Semantics meets Syntax in Coalgebra

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SYSMICS’19
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SYSMICS’19
I will tell you about:

1. Just the absolute basics of coalgebra
2. (Trace) semantics via determinisation…
3. …enabled by algebraic structure
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Mathematical framework based on category theory for state-based systems semantics
I will tell you about:

1. Just the absolute basics of coalgebra
2. (Trace) semantics via determinisation…
3. …enabled by algebraic structure

Mathematical framework based on category theory for state-based systems semantics for nondeterministic, probabilistic… systems
I will tell you about:

1. Just the absolute basics of coalgebra

2. (Trace) semantics via determinisation...

3. …enabled by algebraic structure

Mathematical framework based on category theory for state-based systems semantics

for nondeterministic, probabilistic… systems

systems with algebraic effects
I will tell you about:

1. Just the absolute basics of coalgebra

2. (Trace) semantics via determinisation...

3. …enabled by algebraic structure
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \overset{c}{\rightarrow} FX \]

states

object in the base category \( C \)
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]

- states
- behaviour type
- object in the base category $\mathcal{C}$
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]

- states
- behaviour type
- object in the base category $\mathbf{C}$
- functor on the base category $\mathbf{C}$
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]

- states
- behaviour type
- object in the base category \( \mathcal{C} \)
- functor on the base category \( \mathcal{C} \)
- form a category too
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[
X \overset{c}{\to} FX
\]

- states
- behaviour type
- object in the base category \( \mathcal{C} \)
- functor on the base category \( \mathcal{C} \)
- form a category too

\( \text{CoAlg}_\mathcal{C}(F) \)
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]

states

behaviour type

object in the base category \( C \)

functor on the base category \( C \)

form a category too

\[ \text{CoAlg}_C(F) \]

generic notion of behavioural equivalence (bisimilarity)

\( \approx \)
Examples
Examples

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]
Examples

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Rabin PA

\[ X \rightarrow [0,1] \times (\mathcal{P}_{\leq 1}X)^A \]
Examples

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Simple PA

\[ X \rightarrow ? \times (\mathcal{P}D^\infty X)^A \]

Rabin PA

\[ X \rightarrow [0, 1] \times (\mathcal{P}_{\leq 1} X)^A \]
Examples

NFA
\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Simple PA
\[ X \rightarrow \? \times (\mathcal{P}D_X)^A \]

Rabin PA
\[ X \rightarrow [0,1] \times (\mathcal{P}_{\leq 1}X)^A \]

systems with nondeterminism and probability
In general
In general

Systems

$X \rightarrow (MX)^A$

\[ a \quad \rightarrow \quad b \]
In general

Systems

\[ X \rightarrow (MX)^A \]

Labelled Transition Systems
In general

$$X \rightarrow (MX)^A$$

Labelled Transition Systems

with M-effects
In general

Systems

\( X \rightarrow (MX)^A \)

Labelled Transition Systems

with M-effects
In general

Systems

\[ X \rightarrow (MX)^A \]

Labelled Transition Systems

with M-effects

Automata

\[ X \rightarrow O \times (MX)^A \]
In general

**Systems**

\[ X \rightarrow (MX)^A \]

**Labelled Transition Systems**

with M-effects

**Automata**

\[ X \rightarrow O \times (MX)^A \]

with observations in O
In general

Systems

$X \rightarrow (MX)^A$

Labelled Transition Systems

with M-effects

Automata

$X \rightarrow O \times (MX)^A$

with observations in $O$

with M-effects

$\begin{align*}
  &a \\
  &b
\end{align*}$
For a monad $M$
For a monad $\mathcal{M}$
For a monad $M$ providing algebraic effects

\[ \mu : TT \Rightarrow T \]
\[ \eta : Id \Rightarrow T \]
For a monad $M$

NFA

$$X \rightarrow 2 \times (\mathcal{P}X)^A$$

Rabin PA

$$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$$

Simple PA

$$X \rightarrow ? \times (\mathcal{PD}X)^A$$

Providing algebraic effects

$$\mu: TT \Rightarrow T$$

$$\eta: Id \Rightarrow T$$
For a monad $M$ providing algebraic effects

**NFA**

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

**Rabin PA**

\[ X \rightarrow [0,1] \times (\mathcal{P}_{\leq 1}X)^A \]

**Simple PA**

\[ X \rightarrow ? \times (\mathcal{P}DX)^A \]

