The Power of Convex Algebra

Ana Sokolova

UNIVERSITY of SALZBURG

CONCUR ’17

Filippo Bonchi

ENS DE LYON

Alexandra Silva

UCL

NII Shonan Meeting “Enhanced Coinduction” 15.11.17

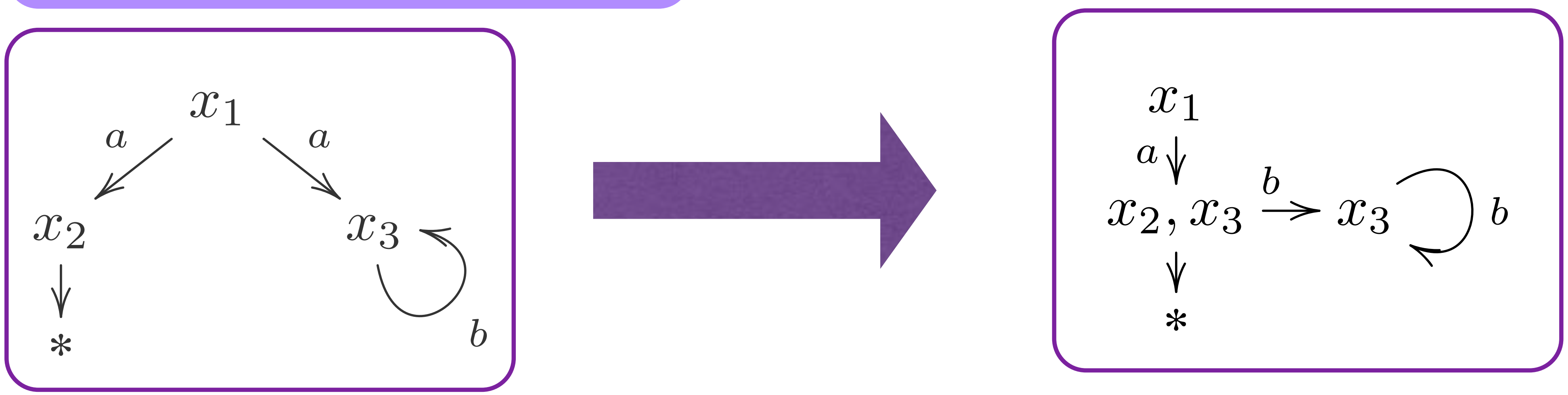
probabilistic automata

The true nature of PA as transformers of belief states

Determinisations

NFA

$X \to 2 \times (\mathcal{P}(X))^A$

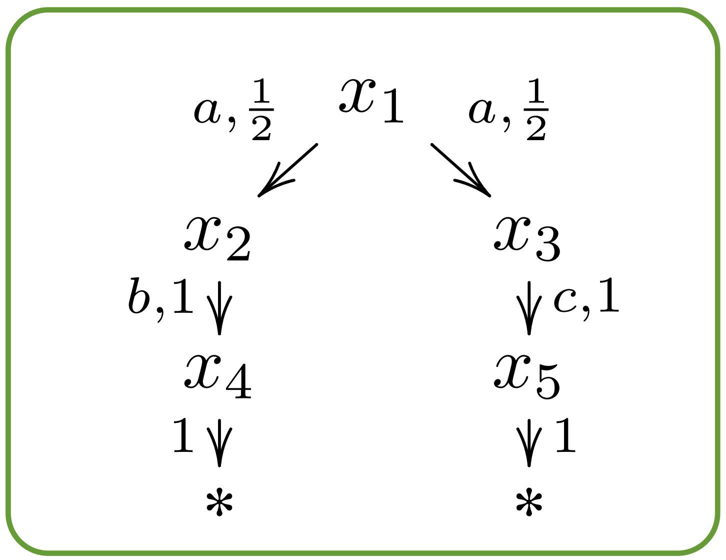


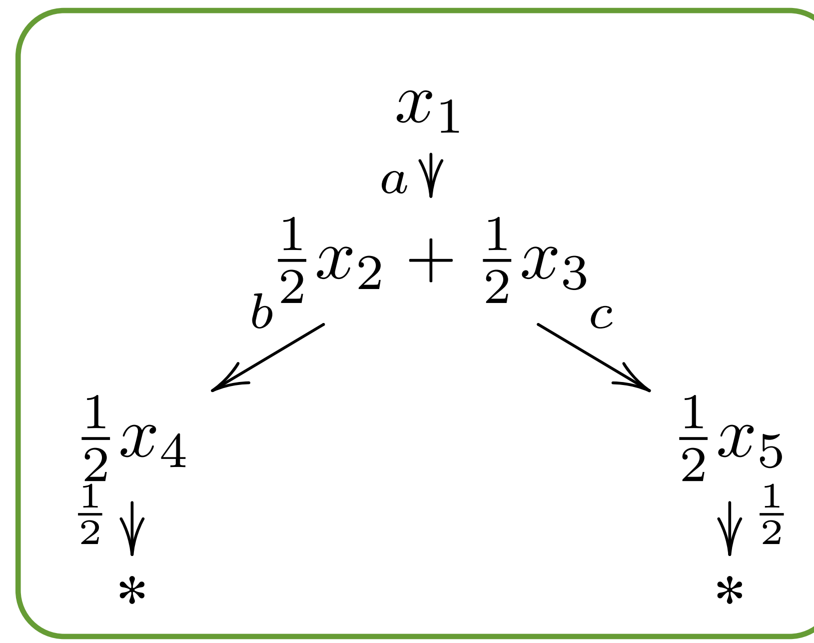
[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations

Generative PTS

$X \to \mathcal{D}(1 + A \times X)$





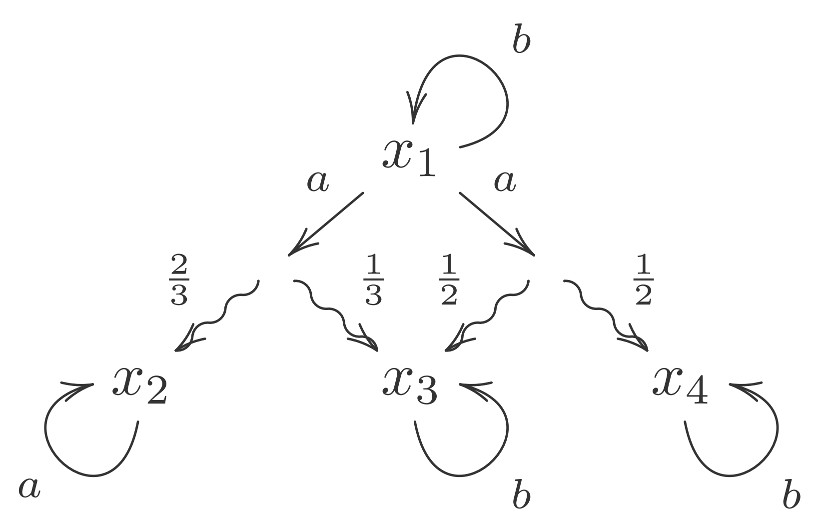
[Silva, S. MFPS'11]

