⑥"] & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
\mathcal{K}\ell(T) \arrow[rr, "E"] \arrow[dd, bend right] \arrow[dr, phantom, "③"] & & \mathcal{EM}(T) \arrow[dd, bend right] \arrow[dl, phantom, "④"] \arrow[dr, phantom, "⑤"] & \\
 & \mathbf{CoAlg}(TF) \arrow[uu, "\mathcal{F}_{\mathcal{K}\ell}"'] \arrow[rr, "\mathcal{E}"] \arrow[dl] \arrow[ur, phantom, "②"] \arrow[dr, phantom, "①"] & & \mathbf{CoAlg}(GT) \arrow[uu, "\mathcal{F}_{\mathcal{EM}}"'] \arrow[dl] \\
C \arrow[rr, equal, "\mathrm{Id}"] \arrow[uu, bend right, "\mathcal{F}"'] \arrow[uu, phantom, "\vdash"] & & C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[uu, phantom, "\vdash"] &
\end{tikzcd}
$$

The big picture

$$
\begin{tikzcd}
 & \mathbf{CoAlg}(\widehat{F}) \arrow[rrr, "\widehat{E}"] \arrow[dl] & & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
\mathcal{K}\ell(T) \arrow[rrr, "E"] \arrow[dd, bend right] \arrow[dd, phantom, "\vdash"] & & & \mathcal{EM}(T) \arrow[dd, bend right] \arrow[dd, phantom, "\vdash"] & \\
 & \mathbf{CoAlg}(TF) \arrow[uu, "\mathcal{F}_{\mathcal{K}\ell}"'] \arrow[rrr, "\mathcal{E}"] \arrow[dl] & & & \mathbf{CoAlg}(GT) \arrow[uu, "\mathcal{F}_{\mathcal{EM}}"'] \arrow[dl] \\
C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[rrr, equal, "\mathrm{Id}"] & & & C \arrow[uu, bend right, "\mathcal{F}"'] &
\arrow[from=1-2, to=2-4, phantom, "\textcircled{6}"]
\arrow[from=2-4, to=3-2, phantom, "\textcircled{4}"]
\arrow[from=2-1, to=3-2, phantom, "\textcircled{3}"]
\arrow[from=2-4, to=3-5, phantom, "\textcircled{5}"]
\arrow[from=3-2, to=2-3, phantom, "\textcircled{2}"]
\arrow[from=3-2, to=4-4, phantom, "\textcircled{1}"]
\end{tikzcd}
$$

The big picture

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\mathcal{K}\ell(T) \arrow[rr, "E"] \arrow[dd, bend right] \arrow[dr, phantom, "③"] & & \mathcal{E}\mathcal{M}(T) \arrow[dd, bend right] \arrow[dr, phantom, "⑤"] \arrow[dl, phantom, "④"] & \\
& \mathbf{CoAlg}(TF) \arrow[uu, "\mathcal{F}_{\mathcal{K}\ell}"'] \arrow[rr, "\mathcal{E}"] \arrow[dl] \arrow[ur, phantom, "②"] \arrow[d, phantom, "①"] & & \mathbf{CoAlg}(GT) \arrow[uu, "\mathcal{F}_{\mathcal{E}\mathcal{M}}"'] \arrow[dl] \\
C \arrow[rr, equal, "\mathrm{Id}"] \arrow[uu, bend right, "\mathcal{F}"'] \arrow[uu, phantom, "\vdash"] & & C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[uu, phantom, "\vdash"] &
\end{tikzcd}
$$

