Coalgebraic Weak Bisimulation
for Action-Type Systems

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Abstract

We propose a coalgebraic definition of weak bisimulation for classes of coalgebras obtained from bifunctors in the category Set. Weak bisimilarity for a system is obtained as strong bisimilarity of a transformed system. The particular transformation consists of two steps: First, the behavior on actions is lifted to behavior on finite words. Second, the behavior on finite words is taken modulo the hiding of internal or invisible actions, yielding behavior on equivalence classes of words closed under silent steps. The coalgebraic definition is validated by two correspondence results: one for the classical notion of weak bisimulation of Milner, another for the notion of weak bisimulation for generative probabilistic transition systems as advocated by Baier and Hermanns.

1 Introduction

We present a definition of weak bisimulation for action type systems based on the general coalgebraic apparatus of bisimulation [1, 21, 38]. Action-type systems are systems that arise from bifunctors in the category Set. A typical and familiar example of an action-type system is a labelled transition system (LTS) (see, e.g., [22, 33]), but also many types of probabilistic systems (see,

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e.g., [24, 40, 16, 7, 39]) fall into this class. Informally, an action-type system in Set is a coalgebra that performs actions from a set A.

For the verification of system properties, behavior equivalences are often employed. One such behavior equivalence is strong bisimilarity. However strong bisimilarity is often too strong an equivalence. Weak bisimilarity, originally defined for LTSs in the work of Milner [28, 30], is a looser equivalence on systems that abstracts away from internal or invisible steps. In fact, weak bisimilarity for a labelled transition system $\mathcal{S}$ amounts to strong bisimilarity on the 'double-arrowed' system $\mathcal{S}'$ induced by $\mathcal{S}$. In fact, the 'double-arrowed' system is the original system saturated with invisible steps. We generalize this idea for a coalgebraic definition of weak bisimulation. Our approach, given a system $\mathcal{S}$, consists of two stages.

1. First, we define a '$*$-extension' $\mathcal{S}'$ of $\mathcal{S}$ which is a system with the same carrier as $\mathcal{S}$, but with action set A^*, the set of all finite words over A. The system $\mathcal{S}'$ captures the behavior of $\mathcal{S}$ on finite traces.

2. Next, given a set of invisible actions $\tau \subseteq A$, we transform $\mathcal{S}'$ into a so-called 'weak τ-extension' $\mathcal{S}''$ which abstracts away from τ steps. Then we define weak bisimilarity on $\mathcal{S}$ as strong bisimilarity on the weak-τ-extension $\mathcal{S}''$.

Defining weak bisimulation for coalgebras has been studied before. There is early work by Rutten on weak bisimulation for while programs [37], succeeded by a syntactic approach to weak bisimulation by Rothe [35]. In the latter paper, weak bisimulation for a particular class of coalgebras was obtained by transforming a coalgebra into an LTS and making use of Milner's weak bisimulation there. This approach also supports a definition of weak homomorphisms and weak simulation relations. Later, in the work of Rothe and Mašulović [36], a complex, but interesting coalgebraic theory was developed leading to weak bisimulation for functors that weakly preserve pullbacks. They also consider a chosen 'observer' and hidden parts of a functor. However, in the case of probabilistic and similar systems, this does not lead to intuitive results and cannot be related to the concrete notions of weak bisimulation. The so-called skip relations used in [36] seem to be the major obstacle as it remains unclear how quantitative information can be incorporated. In the context of open maps, a category theoretical interpretation of weak bisimulation on presheaf models has been proposed in [15]. Recent work [34] shows that weak bisimilarity for LTSs can be captured in a semantic domain involving traces and coalgebraic finality.

Indeed, the two-phase approach of defining weak bisimilarity for general systems is, amplifying Milner's original idea, rather natural. Our proposal for weak bisimilarity of action-type systems builds on the intuition in concrete cases. A drawback of our approach is that the definition of weak bisimulation is parametrized with a notion of a *-extension that does not come from a general categorical construction, but has to be tuned for the concrete type of systems at hand.

In this paper we focus on two particular examples of action-type systems: LTSs and the generative probabilistic systems [17, 16, 42]. The generative systems are closely related to LTSs, the difference is that all non-deterministic choices in an LTS are probabilistic choices in a generative system.