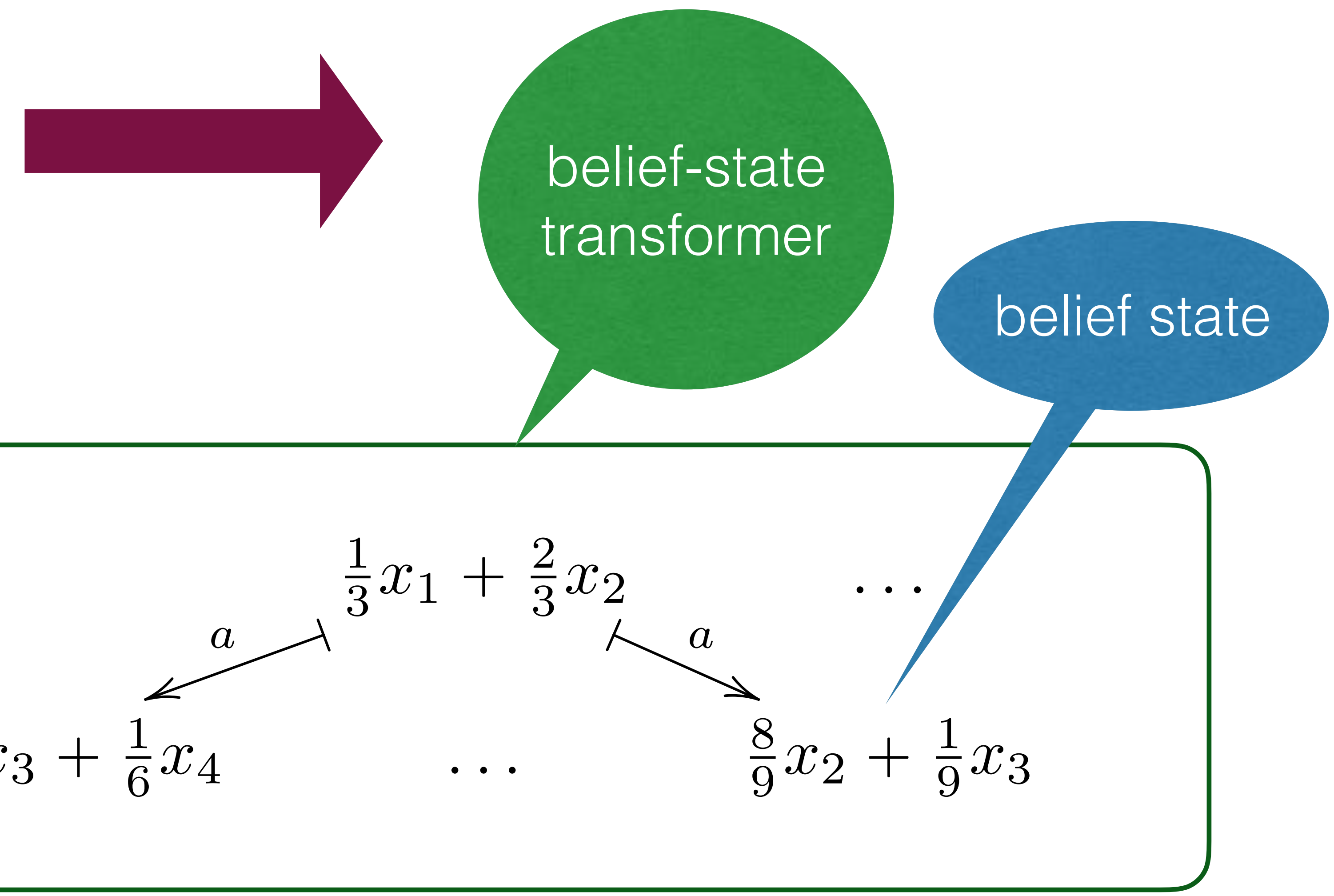
[Jacobs, Silva, S. JCSS'15]

Determinisations

PA

$X \to (\mathcal{PD}(X))^A$



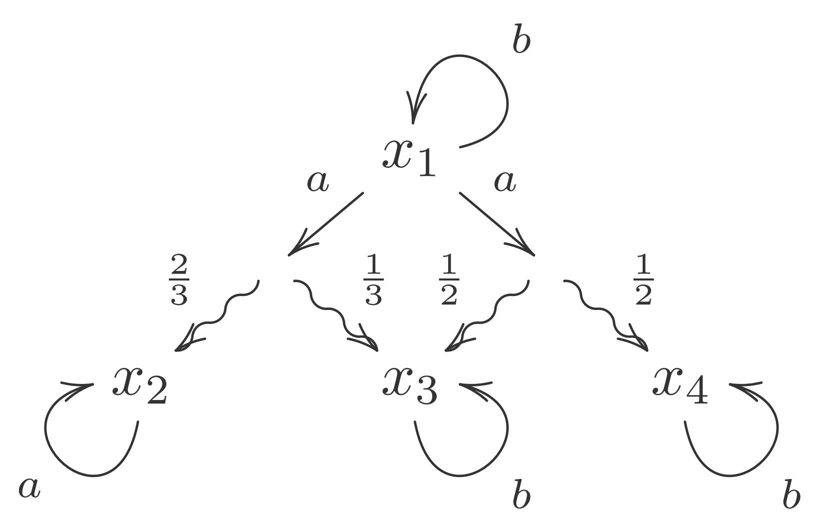


Belief-state transformer

foundation ?

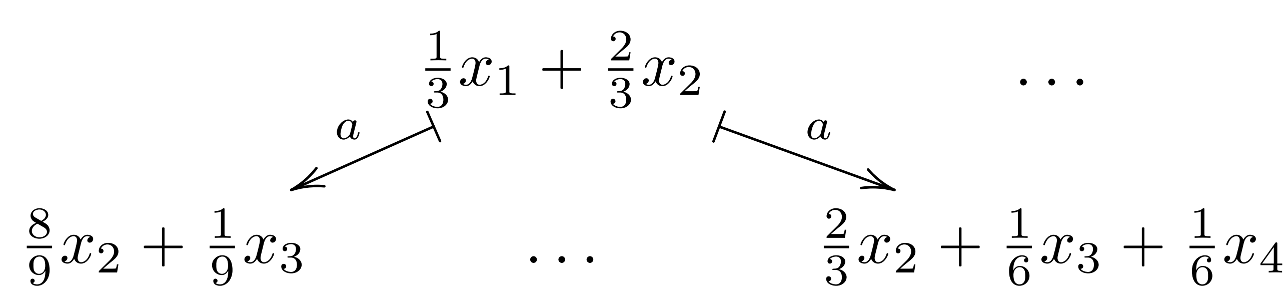
PA

$X \to (\mathcal{PD}(X))^A$

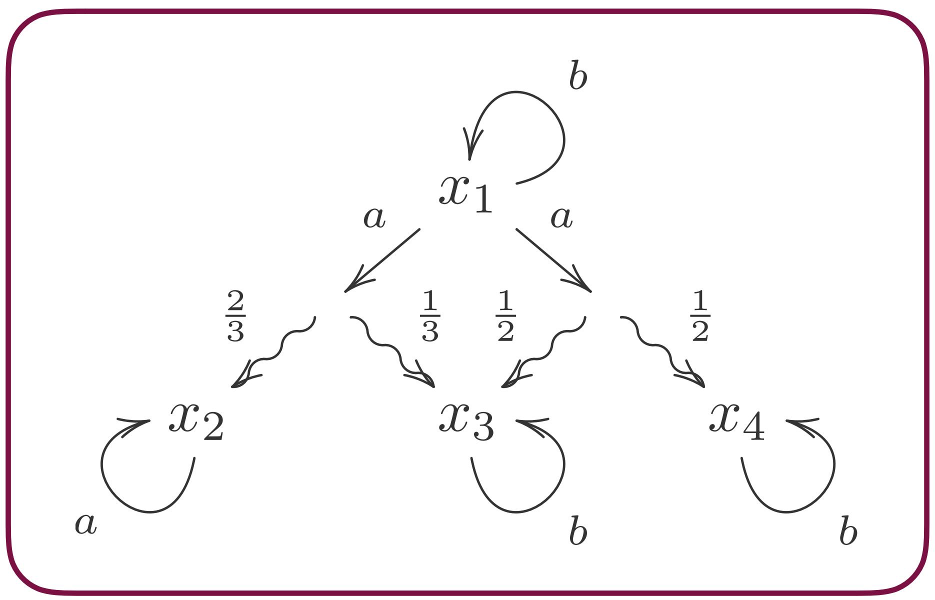


how does it emerge?

