Probabilistic models for verification

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Outline

- formal methods and verification
- probabilistic systems (coalgebras)
- bisimilarity - the strong end of the spectrum
- expressiveness hierarchy
- other semantics - at the weak end of the spectrum

Why formal methods?

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- In hardware and software design
  - trial and error
  - duplication
- Formal methods aim at replacing the ♣ with the ♣

Formal methods

are mathematically based techniques for

- specification
- development
- verification

of software and hardware systems

Formal methods

In general:

- models - transition systems, automata, terms,...
  with a clear semantics
- analysis - model checking
  process algebra
  theorem proving...

Formal methods

Here, now:

- models - transition systems
- semantics - behavior equivalences

Formal methods

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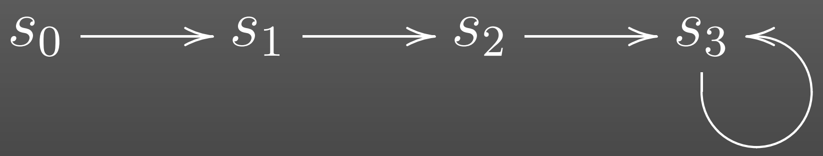
- semantics - behavior equivalences

Aim: one framework for many (probabilistic) models and semantics - compare expressiveness

Example models

deterministic systems



Example models

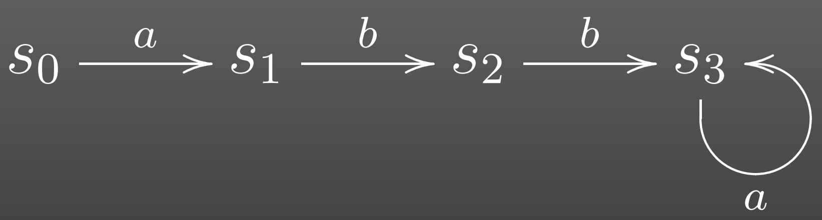
deterministic systems

states $+$ transitions $\alpha : S \to S$

$$\alpha(s_0) = s_1,\ \alpha(s_1) = s_2,\ \dots$$

Example models

labelled deterministic systems A - labels



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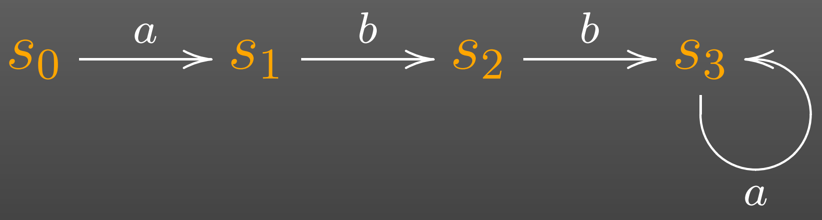

states $+$ transitions $\alpha : S \to A \times S$

$$\alpha(s_0) = \langle a, s_1 \rangle,\ \alpha(s_1) = \langle b, s_2 \rangle,\ \ldots$$

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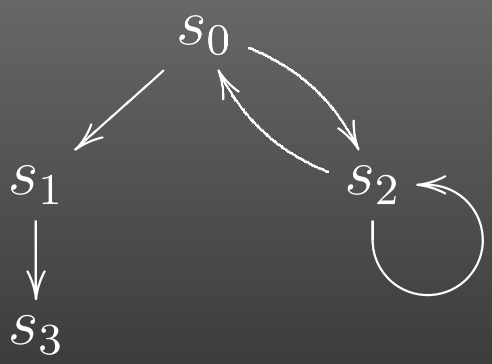
transition systems



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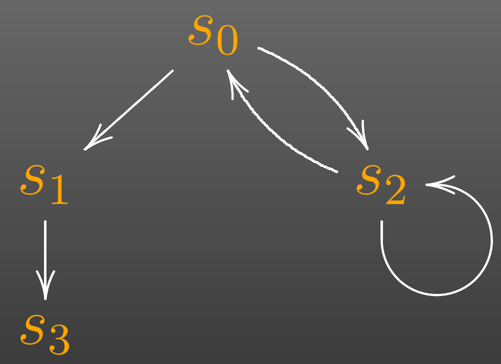

states $+$ transitions $\alpha : S \to \mathcal{P}(S)$

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Example models

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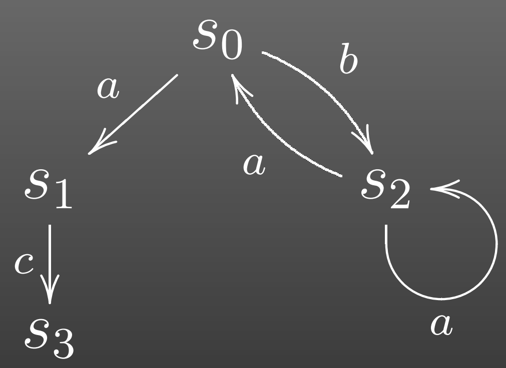

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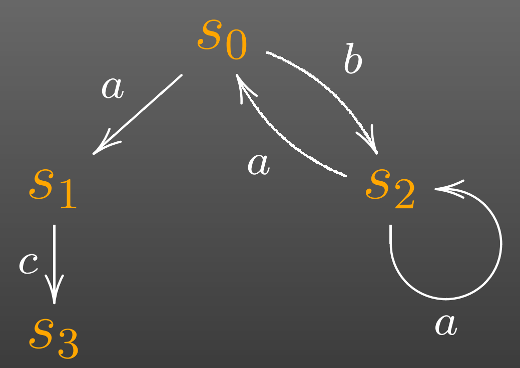

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- rich mathematical structure
- a uniform way for treating transition systems
- general notions and results, generic notion of bisimulation

Probabilistic systems

arise by enriching transition systems with (discrete) probabilities as labels on the transitions.