For LTSs, weak bisimulation is an established notion and the main motivation of the paper is to generalize this notion to coalgebras, as arbitrary as possible. Baier and Hermanns introduced, rather appealingly, the notion of weak bisimulation for generative probabilistic systems [7, 6, 8]. In this paper, we propose a notion of weak bisimulation at a high-level of abstraction that justifies the definition of Baier and Hermanns for generative systems and illuminates the similarity between the notion of weak bisimulation for LTSs and of weak bisimulation for generative systems.

In the context of concrete probabilistic transition systems, there have been several other proposals for a notion of weak bisimulation, often relying on the particular model under consideration. For a detailed study of the different probabilistic models the reader is referred to [10, 11, 43, 42]. Segala [40, 39] proposes four notions of weak relations for his model of simple probabilistic automata. A detailed study of these relations can be found in [45]. It is a topic for further research to see how these notions fit into our general framework. Several groups of authors studied weak equivalences for the so-called alternating model of Hansson [20]. Philippou, Lee and Sokolsky [32] proposed the first notion of weak bisimulation in this setting. This work was extended to infinite systems by Desharnais, Gupta, Jagadeesan and Panangaden [14]. The same authors also provided a metric analogue of weak bisimulation [13]. Recently, Andova and Willemse studied branching bisimulation for the alternating model [4, 5], and together with Baeten [3] provided a complete axiomatization of this process equivalence in a process algebra setting. However, the alternating probabilistic automata are not coalgebras (see [42]) and therefore do not qualify for our definition.

Weak bisimulation was also considered for Markov chains in both discrete time [9, 41] and continuous time [9, 27]. Markov chains are not exactly action type coalgebras, since they are fully probabilistic non-labelled

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systems. However, the notion of weak bisimulation from [41] is based on the notion of weak bisimulation for generative probabilistic systems that is central to our paper. It is interesting to note that the notion of weak bisimulation by Baier and Hermanns has attracted attention in the security community and has been applied to security issues such as non-interference and secure information flow [2, 41, 23]. For the latter paper [23], as we will see for the present paper too, the coincidence of weak bisimulation and branching bisimulation in the setting of generative systems is crucial. Transition systems with both actions and generally distributed time delay occurring as labels are studied in [25] as well as a notion of weak bisimulation taking non-deterministic and sequential composition into account.

Below, we prove, not only for the case of labelled transition systems, but also for generative probabilistic systems that our coalgebraic definition corresponds to the concrete one of [30] and [7]. Despite the appeal of the coalgebraic definition of weak bisimulation, the proofs of correspondence results vary from straightforward to technically involved. For example, the relevant theorem for labelled transition systems takes less than a page, whereas proving the correspondence result for generative probabilistic systems takes in its present form more than twenty pages (additional machinery included).

The paper is organized as follows: Section 2 gathers the preliminary definitions and results. Section 3 is the kernel of the paper presenting the definition of coalgebraic weak bisimulation. We show that our definition of weak bisimilarity leads to Milner's weak bisimilarity for LTSs in Section 4. Section 5 is devoted to the correspondence result for the class of generative systems of the notion of weak bisimilarity of Baier and Hermanns and our coalgebraic definition. This section is a technically involved part of the paper and is divided in several parts, discussing in detail generative probabilistic systems and their concrete and coalgebraic weak bisimulation. In Section 5.1 we study some basic notions, such as paths and cones of generative systems, and their properties. Section 5.2 establishes that the probability distributions defining a generative probabilistic system extend to measures on a certain σ-algebra of paths. In Section 5.3 we present the concrete definitions of weak bisimulation for generative systems by Baier and Hermanns, as well as branching bisimulation, and we gather and prove some properties of these relations (in concrete terms) that we need for our correspondence result. Section 5.4 presents the coalgebraic weak bisimulation for generative probabilistic systems which in Section 5.5 is compared to the concrete notion of weak bisimulation. At the end, Section 6 draws some conclusions. Last, but not least, one will find several appendices. The theme that connects them is the notion of weak pullback preservation—a technical condition

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that is helpful in relating concrete and coalgebraic bisimulations. We recall the definitions of pullbacks and their preservation in Appendix A. We prove weak pullback preservation of the distribution functor (without restricting to finite support) in Appendix B. This is an interesting side-contribution of the paper. Its place is in an appendix in order not to distract the main line of the story. In Appendix C we investigate the weak pullback preservation of the functor appearing in Section 5. Interestingly, this functor does not preserve weak pullbacks, but it preserves total weak pullbacks, a notion that turns out to be important in our investigations.