$M = \mathcal{P}$ for nondeterminism

$\mu: TT \Rightarrow T$

$\eta: Id \Rightarrow T$
For a monad $M$

- **NFA**
  \[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

- **Rabin PA**
  \[ X \rightarrow [0,1] \times (\mathcal{P}_{\leq 1}X)^A \]

- **Simple PA**
  \[ X \rightarrow ? \times (\mathcal{P}D\mathcal{O}X)^A \]

**M = \mathcal{P}**

- Providing algebraic effects
  \[ \mu: TT \Rightarrow T \]
  \[ \eta: Id \Rightarrow T \]

Powerset, subsets
For a monad $M$

For nondeterminism:

- NFA: $\mathcal{X} \rightarrow 2 \times (\mathcal{P}\mathcal{X})^A$

For probability:

- Rabin PA: $\mathcal{X} \rightarrow [0,1] \times (\mathcal{D}^{\leq 1}\mathcal{X})^A$

- Simple PA: $\mathcal{X} \rightarrow ? \times (\mathcal{P}\mathcal{D}\mathcal{X})^A$

$M = \mathcal{P}$ providing algebraic effects

$\mu: TT \Rightarrow T$

$\eta: \text{Id} \Rightarrow T$
For a monad $M$

**NFA**

$X \rightarrow 2 \times (\mathcal{P}X)^A$

- $M = \mathcal{P}$ for nondeterminism

**Rabin PA**

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

- $M = \mathcal{D}$ for probability

**Simple PA**

$X \rightarrow ? \times (\mathcal{PD}X)^A$

- $\mu: TT \Rightarrow T$
- $\eta: Id \Rightarrow T$

Powerset, subsets

Distributions

providing algebraic effects
For a monad $M$

- **NFA**
  $X \rightarrow 2 \times (\mathcal{P}X)^A$
  $M = \mathcal{P}$ for nondeterminism
- **Rabin PA**
  $X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$
  $M = \mathcal{D}$ for probability
- **Simple PA**
  $X \rightarrow ? \times (\mathcal{PD}X)^A$
  $M = \mathcal{PD}$ ??? for nondeterminism and probability

Providing algebraic effects:

- $\mu: TT \Rightarrow T$
- $\eta: Id \Rightarrow T$
For a monad M

NFA
\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

- \( M = \mathcal{P} \) for nondeterminism
- Powerset, subsets

Rabin PA
\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A \]

- \( M = \mathcal{D} \) for probability
- Distributions

Simple PA
\[ X \rightarrow ? \times (\mathcal{C}X)^A \]

- \( M = \mathcal{C} \) for nondeterminism and probability!

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For a monad $M$

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

$M = \mathcal{D}$
for probability

Distributions

Simple PA

$X \rightarrow ? \times (\mathcal{E}X)^A$

$M = \mathcal{E}$
for nondeterminism and probability !

providing algebraic effects
For a monad $M$

- **NFA**
  \[ X \rightarrow 2 \times (\mathcal{P}X)^A \]
  \[ M = \mathcal{P} \text{ for nondeterminism} \]

- **Rabin PA**
  \[ X \rightarrow [0,1] \times (\mathcal{D} \leq 1 X)^A \]
  \[ M = \mathcal{D} \text{ for probability} \]

- **Simple PA**
  \[ X \rightarrow ? \times (\mathcal{C}X)^A \]
  \[ M = \mathcal{C} \text{ for nondeterminism and probability} \]

*Providing algebraic effects*

- \[ \mu : TT \Rightarrow T \]
- \[ \eta : Id \Rightarrow T \]
For a monad $M$

- NFA: $X \rightarrow 2 \times (\mathcal{P}X)^A$ for nondeterminism
- Rabin PA: $X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$ for probability
- Simple PA: $X \rightarrow ? \times (\mathcal{C}X)^A$ for nondeterminism and probability

Providing algebraic effects:

$\mu: T T \Rightarrow T$
$\eta: \text{Id} \Rightarrow T$

Powerset, subsets

Distributions

Convex subsets of distributions
Semantics
Semantics

NFA = LTS + termination

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]
Semantics

NFA = LTS + termination

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Are the (top states of the) following systems equivalent?
Semantics

NFA = LTS + termination

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Semantics

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Are the (top states of the) following systems equivalent?

- no, they are not wrt. bisimilarity
- yes, they are wrt. trace equivalence as
Semantics

NFA = LTS + termination

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Are the (top states of the) following systems equivalent?