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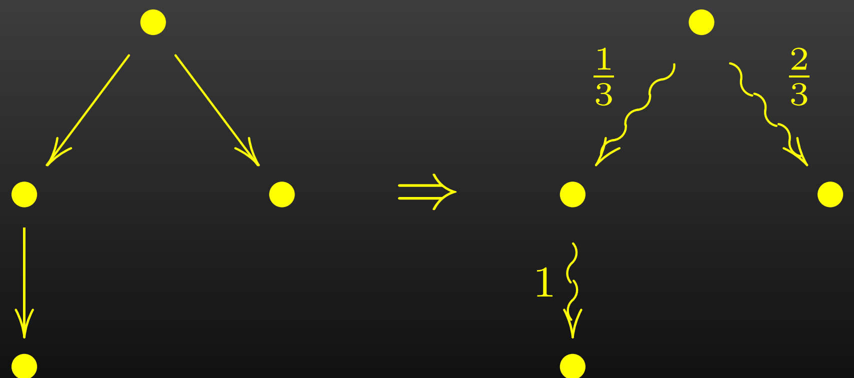

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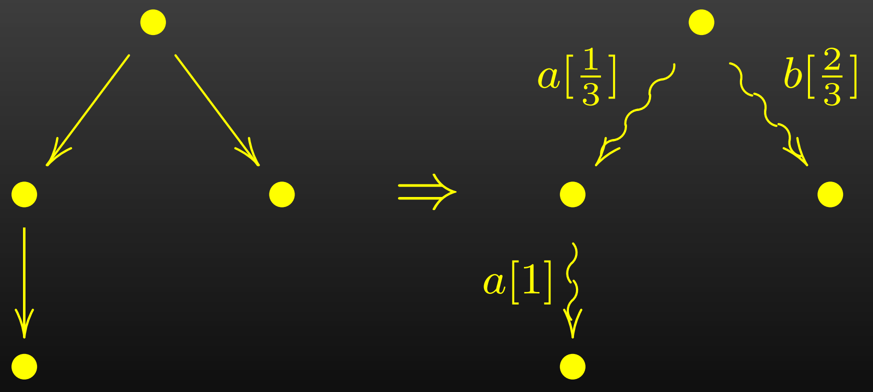

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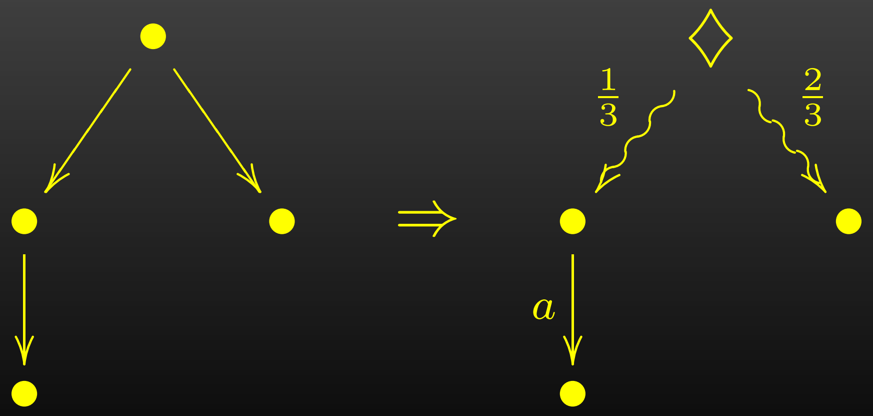

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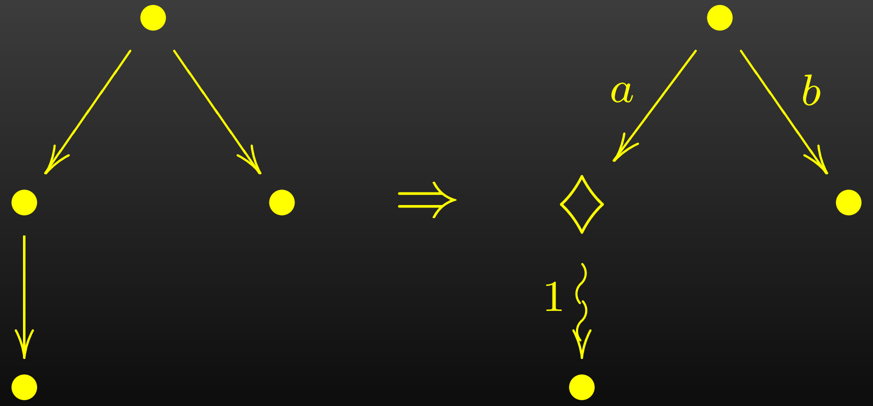

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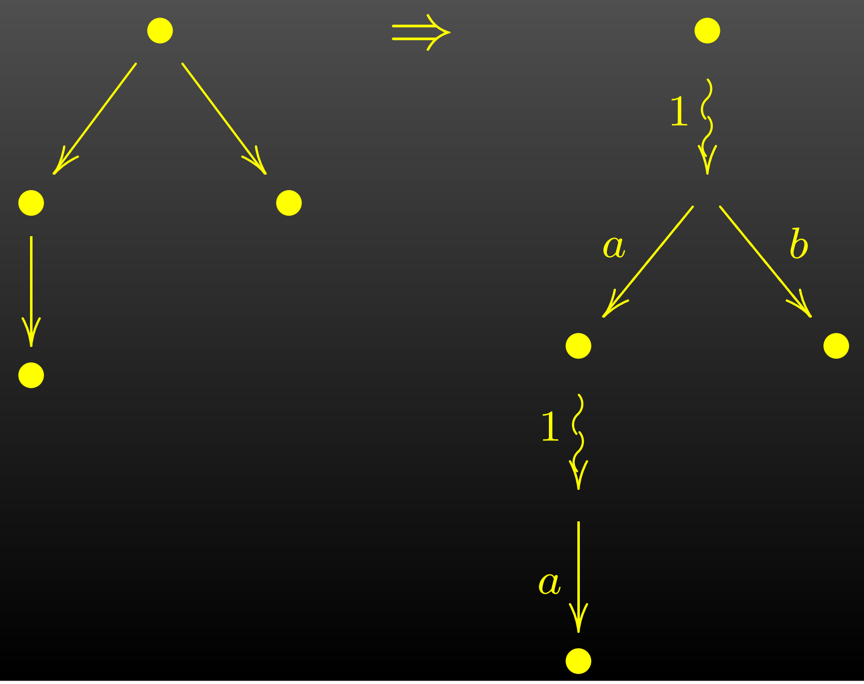