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2   Systems and bisimilarity

We are treating systems from a coalgebraic point of view. Usually, in this context, a system is considered a coalgebra of a given Set endofunctor. For an introduction to the theory of coalgebra the reader is referred to the introductory articles by Rutten, Jacobs, and Gumm [38, 21, 18]. However, in our investigation of weak bisimilarity it is essential to explicitly specify the set of executable actions. Therefore we shall rather start from a so-called bifunctor instead of a Set endofunctor, cf [12, 26].

A *bifunctor* is any functor $\mathcal{F}\colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set}$. If $\mathcal{F}$ is a bifunctor and A is a fixed set, then a Set endofunctor $\mathcal{F}_A$ is defined by

$$\mathcal{F}_A S = \mathcal{F}(A, S), \qquad \mathcal{F}_A f = \mathcal{F}\langle id_A, f\rangle \text{ for } f\colon S \to T. \qquad (1)$$

We formulate the next simple proposition for further reference.

Proposition 1 *Let $\mathcal{F}$ be a bifunctor, and let A_1, A_2 be two fixed sets and $f : A_1 \to A_2$ a mapping. Then f induces a natural transformation $\eta^f : \mathcal{F}_{A_1} \Rightarrow \mathcal{F}_{A_2}$ defined by $\eta^f_S = \mathcal{F}\langle f, id_S\rangle$.* □

We next define action-type coalgebras i.e. action-type systems based on bifunctors.

Definition 1 *Let $\mathcal{F}$ be a bifunctor. If S and A are sets and α is a function, $\alpha : S \to \mathcal{F}_A(S)$, then the triple $\langle S, A, \alpha\rangle$ is called an action type $\mathcal{F}_A$ coalgebra. A homomorphism between two $\mathcal{F}_A$-coalgebras $\langle S, A, \alpha\rangle$ and $\langle T, A, \beta\rangle$ is a function $h : S \to T$ satisfying $\mathcal{F}_A h \circ \alpha = \beta \circ h$. The $\mathcal{F}_A$-coalgebras together with their homomorphisms form a category, which we denote by $\mathsf{Coalg}^A_{\mathcal{F}}$.*

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Next we present two basic types of systems, labelled transition systems and generative systems, which will be treated in more detail in Section 4 and Section 5. We give their concrete definitions first.

Definition 2 *A labelled transition system, or LTS for short, is a triple $\langle S, A, \rightarrow \rangle$ where S and A are sets and $\rightarrow \subseteq S \times A \times S$. We speak of S as the set of states, of A as the set of labels or actions the system can perform and of $\rightarrow$ as the transition relation. As usual we denote $s \xrightarrow{a} s'$ whenever $\langle s, a, s' \rangle \in \rightarrow$.*

When replacing the transition relation of an LTS by a "probabilistic transition relation", the so-called generative probabilistic systems are obtained.

Definition 3 *A generative probabilistic system is a triple $\langle S, A, P \rangle$ where S and A are sets and $P : S \times A \times S \rightarrow [0, 1]$ with the property that for $s \in S$,*

$$\sum_{a \in A, \ s' \in S} P(s, a, s') \in \{0, 1\}. \tag{2}$$

We speak of S as the set of states, of A as the set of labels or actions the system can perform and of P as the probabilistic transition relation. Condition (2) states that for all $s \in S$, $P(s, \_, \_)$ is either a distribution over $A \times S$ or $P(s, \_, \_) = 0$, i.e. s is a terminating state. As usual we denote $s \xrightarrow{a[p]} s'$ whenever $P(s, a, s') = p$, and $s \xrightarrow{a} s'$ for $P(s, a, s') > 0$.

Remark 1 *In order to clarify the condition (2) let us recall that the sum of an arbitrary family $\{x_i \mid i \in I\}$ of non-negative real numbers is defined as*

$$\sum_{i \in I} x_i = \sup\{\sum_{i \in J} x_i \mid J \subseteq I, J \text{ finite}\}.$$

Note that, if $\sum_{i \in I} x_i < \infty$, then the set $\{x_i \mid i \in I, x_i \neq 0\}$ is at most countably infinite.