- no, they are not wrt. bisimilarity
- yes, they are wrt. trace equivalence as

\[ \text{tr}(x_1) = \text{tr}(y_1) = \{ab, ac\} \]
Semantics

NFA = LTS + termination

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\[ \text{tr}(x_1) = \text{tr}(y_1) = \{ab, ac\} \]

\[ \text{tr}: X \rightarrow \mathcal{P}(A^*) \]
Semantics

Rabin PA

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A$
Semantics

Rabin PA

$X \rightarrow [0,1] \times (\exists_{\leq 1} X)^A$

Are the (top states of the) following systems equivalent?
Are the (top states of the) following systems equivalent?

\[
\begin{align*}
X & \rightarrow [0,1] \times (\exists \leq 1 X)^A \\
& \quad \\
\begin{array}{c}
\text{SYSMICS'19}
\end{array}
\end{align*}
\]
Semantics

Rabin PA

\[ X \rightarrow [0,1] \times (\exists \leq 1 X)^A \]

Are the (top states of the) following systems equivalent?

- different wrt. bisimilarity
- equivalent wrt. trace equivalence as
Semantics

Rabin PA

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Are the (top states of the) following systems equivalent?

- different wrt. bisimilarity
- equivalent wrt. trace equivalence as

\[ \text{tr}(x_1) = \text{tr}(y_1) = \left( ab \mapsto \frac{1}{6}, ac \mapsto \frac{1}{8} \right) \]
Semantics

Rabin PA

\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A \]

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- different wrt. bisimilarity
- equivalent wrt. trace equivalence as

\[ \text{tr}(x_1) = \text{tr}(y_1) = \left( ab \mapsto \frac{1}{6}, ac \mapsto \frac{1}{8} \right) \]

\[ \text{tr} : X \rightarrow \mathcal{D}(A^*) \]
Trace semantics coalgebraically?

Two ideas:

1. unfold branching + transitions on words
2. trace = bisimilarity after determinisation
Trace semantics coalgebraically?

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NFA / LTS

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NFA / LTS

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NFA / LTS
Trace semantics coalgebraically?

Two ideas:

(1) unfold branching + transitions on words

(2) trace = bisimilarity after determinisation

NFA / LTS

monads!
Trace semantics coalgebraically
Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category
Trace semantics coalgebraically

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(1) modelling in a Kleisli category

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Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category

algebras of a monad $M$

Hasuo, Jacobs, S. LMCS ’07
Trace semantics coalgebraically

Two approaches:
(1) modelling in a Kleisli category
(2) modelling in an Eilenberg-Moore category
algebras of a monad $M$

Hasuo, Jacobs, S. LMCS ’07
Silva, Bonchi, Bonsangue, Rutten FSTTCS’10
Trace semantics coalgebraically

Two approaches:

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we can relate (1) and (2)

Hasuo, Jacobs, S. LMCS ’07
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algebras of a monad M
Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category

we can relate (1) and (2)
Traces via determinisation
Traces via determinisation

Automaton with M-effects

$$X \rightarrow O \times (MX)^A$$
Traces via determinisation

Automaton with M-effects

\[ X \rightarrow O \times (MX)^A \]
Traces via determinisation

Automaton with M-effects

\[ X \rightarrow O \times (MX)^A \]

Determinisation

\[ MX \rightarrow O \times (MX)^A \]
Traces via determinisation

Automaton with M-effects

\[ X \rightarrow O \times (MX)^A \]

Determinisation

\[ MX \rightarrow O \times (MX)^A \]

Trace = bisimilarity after determinisation
Traces via determinisation

Automaton with $M$-effects

$$X \rightarrow O \times (MX)^A$$

Determinisation

$$MX \rightarrow O \times (MX)^A$$

trace = bisimilarity after determinisation
Traces via determinisation

Automaton with M-effects

\[ X \rightarrow O \times (MX)^A \]

Determinisation

\[ MX \rightarrow O \times (MX)^A \]

trace = bisimilarity after determinisation

ideally we have a presentation

Algebras for M
Traces via determinisation

Automaton with M-effects

$X \rightarrow O \times (MX)^A$

Determinisation

$MX \rightarrow O \times (MX)^A$

O has to be an M-algebra!

trace = bisimilarity after determinisation

Algebras for M

ideally we have a presentation
Traces via determinisation

Automaton with M-effects

\[ X \to O \times (MX)^A \]

Determinisation

\[ MX \to O \times (MX)^A \]

O has to be an M-algebra!

trace = bisimilarity after determinisation

ideally we have a presentation

Algebras for M

Eilenberg-Moore algebras

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Eilenberg-Moore Algebras

abstractly

\[ \mathcal{EM}(M) \]