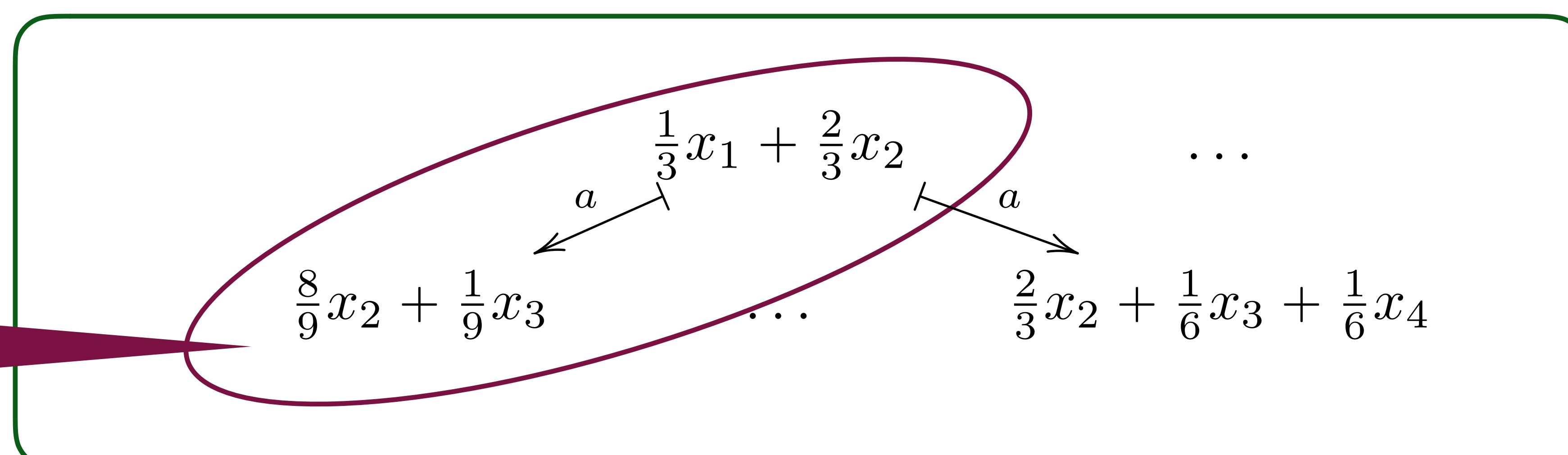
what is it?



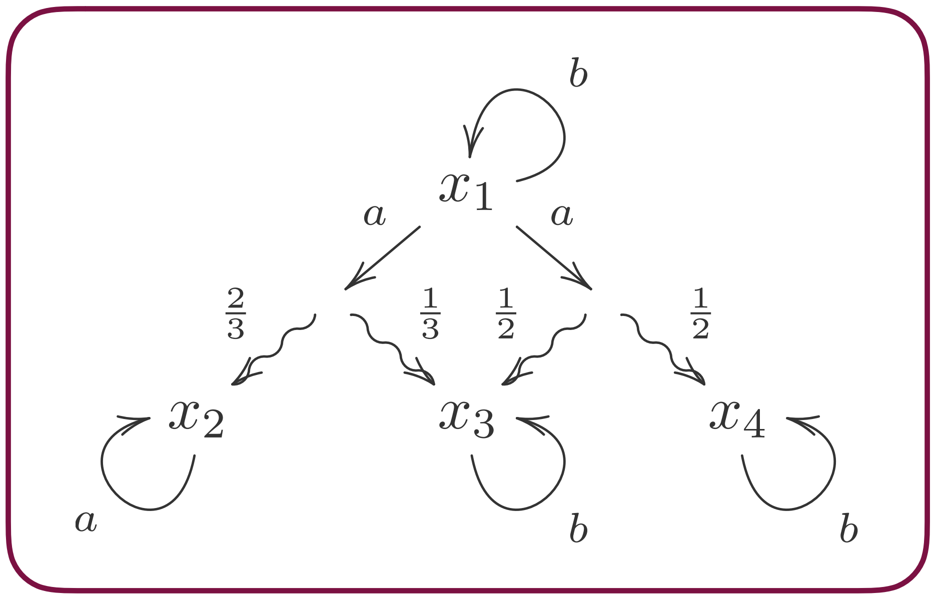
Belief-state transformer

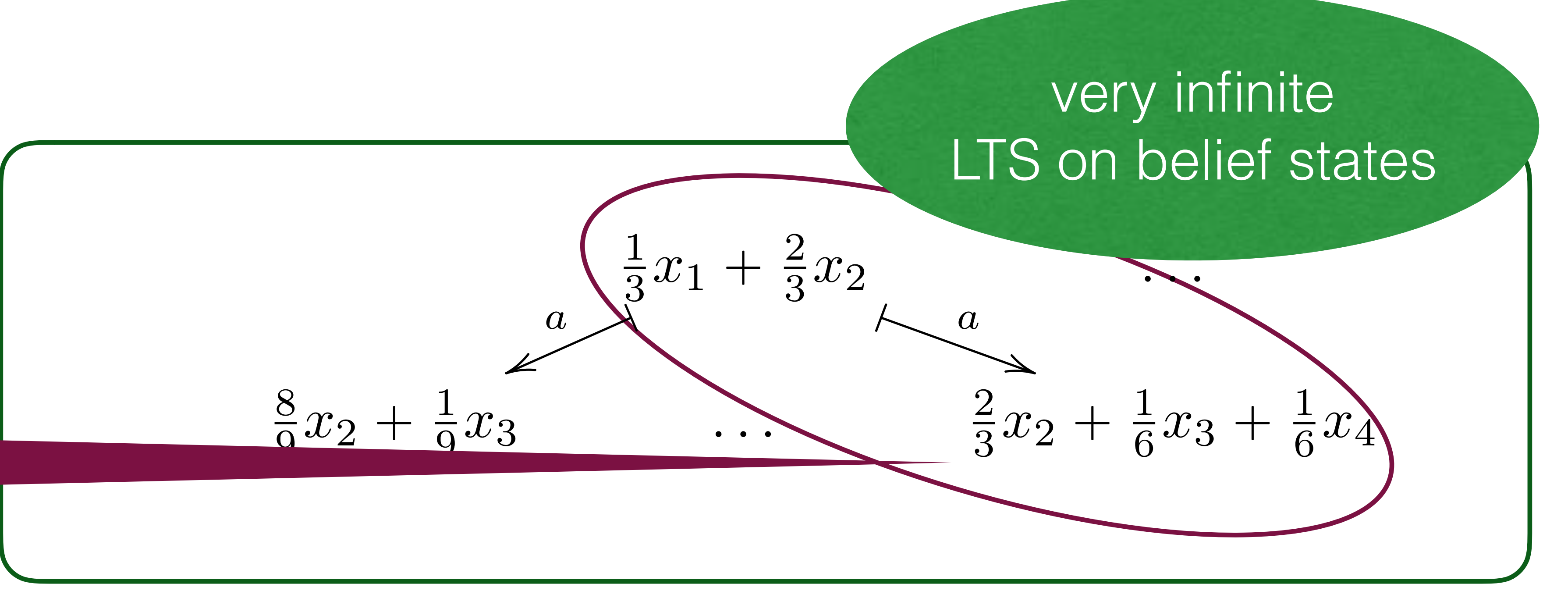


$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$



Belief-state transformer





$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

Probabilistic Automata

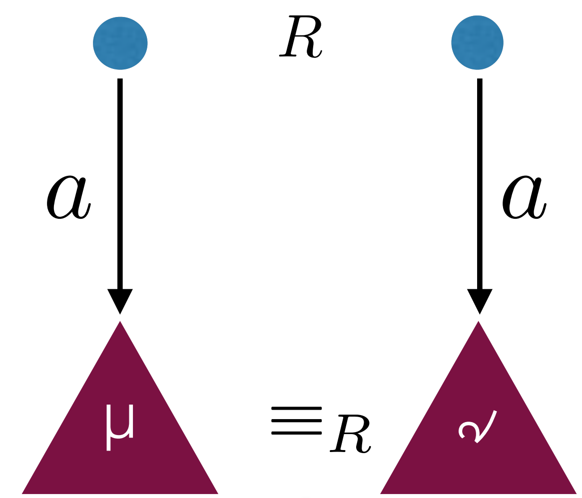
Can be given different semantics:

1. Bisimilarity — strong bisimilarity
2. Convex bisimilarity — probabilistic / combined bisimilarity
3. Distribution bisimilarity — belief-state bisimilarity