The big picture

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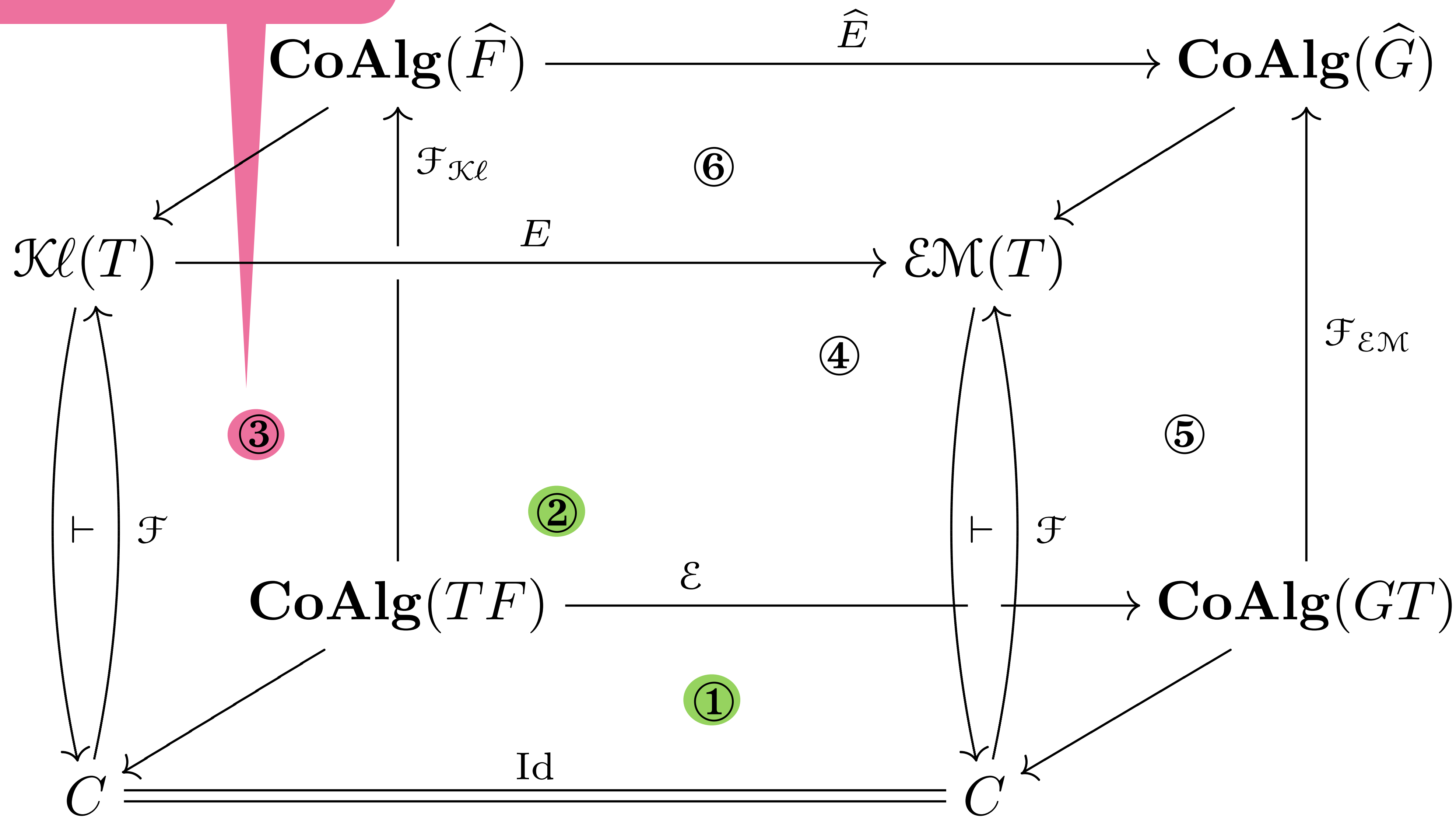
C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[rr, equal, "\mathrm{Id}"] & & C \arrow[uu, bend right, "\mathcal{F}"'] &
\end{tikzcd}
$$

$$E(X) = (\mu\colon T^2(X) \to T(X)). \qquad E(f) = \mu \circ T(f)$$

The big picture

needs Kl-law $\ell : FT \Rightarrow TF$



$\mathcal{F}(X) = X \quad \mathcal{F}(f) = \eta \circ f \qquad \mathcal{F}_{\mathcal{K}\ell}$ - identity on objects $\quad \mathcal{F}_{\mathcal{K}\ell}(f) = \eta \circ f.$

The big picture

$$\begin{pmatrix} TX \\ \downarrow a \\ X \end{pmatrix} \longmapsto \begin{pmatrix} TGX \\ \downarrow G(a)\circ\rho \\ GX \end{pmatrix}$$

$$f \longmapsto G(f)$$

needs EM-law ρ : TG ⇒ GT

$$\begin{tikzcd}
& \mathbf{CoAlg}(\widehat{F}) \arrow[rr, "\widehat{E}"] \arrow[dl] \arrow[drr, phantom, "6"] & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
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\end{tikzcd}$$

$$\mathcal{F}_{\mathcal{EM}}(f) = T(f)$$

$$\mathcal{F}_{\mathcal{EM}}\left(X \xrightarrow{c} GTX\right) = \left(TX \xrightarrow{T(c)} TGTX \xrightarrow{\rho_{TX}} GT^2X \xrightarrow{G(\mu)} GTX\right)$$