Let us turn to the coalgebraic side. LTSs can be viewed as coalgebras corresponding to the bifunctor

$$\mathcal{L} = \mathcal{P}(\mathcal{I}d \times \mathcal{I}d).$$

Namely, if $\langle S, A, \rightarrow \rangle$ is an LTS, then $\langle S, A, \alpha \rangle$, where $\alpha : S \rightarrow \mathcal{L}_A(S)$ is defined by

$$\langle a, s' \rangle \in \alpha(s) \iff s \xrightarrow{a} s'$$

is an $\mathcal{L}_A$-coalgebra, and vice-versa. Further on, we will freely use $\xrightarrow{a}$ notation when talking about $\mathcal{L}_A$-coalgebras. Also the generative systems can be considered as coalgebras corresponding to the bifunctor

$$\mathcal{G} = \mathcal{D}(\mathcal{I}d \times \mathcal{I}d) + 1.$$

Here $\mathcal{D}$ denotes the distribution functor, that is, $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$

$$\mathcal{D}X = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

$$(\mathcal{D}f)(\mu)(y) = \sum_{f(x)=y} \mu(x), \quad f : X \to Y, \mu \in \mathcal{D}X, y \in Y.$$

If $\langle S, A, P \rangle$ is a generative system, then $\langle S, A, \alpha \rangle$ is a $\mathcal{G}_A$-coalgebra where $\alpha : S \to \mathcal{G}_A(S)$ is given by

$$\alpha(s)(a, s') = P(s, a, s'),$$

and vice-versa. Thereby we interpret the singleton set 1 as the set containing the zero-function on $A \times S$. Note that $\alpha(s)$ is the zero-function if and only if s is a terminating state.

In the literature it is common to restrict to generative systems $\langle S, A, \alpha \rangle$ where for any state s the function $\alpha(s)$ has finite support. The restriction to finite support guarantees existence of a final coalgebra. However, in many respects, in particular when the existence of a final coalgebra is not needed, this restriction is not necessary.

An important notion in this paper is that of a bisimulation relation between two systems. We recall here the general definition of bisimulation in coalgebraic terms.

Definition 4 *Let $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ be two $\mathcal{F}_A$-coalgebras. A bisimulation between $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ is a relation $R \subseteq S \times T$, such that there exists a map $\gamma : R \to \mathcal{F}_A R$ making the projections π_1 and π_2 coalgebra homomorphisms between the respective coalgebras, i.e. making the following diagram commute:*

$$
\begin{array}{ccccc}
S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & T \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\beta} \\
\mathcal{F}_A S & \xleftarrow[\mathcal{F}_A\pi_1]{} & \mathcal{F}_A R & \xrightarrow[\mathcal{F}_A\pi_2]{} & \mathcal{F}_A T
\end{array}
$$

Two states $s \in S$ and $t \in T$ are bisimilar, notation $s \sim t$ if they are related by some bisimulation between $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$.

Often we will consider bisimulations that are equivalence relations on a single coalgebra $\langle S, A, \alpha \rangle$.

In general, hence also for functors $\mathcal{F}_A$ and $\mathcal{G}_A$ arising from bifunctors $\mathcal{F}$ and $\mathcal{G}$, it holds that a natural transformation $\eta : \mathcal{F}_A \Rightarrow \mathcal{G}_A$ determines a functor $\mathcal{T} : \mathsf{Coalg}_{\mathcal{F}}^A \rightarrow \mathsf{Coalg}_{\mathcal{G}}^A$ defined by

$$\mathcal{T}(\langle S, A, \alpha \rangle) = \langle S, A, \eta_S \circ \alpha \rangle, \quad \mathcal{T}f = f. \tag{3}$$

We will refer to the functor $\mathcal{T}$ as the *functor induced* by the natural transformation η. Functors induced by natural transformations preserve homomorphisms and thus preserve bisimulation relations, in particular bisimilarity (cf. [38]).

LTSs and generative systems come equipped with their concrete notions of bisimulation relations, cf. [29] and [24, 16], respectively, which we present next.

Definition 5 *Let* $\langle S, A, \rightarrow \rangle$ *be an LTS. An equivalence relation* $R \subseteq S \times S$ *is a (strong) bisimulation on* $\langle S, A, \rightarrow \rangle$ *if and only if whenever* $\langle s, t \rangle \in R$ *then for all* $a \in A$ *the following holds:*

$s \xrightarrow{a} s'$ *implies that there exists* $t' \in S$ *with* $t \xrightarrow{a} t'$ *and* $\langle s', t' \rangle \in R$.

Two states s *and* t *of an LTS are called bisimilar if and only if they are related by some bisimulation relation. Notation* $s \sim_\ell t$.