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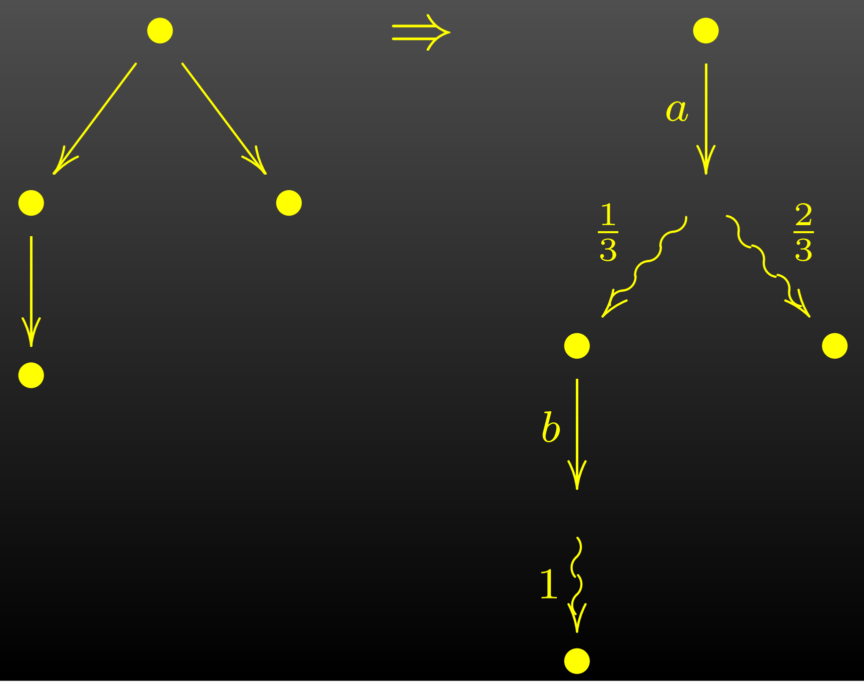

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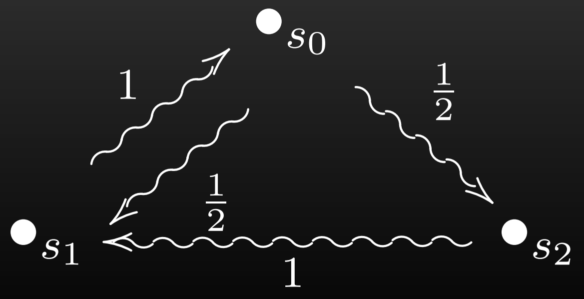
Example: $\alpha : S \to \mathcal{D}S$



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the probabilistic systems are also coalgebras
... of functors built by the following syntax

$$\mathcal{F} ::= \_ \mid A \mid \mathcal{P} \mid \mathcal{D} \mid \mathcal{G} + \mathcal{H} \mid \mathcal{G} \times \mathcal{H} \mid \mathcal{G}^A \mid \mathcal{G} \circ \mathcal{H}$$

Probabilistic system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
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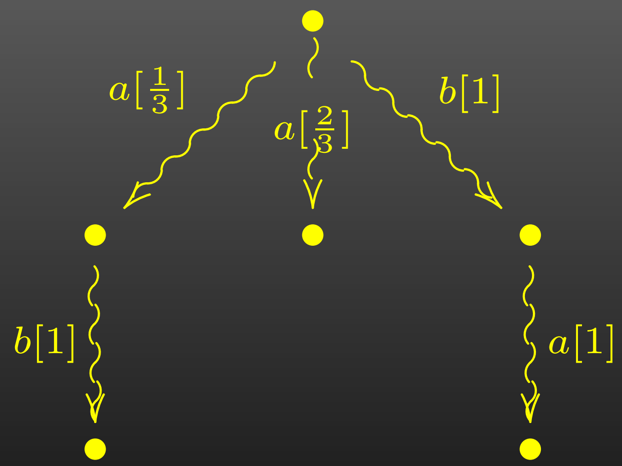

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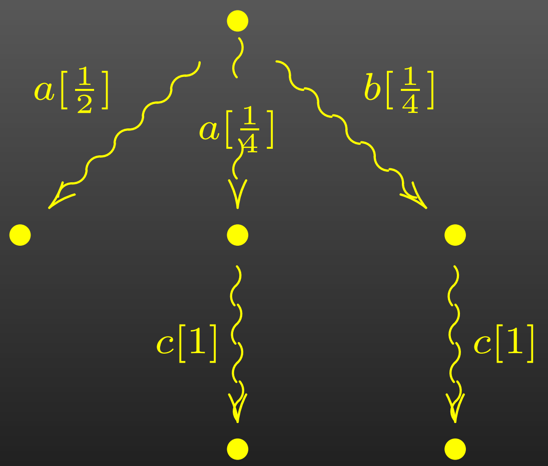

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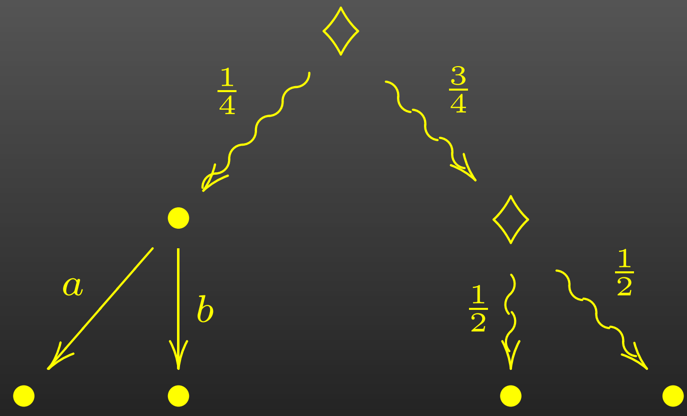

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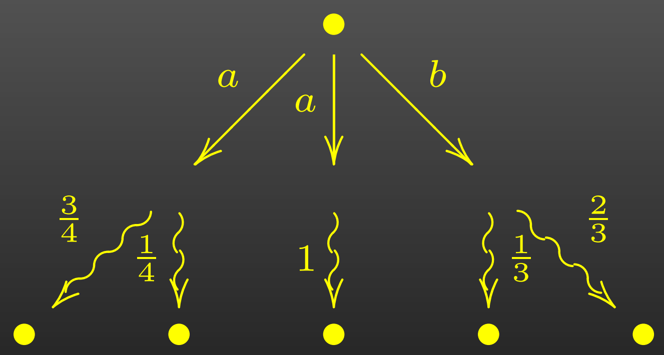