GPC [SBBR'10]
determinization

The big picture

$$\mathcal{E}(c) = e \circ c$$

$$\mathcal{E}(f) = f$$

needs extension-law e :TF ⇒ GT

$$\begin{tikzcd}
& \mathbf{CoAlg}(\widehat{F}) \arrow[rr, "\widehat{E}"] \arrow[dl] \arrow[dr, phantom, "⑥"] & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
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C \arrow[rr, equal, "\mathrm{Id}"] \arrow[uu, bend right, "\mathcal{F}"'] & & C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[ul, phantom, "①"] &
\end{tikzcd}$$

④

$$\left(X \xrightarrow{c} \widehat{F}X\right) \xmapsto{\widehat{E}} \left(TX \xrightarrow{T(c)} T^2FX \xrightarrow{\mu} TFX \xrightarrow{e} GTX\right) \qquad f \xmapsto{\widehat{E}} E(f) = \mu \circ T(f)$$

The big picture

Final via [HJS'07] if ...

The G-final lifts !

$$
\begin{tikzcd}
 & \mathbf{CoAlg}(\widehat{F}) \arrow[rr, "\widehat{E}"] \arrow[dl] \arrow[dr, phantom, "⑥"] & & \mathbf{CoAlg}(\widehat{G}) \arrow[dl] \\
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C \arrow[uu, bend right, "\mathcal{F}"'] \arrow[rr, equal, "\mathrm{Id}"] & & C \arrow[uu, bend right, "\mathcal{F}"'] &
\end{tikzcd}
$$

The semantics coincide (almost)

End of story?

(Un)fortunately not!

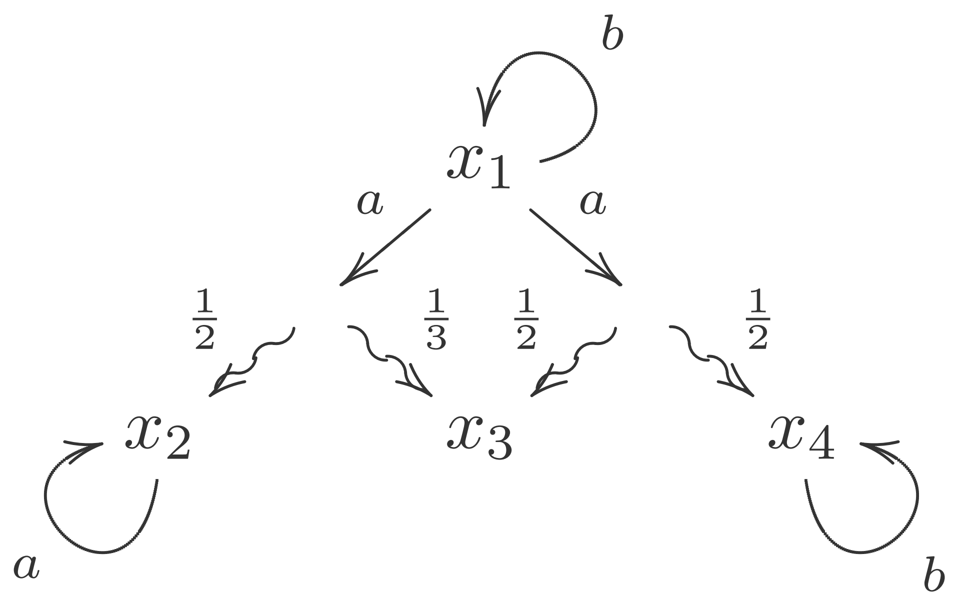
Wanted: good non-determinization for PA

$$
\boxed{\begin{tikzcd} X \arrow[d, "c"'] \\ \mathcal{P}(A \times \mathcal{D}(X)) \end{tikzcd}}
\Longrightarrow
\boxed{\begin{tikzcd} \mathcal{D}(X) \arrow[d, "\mathrm{det}_c"] \\ \mathcal{P}(A \times \mathcal{D}(X)) \end{tikzcd}}
$$