- **objects**
  
  \[
  MA \xrightarrow{a} MA
  \]

- **morphisms**
  
  satisfying

  \[
  A \xrightarrow{\eta} MA \\
  \mu : \text{natural transformation}
  \]

  \[
  A \xrightarrow{a} MA \\
  MA \xrightarrow{a} A
  \]

  \[
  MA \xrightarrow{a} MA \\
  MB \xrightarrow{a} MB
  \]

  \[
  A \xrightarrow{h} B
  \]
Traces via determinisation
Traces via determinisation

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]
Traces via determinisation

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$
Traces via determinisation

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

DFA

\[ \mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A \]
Traces via determinisation

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

\[ x_2 \xrightarrow{a} x_1 \]
\[ x_1 \xrightarrow{a} x_3 \]
\[ x_3 \xrightarrow{b} x_3 \]
\[ \star \]

DFA

\[ \mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A \]

\[ x_1 \xrightarrow{a} \]
\[ x_2 \xrightarrow{\oplus} x_3 \xrightarrow{b} x_3 \]
\[ \star \]

trace = bisimilarity after determinisation
Traces via determinisation

NFA
\[ \mathcal{X} \rightarrow 2 \times (\mathcal{P}X)^A \]

DFA
\[ \mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A \]

\[ \xrightarrow{a} x_1 \leftarrow x_2 \quad \xrightarrow{a} x_3 \leftarrow x_3 \]

\[ \xrightarrow{b} x_2 \oplus x_3 \rightarrow x_3 \]

trace = bisimilarity after determinisation

Algebras for \( \mathcal{P} \)
Traces via determinisation

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$

DFA

$\mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A$

trace = bisimilarity after determinisation

Algebras for $\mathcal{P}$

join semilattices with bottom
Traces via determinisation

Traces = bisimilarity after determinisation

Algebras for $P$ join semilattices with bottom

finite powerset!
Traces via determinisation

Rabin PA

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

\[
a, \frac{1}{2} \quad x_2 \quad b, \frac{1}{3} \quad x_4 \\
a, \frac{1}{4} \quad x_1 \quad x_3 \quad c, \frac{1}{2} \\
1 \quad x_2 \quad x_3 \quad x_5 \quad 1 \\
* \quad * \quad * \quad * 
\]
Traces via determinisation

Rabin PA

\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

- \( a, \frac{1}{2} \) \( \xrightarrow{x_1} \) \( a, \frac{1}{4} \)
- \( b, \frac{1}{3} \) \( \downarrow \)
- \( x_2 \)
- \( x_4 \)
- \( 1 \) \( \downarrow \) \( * \)

- \( x_3 \)
- \( x_5 \)
- \( 1 \) \( \downarrow \) \( * \)
Traces via determinisation

**Rabin PA**

\[ X \rightarrow [0, 1] \times (\exists_\leq X)^A \]

**DFA**

\[ \exists_\leq X \rightarrow [0, 1] \times (\exists_\leq X)^A \]
Traces via determinisation

Rabin PA
\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

DFA
\[ \mathcal{D}_{\leq 1} X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

trace = bisimilarity after determinisation
Traces via determinisation

Rabin PA

\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

DFA

\[ \mathcal{D}_{\leq 1} X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

trace = bisimilarity after determinisation

Algebras for \( \mathcal{D}_{(\leq 1)} \)
Traces via determinisation

Rabin PA
\[ \mathcal{X} \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}\mathcal{X})^A \]

DFA
\[ \mathcal{D}_{\leq 1}\mathcal{X} \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}\mathcal{X})^A \]

trace = bisimilarity after determinisation
Traces via determinisation

\[ X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

\[ \mathcal{D}_{\leq 1} X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1} X)^A \]

Rabin PA

DFA

\[ x_1 \Downarrow a \]
\[ \frac{1}{2} x_2 + \frac{1}{4} x_3 \Downarrow c \]
\[ \frac{1}{6} x_4 \Downarrow \frac{1}{8} \]
\[ * \]

\[ a, \frac{1}{2} \rightarrow x_1 \]
\[ b, \frac{1}{3} \Downarrow x_2 \]
\[ \frac{1}{6} x_4 \Downarrow * \]

(positive) convex algebras

finitely supported (sub)distributions!