For generative systems we have the following definition of bisimulation.

Definition 6 *Let* $\langle S, A, P \rangle$ *be a generative system. An equivalence relation* $R \subseteq S \times S$ *is a (strong) bisimulation on* $\langle S, A, P \rangle$ *if and only if whenever* $\langle s, t \rangle \in R$ *then for all* $a \in A$ *and for all equivalence classes* $C \in S/R$

$$P(s, a, C) = P(t, a, C). \tag{4}$$

Here we have put $P(s, a, C) = \sum_{s' \in C} P(s, a, s')$. *Two states* s *and* t *of a generative system are bisimilar if and only if they are related by some bisimulation relation. Notation* $s \sim_g t$.

The concrete notion of bisimilarity for LTSs and generative systems and the respective notions of bisimilarity obtained from Definition 4 coincide. For the case of LTSs a direct proof was given, for example, by Rutten

[38]. For generative systems this fact goes back to the work of De Vink and Rutten [46] where Markov systems were considered, and was treated in [10] for generative systems with finite support.

We will now describe a general procedure to obtain coincidence results of this kind. This method already appeared implicitly in [11]. It applies to LTSs as well as to generative systems in their full generality. We will also use the method to obtain a concrete characterization of bisimilarity for another, more complex, functor, in Section 5.

Definition 7 *Let $R \subseteq S \times T$ be a relation, and $\mathcal{F}$ a* Set *functor. The relation R can be lifted to a relation $\equiv_{\mathcal{F},R} \subseteq \mathcal{F}S \times \mathcal{F}T$ defined by*

$$x \equiv_{\mathcal{F},R} y \iff \exists z \in \mathcal{F}R \colon \mathcal{F}\pi_1(z) = x, \ \mathcal{F}\pi_2(z) = y.$$

The following lemma is obvious from Definition 4.

Lemma 1 *A relation $R \subseteq S \times T$ is a bisimulation between the $\mathcal{F}_A$ systems $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ if and only if*

$$\langle s, t \rangle \in R \implies \alpha(s) \equiv_{\mathcal{F}_A,R} \beta(t). \tag{5}$$

□

Note that the condition (5) is an abstract formulation of what is commonly referred to as a transfer condition.

For the sequel, weak pullback preservation will be of some importance. We recall the definitions of (weak) pullbacks and some needed properties concerning their preservation in Appendix A. One particular kind of pullbacks, *total pullbacks*, are important for our investigations. A total pullback is a weak pullback with surjective legs.
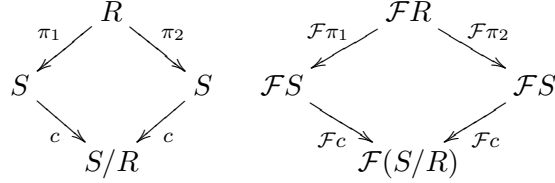
A characterization of bisimilarity will follow from the next lemma.

Lemma 2 *If the functor $\mathcal{F}$ weakly preserves total pullbacks and R is an equivalence on S, then $\equiv_{\mathcal{F},R}$ is the pullback in* Set *of the cospan*

$$\mathcal{F}S \xrightarrow{\mathcal{F}c} \mathcal{F}(S/R) \xleftarrow{\mathcal{F}c} \mathcal{F}S \tag{6}$$

where $c \colon S \to S/R$ is the canonical morphism mapping each element to its equivalence class.
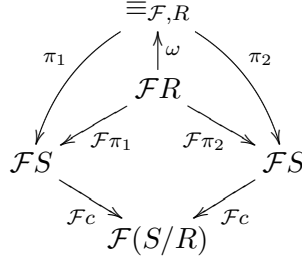
Proof: Since R is an equivalence relation and therefore reflexive, the left diagram below is a pullback diagram with epi legs, i.e., a total pullback.

$$
\begin{array}{ccc}
& R & \\
{}^{\pi_1}\swarrow & & \searrow{}^{\pi_2} \\
S & & S \\
{}_{c}\searrow & & \swarrow{}_{c} \\
& S/R &
\end{array}
\qquad
\begin{array}{ccc}
& \mathcal{F}R & \\
{}^{\mathcal{F}\pi_1}\swarrow & & \searrow{}^{\mathcal{F}\pi_2} \\
\mathcal{F}S & & \mathcal{F}S \\
{}_{\mathcal{F}c}\searrow & & \swarrow{}_{\mathcal{F}c} \\
& \mathcal{F}(S/R) &
\end{array}
$$