Bisimilarity

$\sim$ largest bisimulation

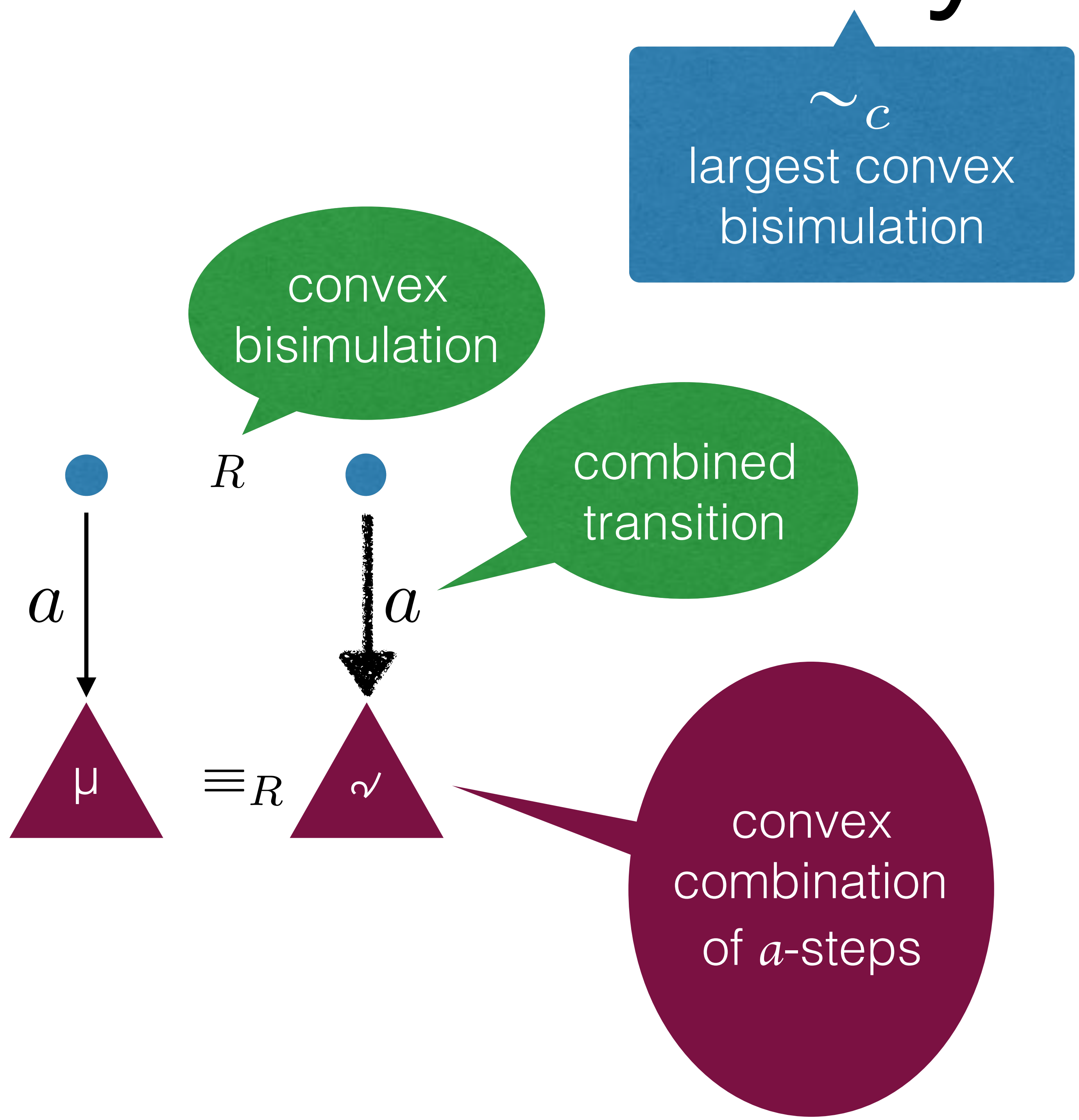
bisimulation



lifting of R to distributions

assign the same probability to “R-classes”

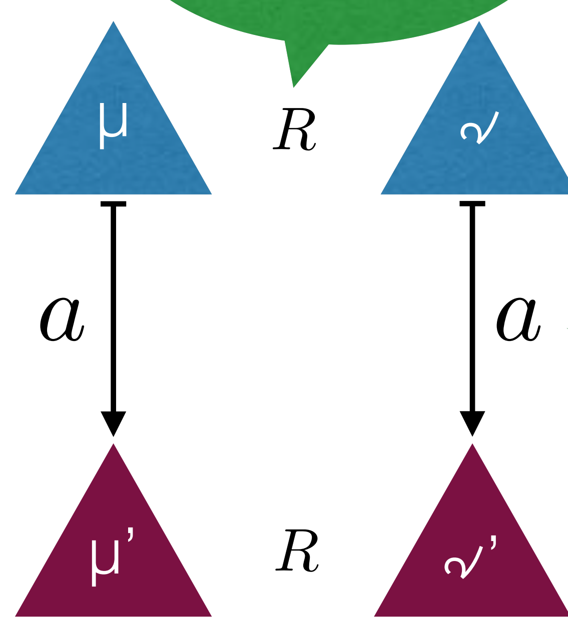
Convex bisimilarity



Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation



transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

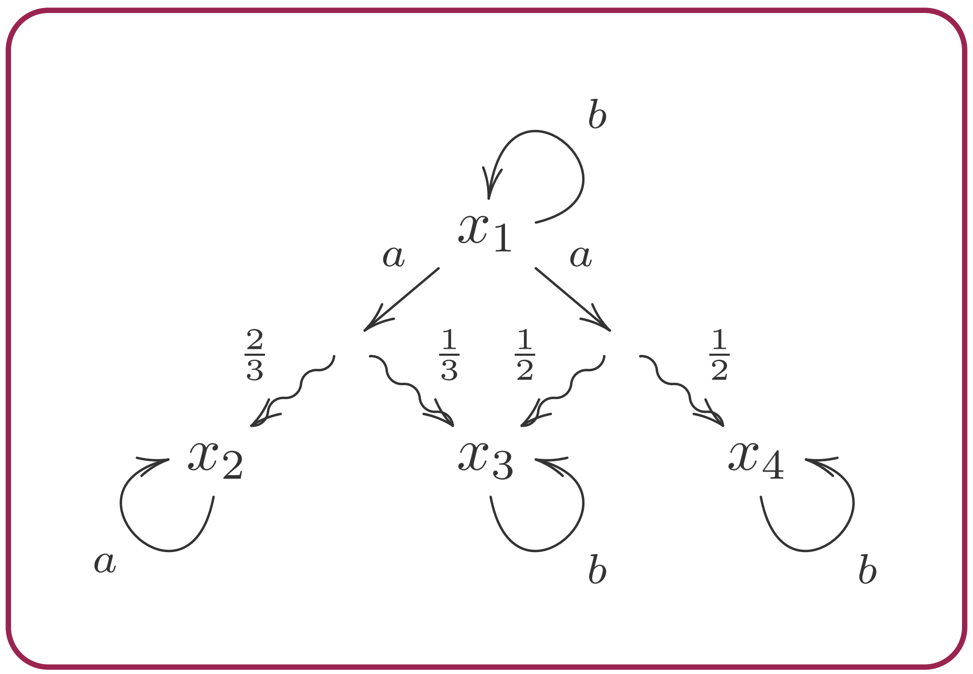
[Hermanns, Krcal, Kretinsky CONCUR'13]

Belief-state transformer

foundation ?

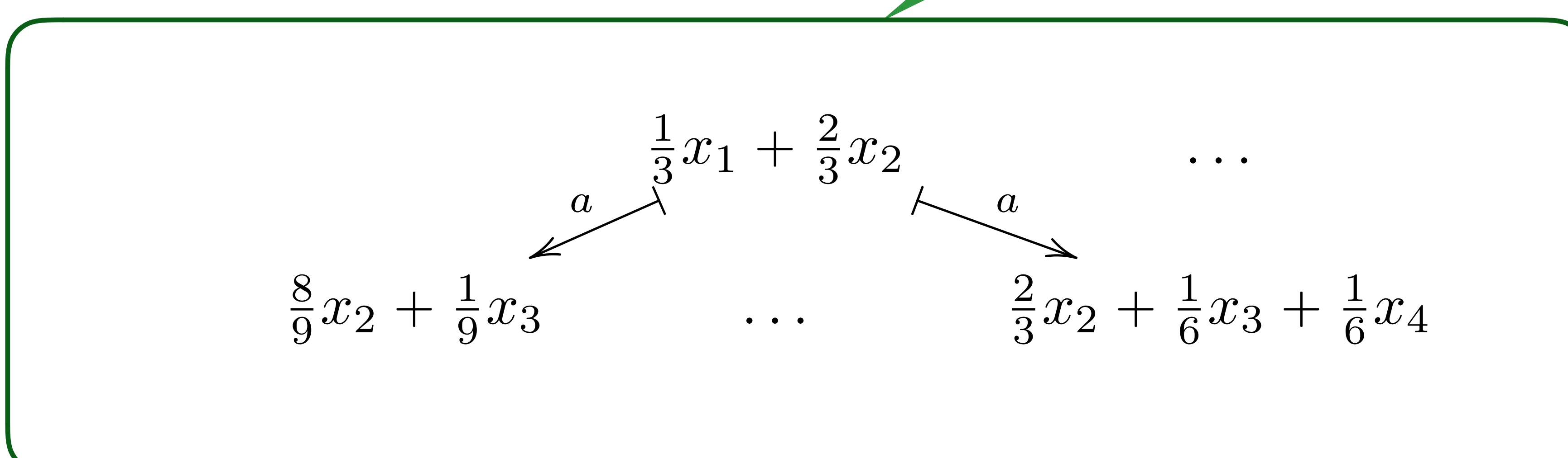
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

what is it?



Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$X \xrightarrow{c} FX$

generic notion of behavioural equivalence $\approx$

states

object in the base category $\boldsymbol{C}$

behaviour type

functor on the base category $\boldsymbol{C}$

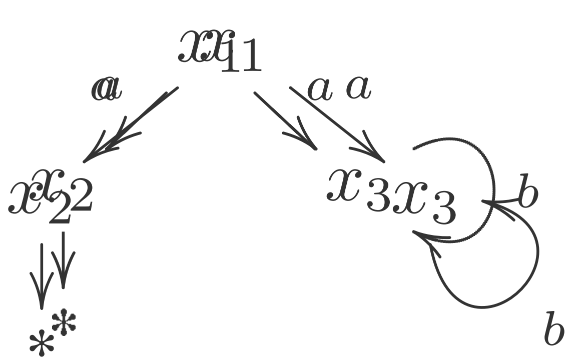
form a category too

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

Examples

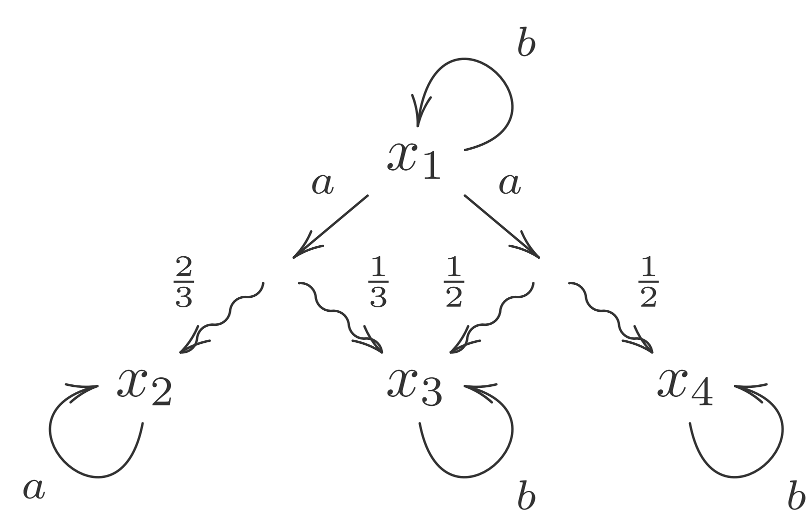
NFA

$X \to 2 \times (\mathcal{P}(X))^A$



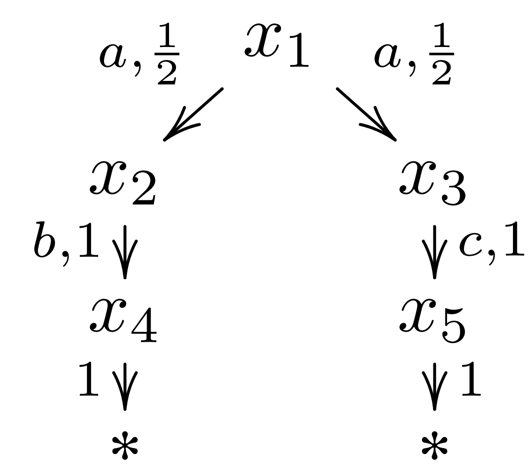
PA

$X \to (\mathcal{P}\mathcal{D}(X))^A$



Generative PTS

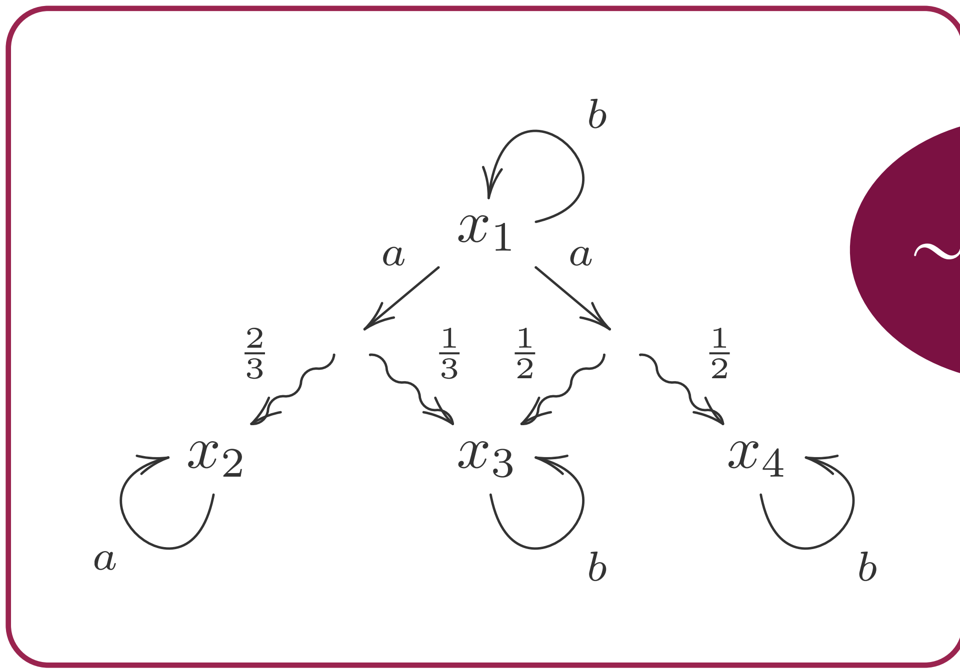
$X \to \mathcal{D}(1 + A \times X)$



all on **Sets**

PA coalgebraically

$X \to (\mathcal{P}\,\mathcal{D}(X))^A$



on **Sets**

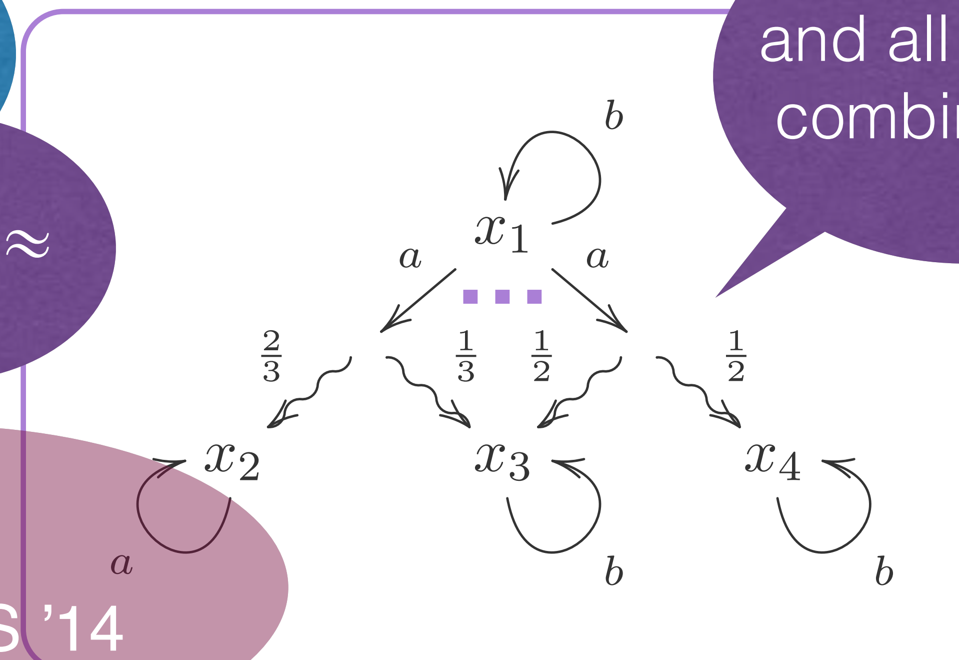
$\sim \, = \, \approx$

$\sim_c \, = \, \approx$

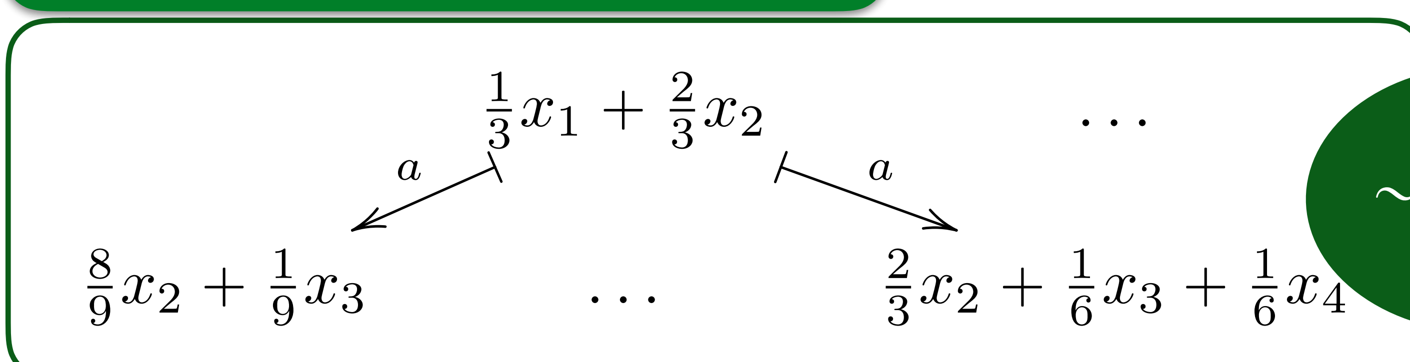
Mio FoSSaCS '14

$X \to (\mathcal{C}(X))^A$

and all convex combinations



$X \to (\mathcal{P}_c(X)+1)^A$



on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \, = \, \approx$

Convex Algebras

infinitely many finitary operations

convex combinations

binary ones "suffice"

- algebras

$$(A, \sum_{i=1}^{n} p_i(-)_i)$$

$$p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$$

- convex (affine) maps

$$h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)$$

satisfying

- Projection $\sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

Eilenberg-Moore Algebras

convex algebras abstractly

$\mathcal{EM}(\mathcal{D})$