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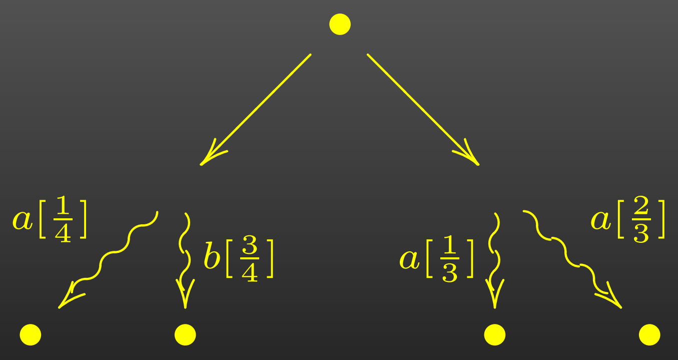

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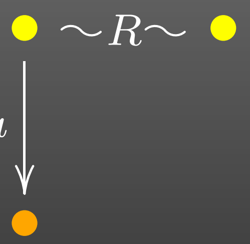

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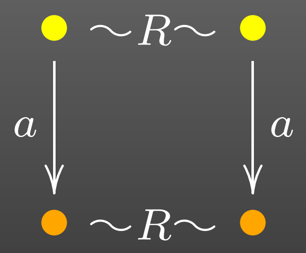

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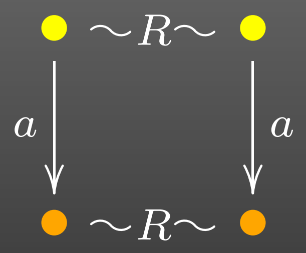

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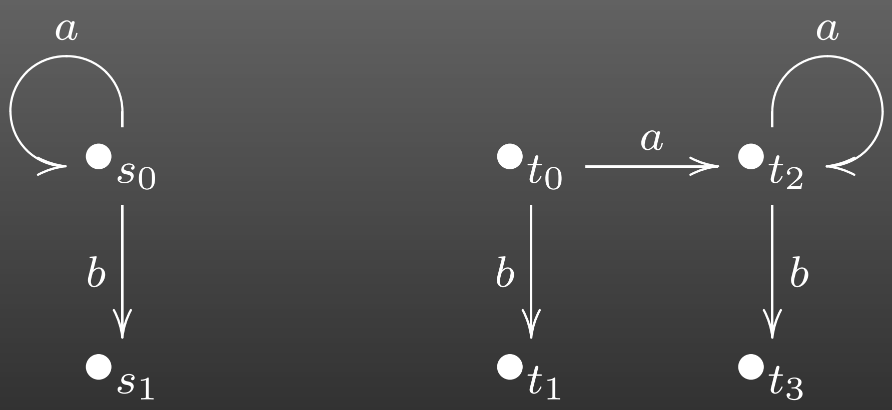
Bisimulation - LTS

Example: Consider the LTS



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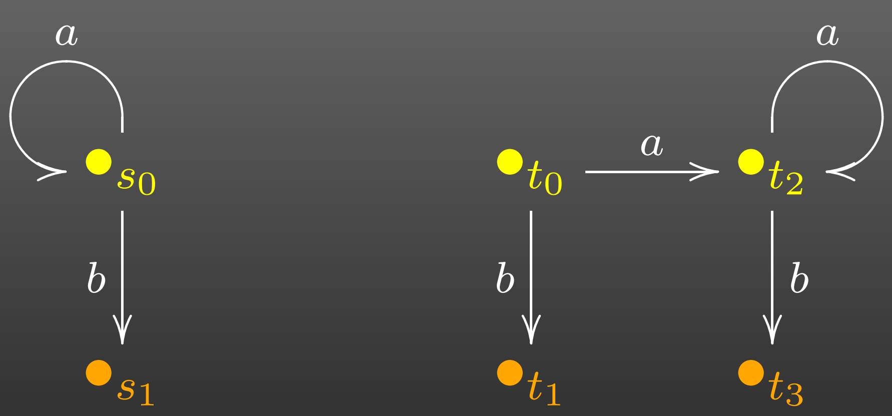

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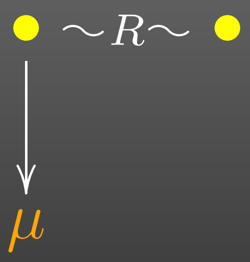

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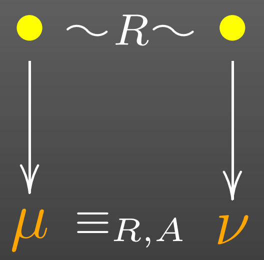

$\equiv_{R,A}$ relates distributions that assign the same probability to each label and each R-class

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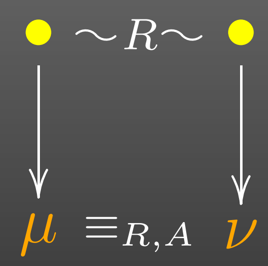

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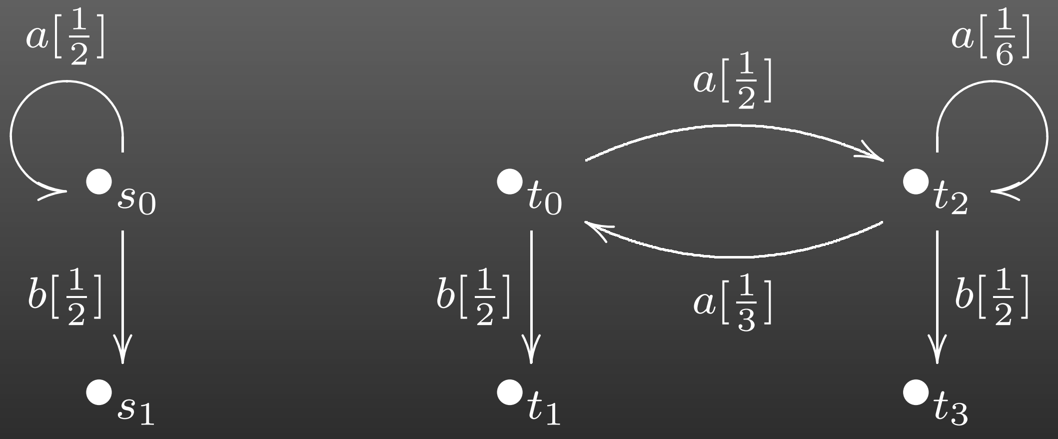