A non-determinization of PA

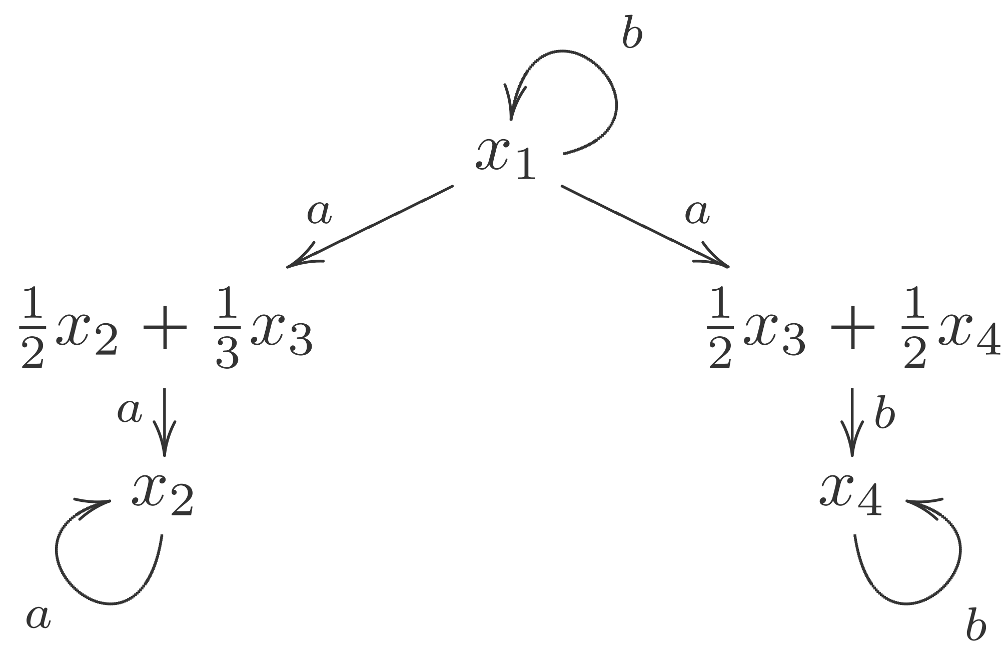
PA

$\mathcal{P}(A \times \mathcal{D}(-))$



LTS

$\mathcal{P}(A \times (-))$ states $\mathcal{D}(-)$



Another one... [DvGHMZ'07]

Let $\mathcal{R} \subseteq S \times \mathcal{D}(S)$ be a relation from states to distributions. We lift it to a relation $\overline{\mathcal{R}} \subseteq \mathcal{D}(S) \times \mathcal{D}(S)$ by letting $\Delta_1 \; \overline{\mathcal{R}} \; \Delta_2$ whenever

(i) $\Delta_1 = \sum_{i \in I} p_i \cdot \overline{s_i}$, where I is a finite index set and $\sum_{i \in I} p_i = 1$

(ii) For each $i \in I$ there is a distribution Φ_i such that $s_i \; \mathcal{R} \; \Phi_i$

(iii) $\Delta_2 = \sum_{i \in I} p_i \cdot \Phi_i$.

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(iii) $\Delta_2 = \sum_{i \in I} p_i \cdot \Phi_i$.

$$X \xrightarrow{c} \mathcal{P}(\mathcal{D}(X))$$

$R \subseteq X \times \mathcal{D}(X))$

$R' \subseteq \mathcal{D}(X) \times \mathcal{D}^2(X)$

$R'' \subseteq \mathcal{D}(X) \times \mathcal{D}(X)$

$$\mathcal{D}(X) \xrightarrow{det_c} \mathcal{P}(\mathcal{D}(X))$$

Does it fit?

Yes and no

Does it fit?

Yes and no

but not a distributive law

$$
\begin{tikzcd}
X \arrow[d, "c"'] \\
\mathcal{P}(\mathcal{D}(X))
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
\mathcal{D}(X) \arrow[d, "\det_c = \mathcal{P}(\mu) \circ \mathbf{r} \circ \mathcal{D}(c)"] \\
\mathcal{P}(\mathcal{D}(X))
\end{tikzcd}
$$

still a lot to be done