- objects

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array}$$

satisfying

$$\begin{array}{ccc} A & \xrightarrow{\eta} & \mathcal{D}A \\ & \overset{a}{\searrow\!\!\!=} & \downarrow a \\ & & A \end{array} \qquad \begin{array}{ccc} \mathcal{D}\mathcal{D}A & \xrightarrow{\mu} & \mathcal{D}A \\ \mathcal{D}a \downarrow & & \downarrow a \\ \mathcal{D}A & \xrightarrow{a} & A \end{array}$$

- morphisms

$$\begin{array}{c} \mathcal{D}A \\ \downarrow a \\ A \end{array} \xrightarrow{h} \begin{array}{c} \mathcal{D}B \\ \downarrow b \\ B \end{array}$$

$$\begin{array}{ccc} \mathcal{D}A & \xrightarrow{\mathcal{D}h} & \mathcal{D}B \\ a \downarrow & & \downarrow b \\ A & \xrightarrow{h} & B \end{array}$$

Belief-state transformers

$$\mathbb{D}_X = \begin{array}{c}\mathcal{D}\mathcal{D}X \\ \downarrow \mu \\ \mathcal{D}X\end{array}$$

convex combinations

free convex algebra

coalgebras on free convex algebras

$$\mathbb{D}_S \to (\mathcal{P}_c(\mathbb{D}_S) + 1)^A$$

constant exponent

nonempty convex powerset

termination

$$pA_1 + (1-p)A_2 = \{pa_1 + (1-p)a_2 \mid a_1 \in A_1, a_2 \in A_2\}$$

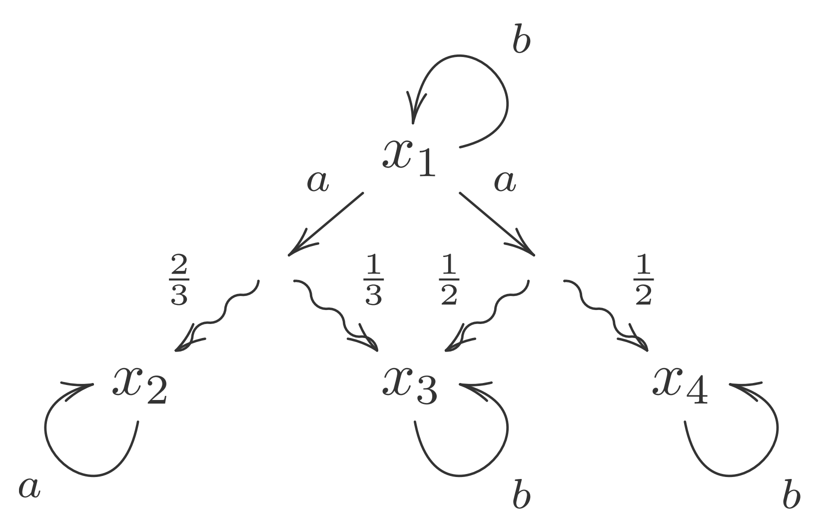
Minkowski sum

Belief-state transformer

foundation ?

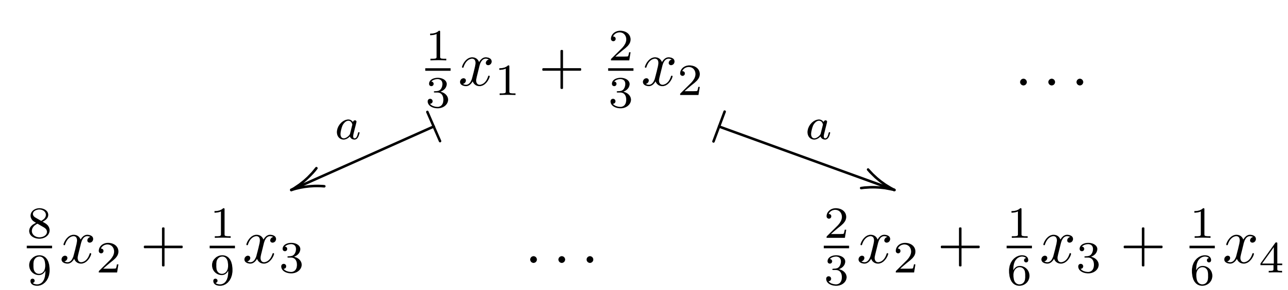
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

what is it?



Belief-state transformer

foundation ?

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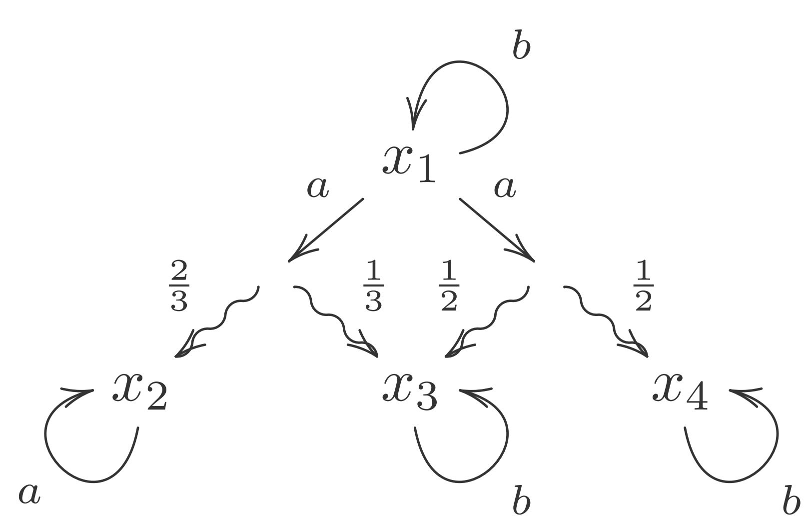
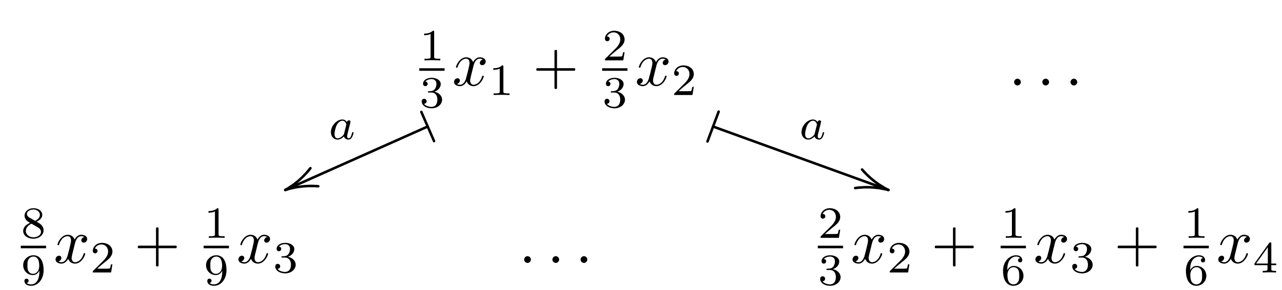


how does it emerge?