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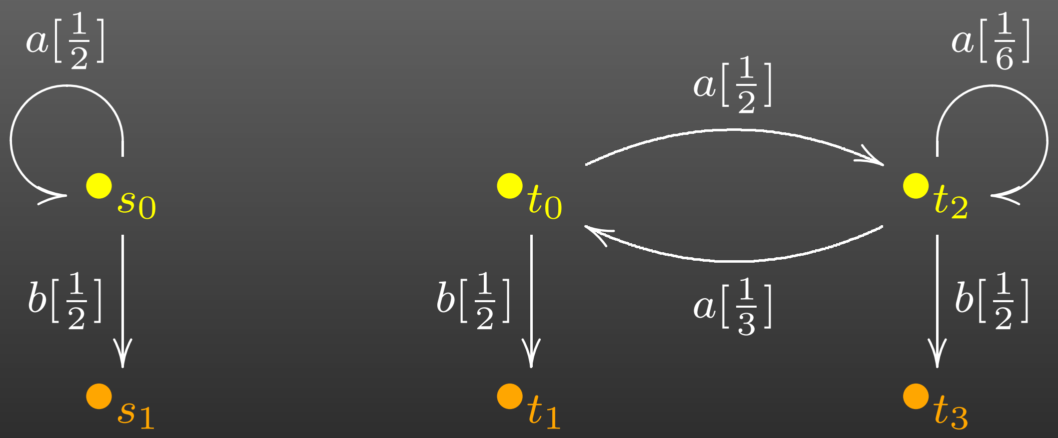

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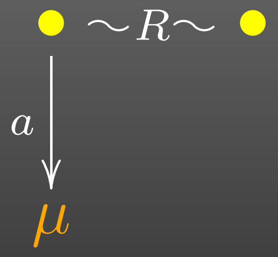

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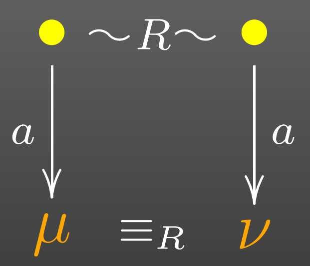

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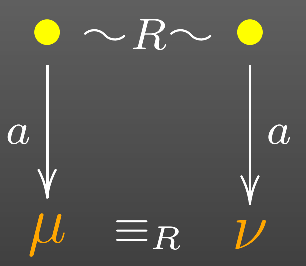

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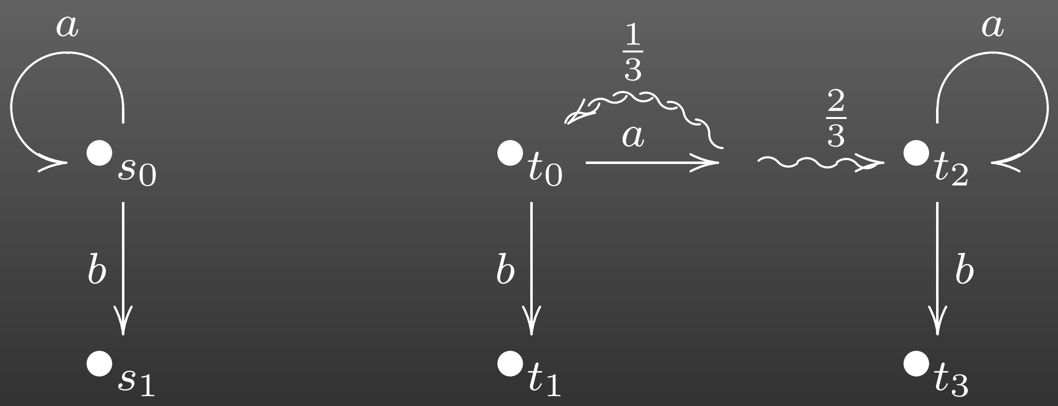

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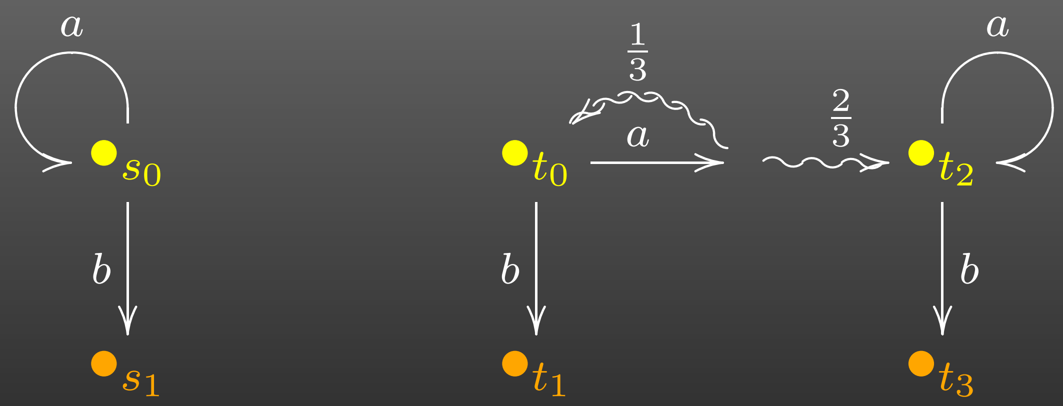

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Coalgebraic bisimulation

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle$$

is $R \subseteq S \times S$ such that

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is $R \subseteq S \times S$ such that γ exists:

$$\begin{array}{ccccc}
S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & S \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\
\mathcal{F}S & \xleftarrow[\mathcal{F}\pi_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}\pi_2]{} & \mathcal{F}S
\end{array}$$

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$$\bullet_s \rightsquigarrow R \rightsquigarrow \bullet_t$$

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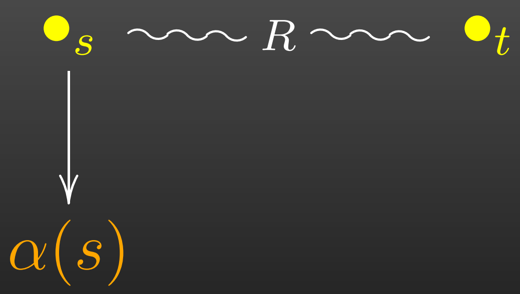

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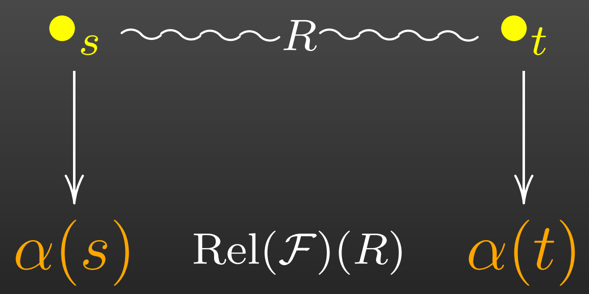