Algebras for \( \mathcal{D}_{\leq 1} \)

trace = bisimilarity after determinisation
Traces via determinisation

Simple PA

\( X \rightarrow ? x (\xi x)^A \)
Traces via determinisation

Simple PA

\[ X \rightarrow ? x (\mathcal{E}X)^A \]
Traces via determinisation

Simple PA

\[ X \rightarrow ? \times (cX)^A \]

DFA

\[ cX \rightarrow ? \times (cX)^A \]

\[ x_1 \xrightarrow{\frac{1}{3}} x_2 \xrightarrow{\frac{2}{3}} \]
\[ b \xrightarrow{\frac{1}{2}} a \]

\[ \frac{1}{3} x_2 + \frac{2}{3} x_3 \oplus \frac{1}{2} x_3 + \frac{1}{2} x_4 \]
Traces via determinisation

Simple PA

\[ X \rightarrow ? \times (\varepsilon X)^A \]

DFA

\[ eX \rightarrow ? \times (\varepsilon X)^A \]

\[ (\frac{1}{3} x_2 + \frac{2}{3} x_3) \oplus (\frac{1}{2} x_3 + \frac{1}{2} x_4) \]

trace = bisimilarity after determinisation
Traces via determinisation

Simple PA

\[ X \rightarrow ? \times (eX)^A \]

DFA

\[ eX \rightarrow ? \times (eX)^A \]

\[ x_1 \xrightarrow{b} \]

\[ (\frac{1}{3}x_2 + \frac{2}{3}x_3) \oplus (\frac{1}{2}x_3 + \frac{1}{2}x_4) \]

trace = bisimilarity after determinisation
Traces via determinisation

Simple PA

\[ X \to ? \times (\epsilon X)^A \]

DFA

\[ \epsilon X \to ? \times (\epsilon X)^A \]

\[ (\frac{1}{3}x_2 + \frac{2}{3}x_3) \oplus (\frac{1}{2}x_3 + \frac{1}{2}x_4) \]

trace = bisimilarity after determinisation

Algebras for C

convex semilattices
Traces via determinisation

Simple PA

$X \rightarrow ? \times (\epsilon X)^A$

DFA

$\epsilon X \rightarrow ? \times (\epsilon X)^A$

trace = bisimilarity after determinisation

Algebras for $C$

convex semilattices

finitely generated convex sets of distributives
Presentation for $C$

- Algebras for $C$
- Convex semilattices
- Finitely generated convex sets of distr…
Presentation for $\mathcal{C}$

- Algebras for $\mathcal{C}$
- Convex semilattices
- Finitely generated convex sets of distr...

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Algebras for \( c \)

convex semilattices

finitely generated convex sets of distr…

\[ \Delta = (A, \oplus, +_p) \]

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Algebras for $c$ - convex semilattices

Finitely generated convex sets of distributions

$\mathbb{A} = (A, \oplus, +_p)$

$p \in (0, 1)$

Bonchi, S., Vignudelli '19
### Presentation for $c$

**Algebras for $c$**

- **convex semilattices**
- finitely generated convex sets of distributions

**Bonchi, S., Vignudelli ‘19**

**Example Algebras**

\[ \mathbb{A} = (A, \oplus, +_p) \]

- \( p \in (0, 1) \)

**Axioms**

\[
\begin{align*}
(x \oplus y) \oplus z &\quad \overset{(A)}{=} \quad x \oplus (y \oplus z) \\
x \oplus y &\quad \overset{(C)}{=} \quad y \oplus x \\
x \oplus x &\quad \overset{(I)}{=} \quad x
\end{align*}
\]

\[
\begin{align*}
(x +_p y) +_p z &\quad \overset{(A_p)}{=} \quad x +_p (y +_p (1-\frac{1}{p}) \cdot z) \\
x +_p y &\quad \overset{(C_p)}{=} \quad y +_p (1-p) \cdot x \\
x +_p x &\quad \overset{(I_p)}{=} \quad x
\end{align*}
\]

\[
(x \oplus y) +_p z \quad \overset{(D)}{=} \quad (x +_p z) \oplus (y +_p z)
\]
Presentation for $C$

Algebras for $C$

convex semilattices

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finitely generated convex sets of distributions

$\Delta = (A, \oplus, +_p)$

$p \in (0, 1)$

\[(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)\]
\[x \oplus y \overset{(C)}{=} y \oplus x\]
\[x \oplus x \overset{(I)}{=} x\]

\[(x +_q y) +_p z \overset{(A_p)}{=} x +_p (y +_p (1-p)z)\]
\[x +_p y \overset{(C_p)}{=} y +_1 x\]
\[x +_p x \overset{(I_p)}{=} x\]

\[(x \oplus y) +_p z \overset{(D)}{=} (x +_p z) \oplus (y +_p z)\]
Presentation for $\mathcal{C}$