coalgebra over free convex algebra



Determinisations I

determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \to \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \to \mathbf{CoAlg}_{\mathbf{C}}(F)$$

forget

needed:
a lifting

works for NFA

not for generative PTS
not for PA / belief-state
transformer

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations II

determinise

forget

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$$

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

needed:
a lifting

works for generative PTS

not for PA / belief-state transformer

[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

Determinisations III

determinise

forget

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$

$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

needed:
a quasi lifting
and
a lax lifting

works for PA /belief-state transformer

$(\mathcal{P}_c+1)^A$ on $\mathcal{EM}(\mathcal{D})$

is a quasi lifting and lax lifting of

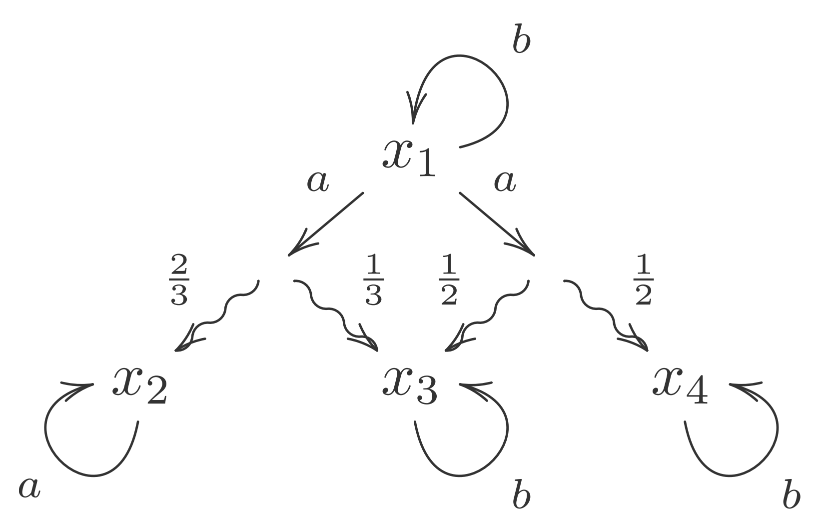
$\mathcal{C}^A$ and $\mathcal{P}^A$ on $\mathbf{Sets}$

Belief-state transformer

foundation ?

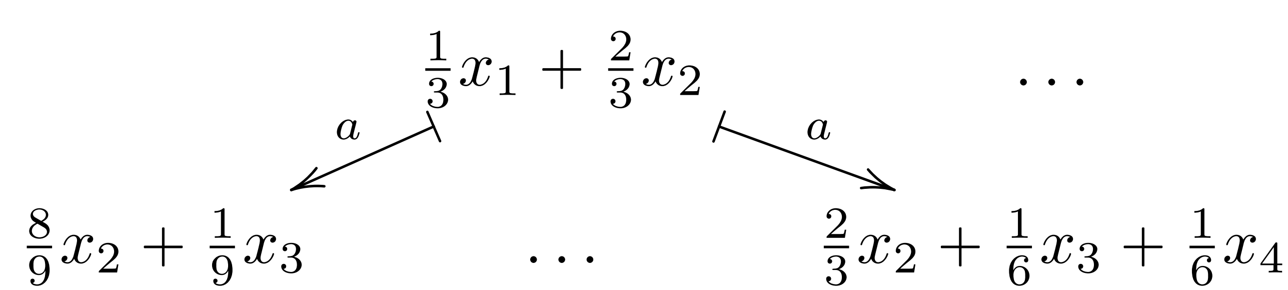
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

coalgebra over free convex algebra



Belief-state transformer

foundation ?

PA

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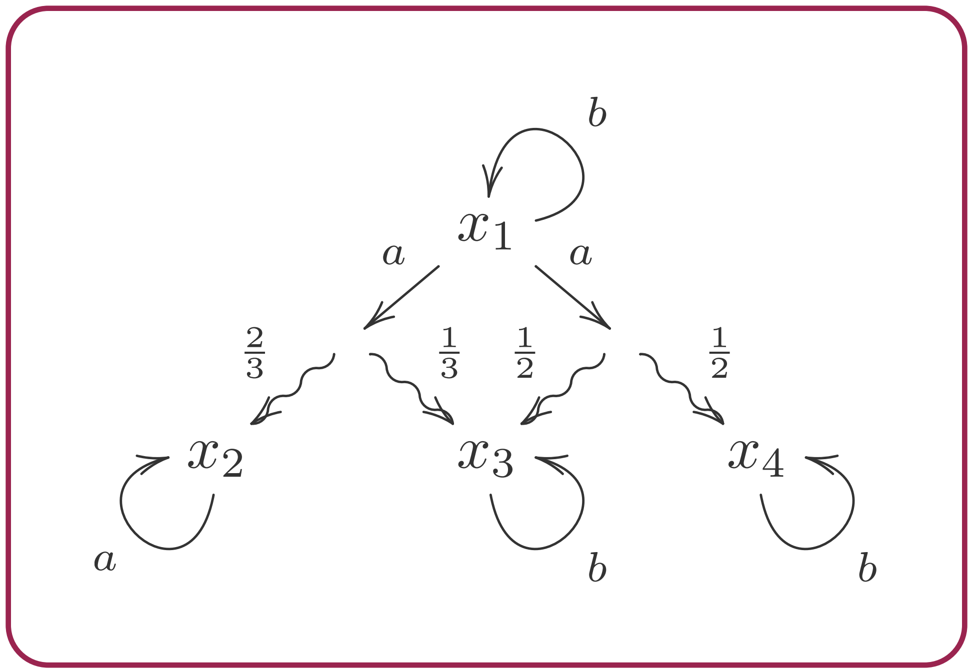
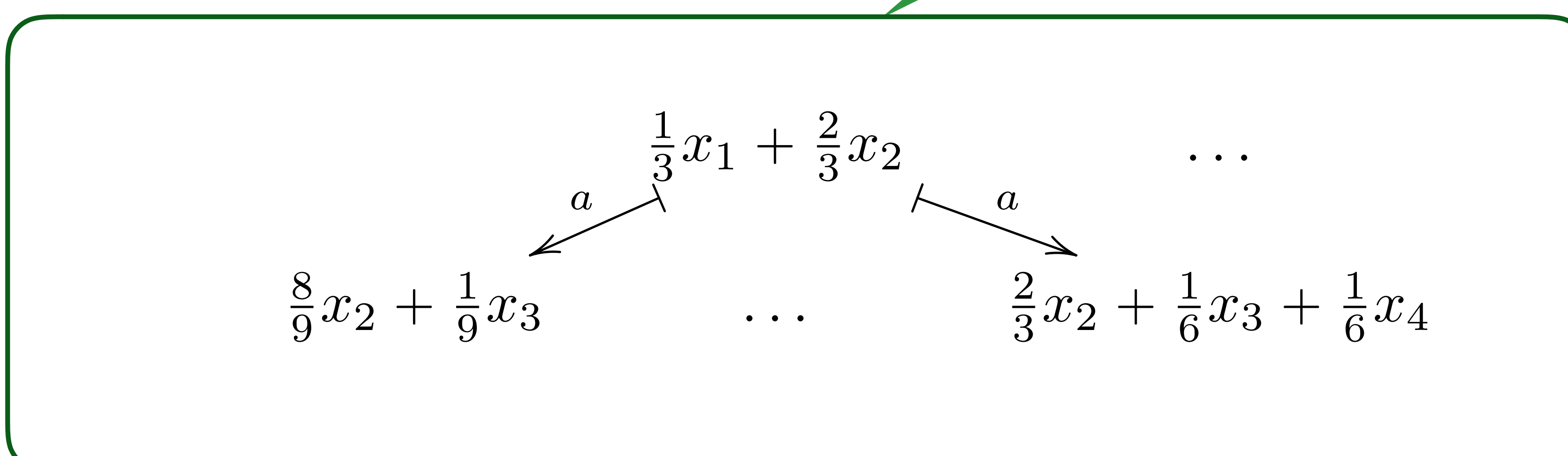