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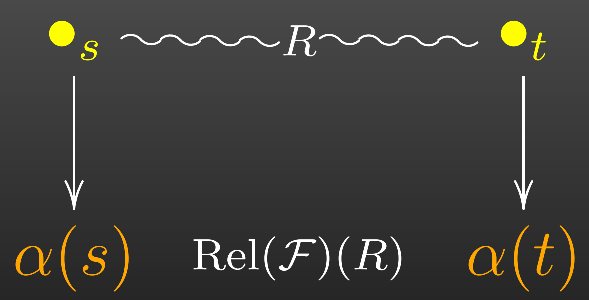

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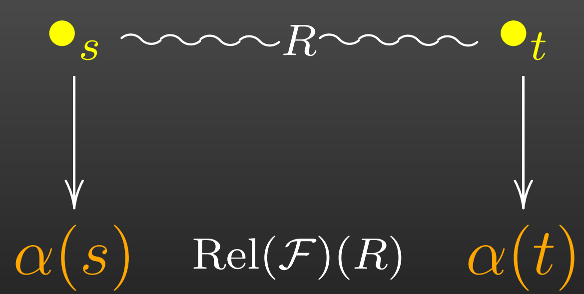
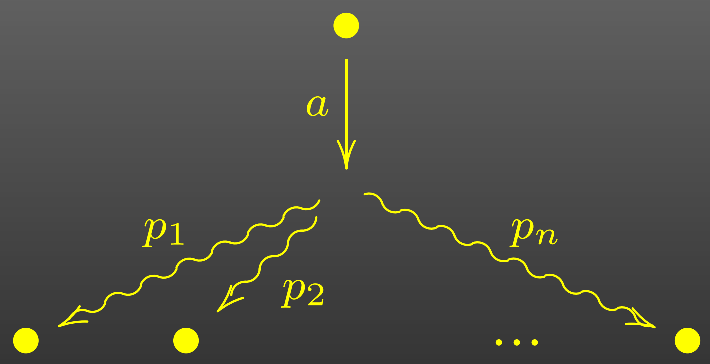


Theorem: Coalgebraic and concrete bisimilarity coincide !

Expressiveness

simple Segala system $\longrightarrow$ Segala system



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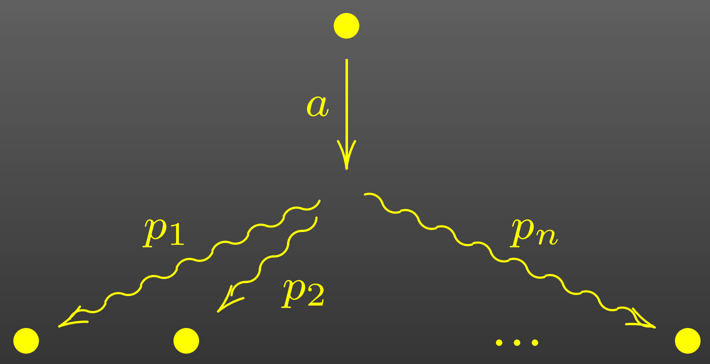

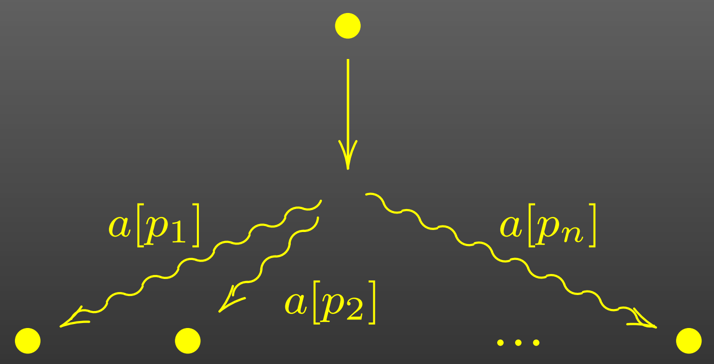

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simple Segala system $\longrightarrow$ Segala system

When do we consider one type of systems more expressive than another?

Comparison criterion

$$\mathsf{Coalg}_{\mathcal{F}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}$$

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states are bisimilar in the original system iff they are bisimilar in the translation

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proof via cocongruences - behavioral equivalence

Example translation

Indeed $\mathbf{SSeg} \rightarrow \mathbf{Seg}$ since

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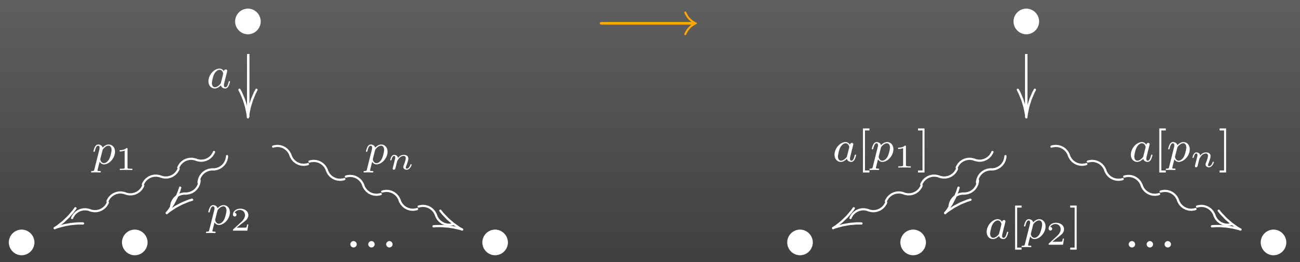