Algebras for $\mathcal{C}$

convex semilattices

Bonchi, S., Vignudelli ‘19

finitely generated convex sets of distributions

$p \in (0, 1)$

$\Delta = (A, \oplus, +p)$

<table>
<thead>
<tr>
<th>Equation</th>
<th>Significance</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x \oplus y) \oplus z \overset{(A)}{=} x \oplus (y \oplus z)$</td>
<td>Additive closure</td>
</tr>
<tr>
<td>$x \oplus y \overset{(C)}{=} y \oplus x$</td>
<td>Commutativity</td>
</tr>
<tr>
<td>$x \oplus x \overset{(I)}{=} x$</td>
<td>Identity</td>
</tr>
</tbody>
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<tr>
<td>$(x \oplus q \cdot y) \oplus p \cdot z \overset{(A_p)}{=} x \oplus p \cdot (y + p \cdot \frac{(1-q)\cdot z}{1-pq})$</td>
<td>Additive closure over probability</td>
</tr>
<tr>
<td>$x \oplus_p y \overset{(C_p)}{=} y + (1-p) \cdot x$</td>
<td>Convex combination</td>
</tr>
<tr>
<td>$x \oplus_p x \overset{(I_p)}{=} x$</td>
<td>Identity</td>
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<td>$(x \oplus y) \oplus_p z \overset{(D)}{=} (x \oplus_p z) \oplus (y \oplus_p z)$</td>
<td>Distributivity</td>
</tr>
</tbody>
</table>

semilattice

convex algebra
In this paper we care for both categorical algebra, algebras is isomorphic to \( h_A \). Hence, an algebra is the Minkowski sum of two convex sets.

The theory for nondeterminism and probability terms in \( \varsigma \), cf. [26, 27], provides a presentation for the monad \( \mu \). We refer to this theory leading to the theory of \( \delta \) for nondeterminism and probability. However, we could not find a proof of \( \nu \). In such a case we also say that \( \lambda \) is generated by its subset \( \gamma \), i.e., the smallest convex set that \( \zeta \) can be seen as the quotient of monads \( \theta \) and all \( \epsilon \). Hence, \( \omega \) is a monad map, also \( \zeta \) makes finding examples of algebras and working with them particularly useful in examples and our further developments. Note that commutes appropriately with the \( \zeta \) and \( \beta \).

\[
\Delta = (A, \oplus, +p)
\]

\( p \in (0, 1) \)

Algebras for \( \mathcal{C} \)

convex semilattices

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S., Woracek '15, '17, '18

Fig. 1. NPLTS, cf. [30, 31].

\[
\begin{align*}
(x \oplus y) \oplus z & \overset{(A)}{=} x \oplus (y \oplus z) \\
x \oplus y & \overset{(C)}{=} y \oplus x \\
x \oplus x & \overset{(I)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x +_q y) +_p z & \overset{(A_p)}{=} x +_p q (y + p(1-q) z) \\
x +_p y & \overset{(C_p)}{=} y +_1-p x \\
x +_p x & \overset{(I_p)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x \oplus y) +_p z & \overset{(D)}{=} (x +_p z) \oplus (y +_p z)
\end{align*}
\]
**Presentation for** \( C \)

**Algebras for** \( C \)

\[ \Delta = (A, \oplus, +_p) \]

\[ p \in (0, 1) \]

**convex semilattices**

\[
\begin{align*}
(x \oplus y) \oplus z & \overset{(A)}{=} x \oplus (y \oplus z) \\
 x \oplus y & \overset{(C)}{=} y \oplus x \\
 x \oplus x & \overset{(I)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x +_q y) +_p z & \overset{(A_p)}{=} x +_p (y +_{p(1-q)} z) \\
x +_p y & \overset{(C_p)}{=} y +_{1-p} x \\
x +_p x & \overset{(I_p)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x \oplus y) +_p z & \overset{(D)}{=} (x +_p z) \oplus (y +_p z)
\end{align*}
\]

**Bonchi, S., Vignudelli ‘19**

**S., Woracek ‘15, ’17, ‘18**

**Semitecture**

**Convex algebra**

**Distributivity**
Three things to take home:

1. Semantics via determinisation is easy for systems / automata with M-effects

2. Having a presentation for M gives us syntax

3. Having the syntax makes determinisation natural!

Many general properties follow also a sound up-to context proof technique.
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Thank You!