via a generalised[3] determinisation

coalgebra over free convex algebra

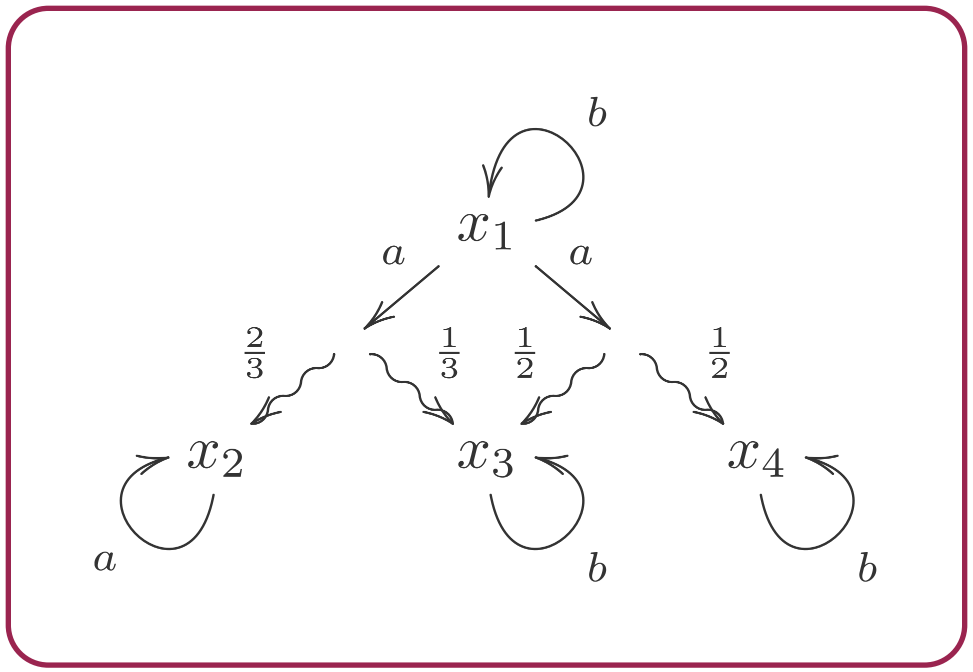


Belief-state transformer

are natural indeed

PA

$X \to (\mathcal{PD}(X))^A$



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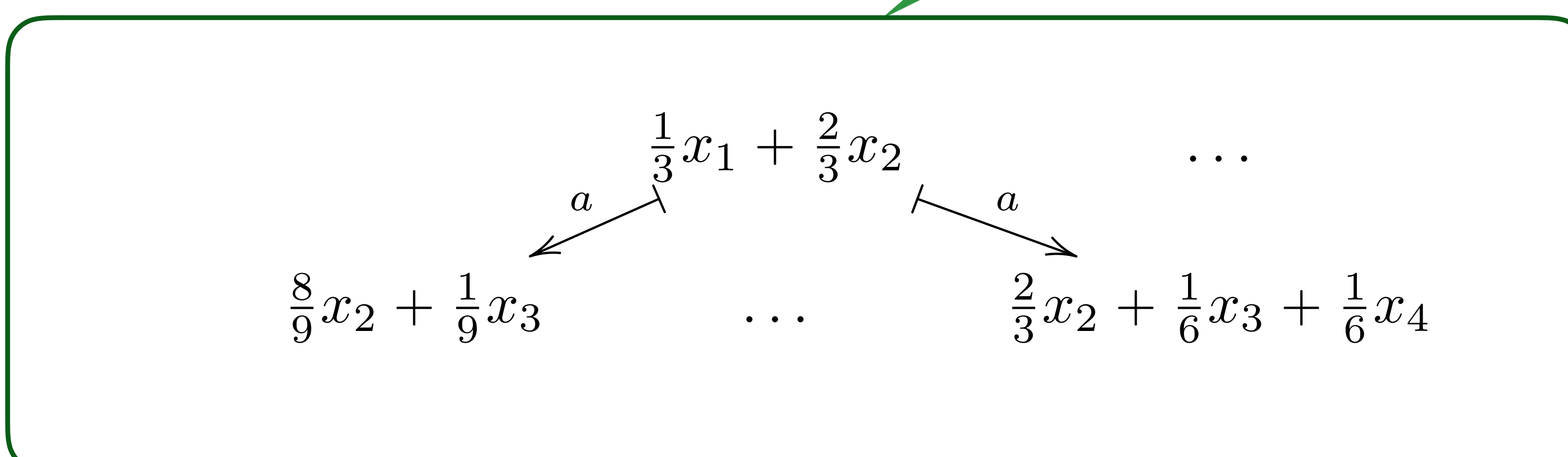
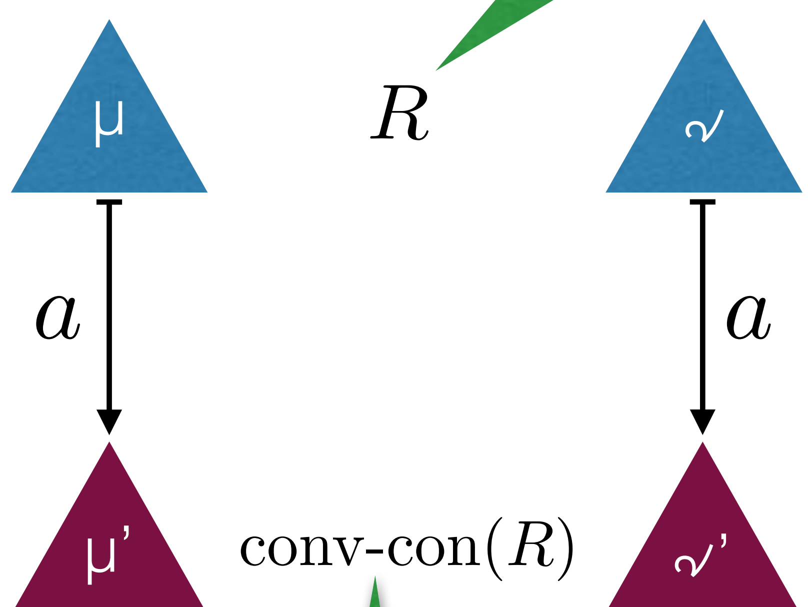


Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull



convex and equivalence closure of *R*

to prove μ $\sim_d$ ν it suffices to find a bisimulation up-to convex hull *R* with μ *R* ν

f.p. = f.g. for (positive) convex algebras

there always exists a finite one!

[S., Woracek JPAA'15]

Belief-state transformer

are natural indeed

PA

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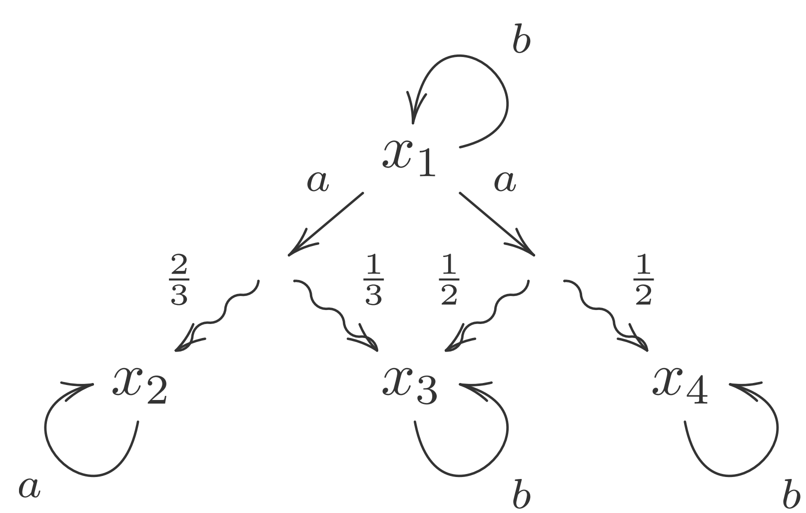

Thank You!

sound proof method for distribution bisimilarity