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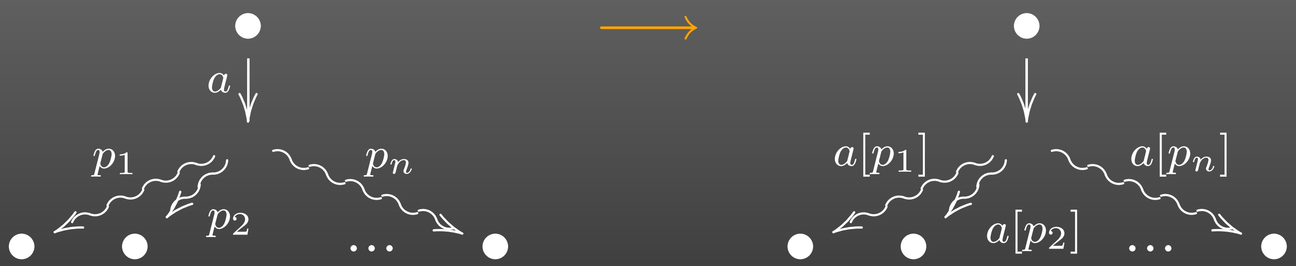
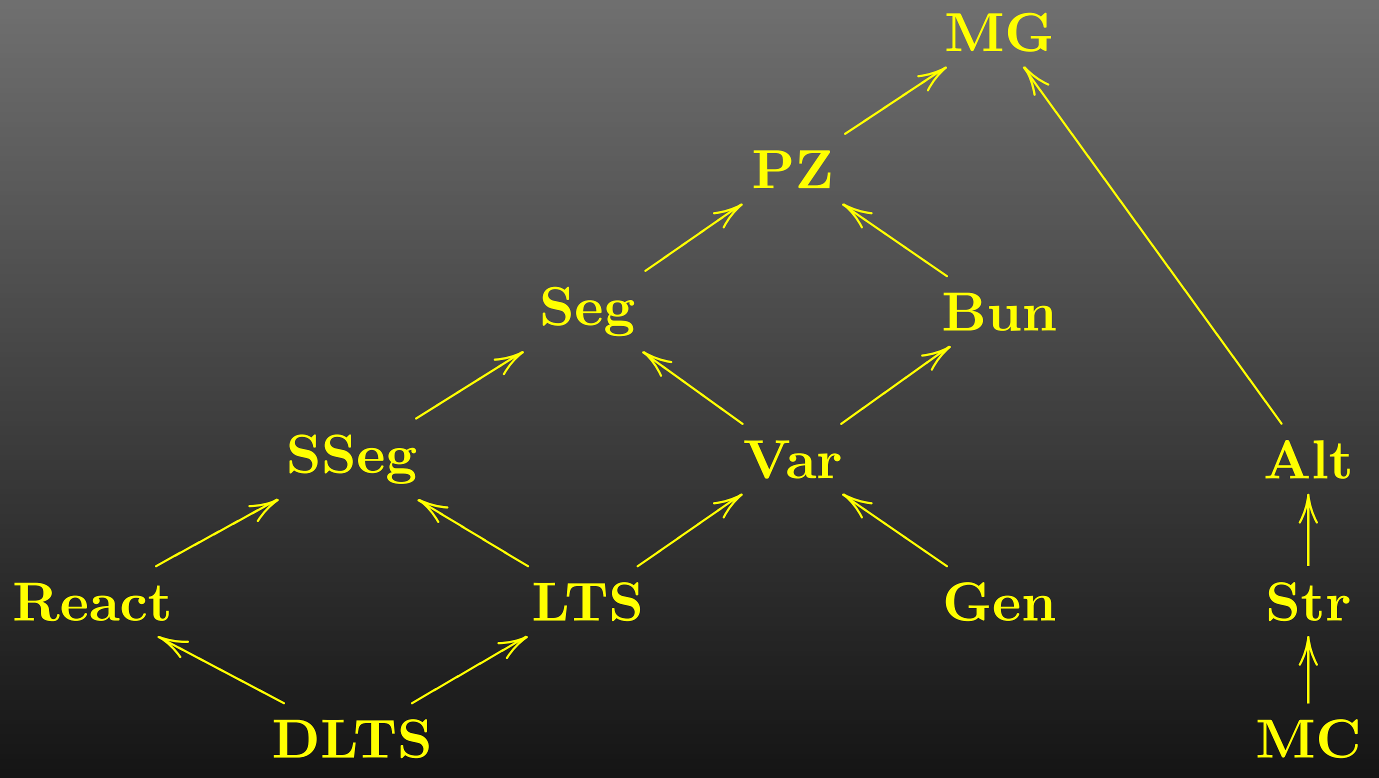


gives us an injective natural transformation

$$\mathcal{P}(A \times \mathcal{D}) \Longrightarrow \mathcal{P}\mathcal{D}(A \times \_\,)$$

The hierarchy...



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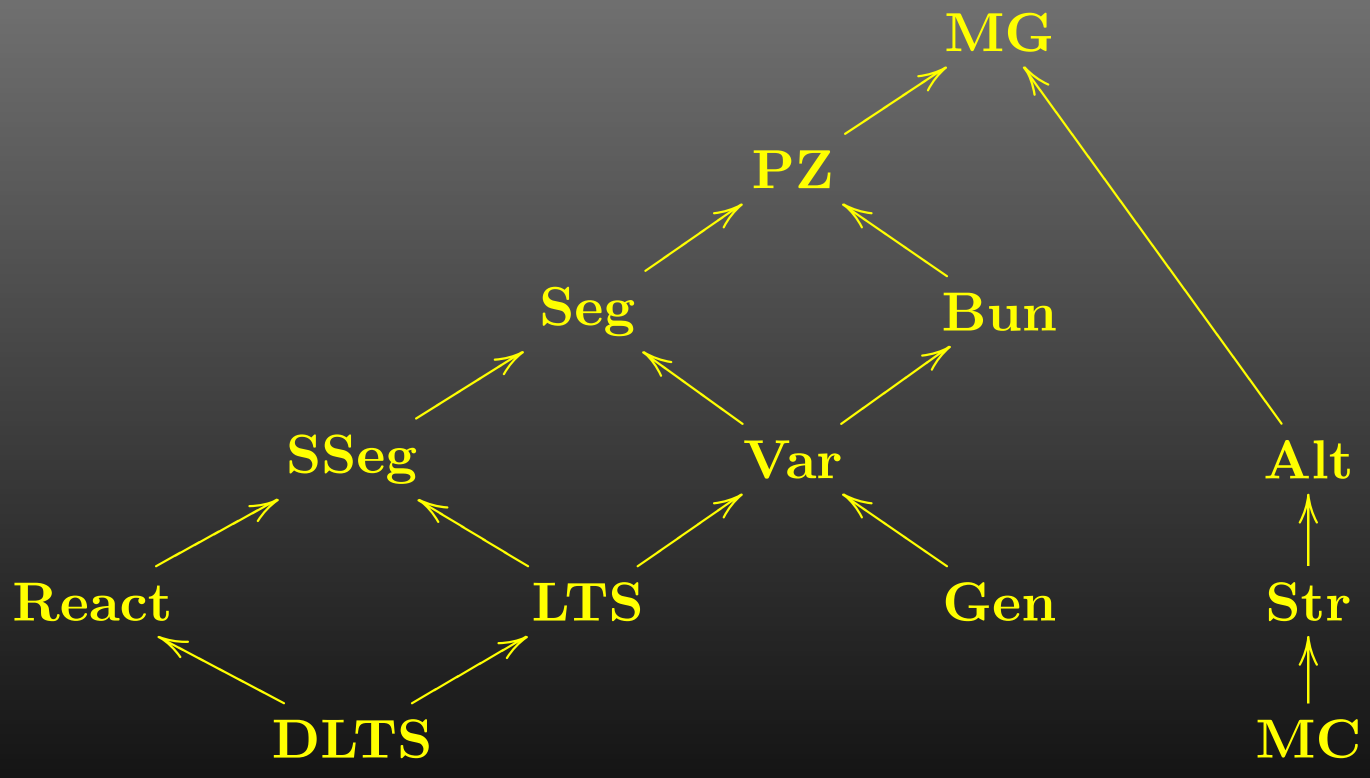


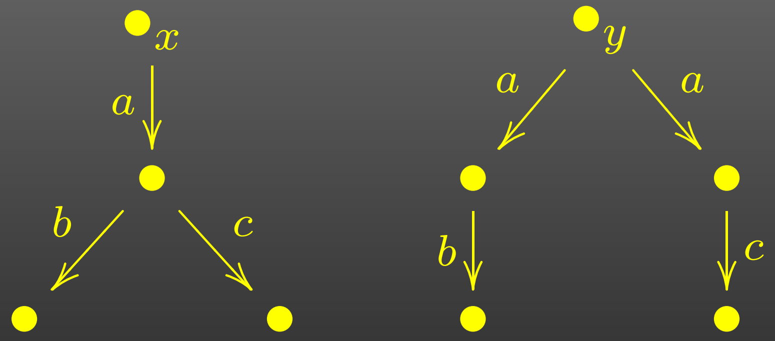
* Falk Bartels, AS, Erik de Vink

LT/BT spectrum

Bisimilarity is not the only semantics...

LT/BT spectrum

Are these non-deterministic systems equal ?



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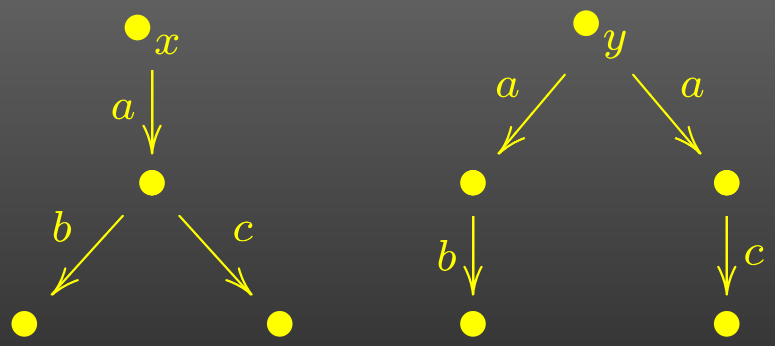

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- different wrt. bisimilarity

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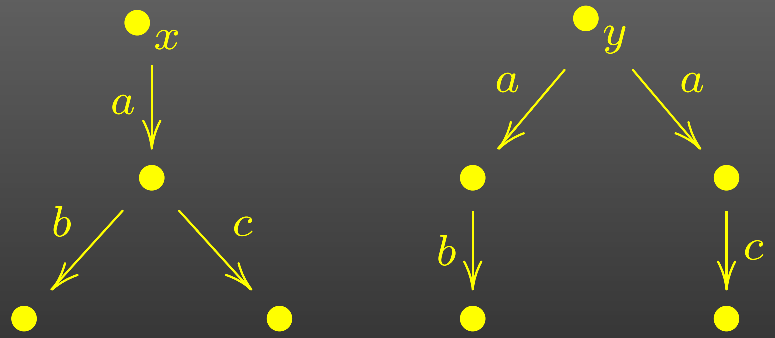

x and y are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

Traces - LTS

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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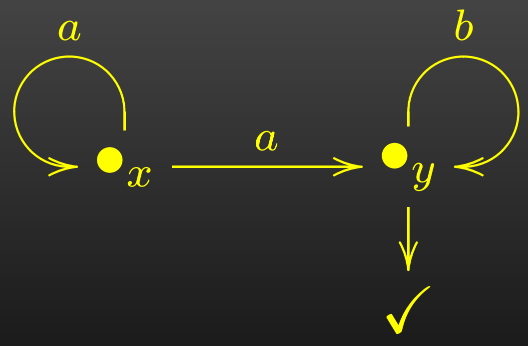

$$\mathrm{tr}(y) \;=\; b^*, \qquad \mathrm{tr}(x) \;=\; a^+ \cdot \mathrm{tr}(y) \;=\; a^+ \cdot b^*$$

Traces - generative

For generative probabilistic systems with ex. termination

trace = sub-probability distribution over possible linear behaviors

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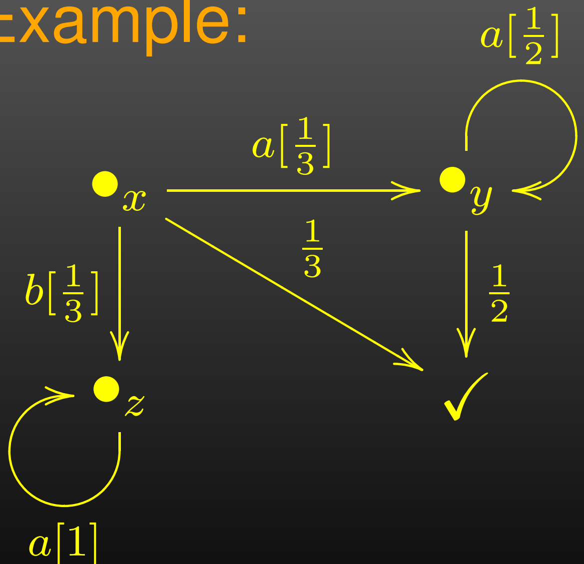

$$\mathrm{tr}(x) : \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3} \cdot \frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\cdots$$

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main idea: coinduction in a Kleisli category

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- comparison of systems is then easy
- we have built an expressiveness hierarchy w.r.t bisimulation semantics
- trace semantics can also be captured coalgebraically