Since $\mathcal{F}$ weakly preserves total pullbacks, the right diagram is a weak pullback diagram. By Definition 7 the map

$$\omega\colon \mathcal{F}R \to\, \equiv_{\mathcal{F},R}, \quad \omega(z) = \langle \mathcal{F}\pi_1(z), \mathcal{F}\pi_2(z)\rangle$$

is well-defined, surjective, and it makes the two upper triangles of the next diagram commute:

$$
\begin{array}{ccc}
& \equiv_{\mathcal{F},R} & \\
{}^{\pi_1}\nearrow \quad \uparrow\omega & & \nwarrow{}^{\pi_2} \\
& \mathcal{F}R & \\
{}^{\mathcal{F}\pi_1}\swarrow & & \searrow{}^{\mathcal{F}\pi_2} \\
\mathcal{F}S & & \mathcal{F}S \\
{}_{\mathcal{F}c}\searrow & & \swarrow{}_{\mathcal{F}c} \\
& \mathcal{F}(S/R) &
\end{array}
$$

As the lower square commutes and ω is surjective, the outer square of the above diagram also commutes, and by the existence of ω from the weak pullback $\mathcal{F}R$ to $\equiv_{\mathcal{F},R}$, $\equiv_{\mathcal{F},R}$ is a weak pullback as well. However, since it has projections as legs it is a pullback. $\square$

Suppose that a functor $\mathcal{F}$ weakly preserves total pullbacks and assume that R is an equivalence bisimulation on S, i.e., R is both an equivalence relation and a bisimulation on S, such that $\langle s, t\rangle \in R$. The pullback in Set of the cospan (6) is the set $\{\,\langle x, y\rangle \mid \mathcal{F}c(x) = \mathcal{F}c(y)\,\}$. By Lemma 2 this set coincides with the lifted relation $\equiv_{\mathcal{F},R}$. Thus $x \equiv_{\mathcal{F},R} y \iff \mathcal{F}c(x) = \mathcal{F}c(y)$. Therefore, we obtain the transfer condition for the particular notion of bisimulation if we succeed in expressing concretely $(\mathcal{F}c\circ\alpha)(s) = (\mathcal{F}c\circ\alpha)(t)$ in terms of the representation of $\alpha(s)$ and $\alpha(t)$.

To illustrate the method, we will use it in showing the well-known correspondence of coalgebraic and concrete bisimulation for LTSs.

Lemma 3 *An equivalence relation R on a set S is a coalgebraic bisimulation on the LTS $\langle S, A, \alpha\rangle$ according to Definition 4 for the functor $\mathcal{L}_A$ if and only if it is a concrete bisimulation according to Definition 5.*

Proof: It is easy to show that the LTS functor $\mathcal{L}_A$ preserves weak pullbacks (see e.g. [42]). For $X \in \mathcal{L}_A(S)$, i.e. $X \subseteq A \times S$, we have $\mathcal{L}_A(c)(X) = \mathcal{P}\langle id_A, c\rangle(X) = \langle id_A, c\rangle(X) = \{\langle a, c(x)\rangle \mid \langle a, x\rangle \in X\}$. Using Lemma 1 we get that an equivalence $R \subseteq S \times S$ is a coalgebraic bisimulation for an LTS $\langle S, A, \alpha\rangle$ if and only if

$$\langle s, t\rangle \in R \implies \{\langle a, c(s')\rangle \mid \langle a, s'\rangle \in \alpha(s)\} = \{\langle a, c(t')\rangle \mid \langle a, t'\rangle \in \alpha(t)\}$$

or, equivalently

$$\langle s, t\rangle \in R \implies (\ s \xrightarrow{a} s' \implies \exists t' \in S \colon t \xrightarrow{a} t' \wedge \langle s', t'\rangle \in R\ ).$$

which is the transfer condition from Definition 5. $\square$

The most difficult part in establishing the correspondence result for generative systems is proving the weak pullback preservation for the distribution functor.

Proposition 2 *The functor $\mathcal{D}$ preserves weak pullbacks.* $\square$

Appendix B is dedicated to the proof of this proposition. As a consequence we get that the functor for generative systems $\mathcal{G}_A$ preserves weak pullbacks. An application of Lemma 1 and some simple derivations now suffice to show the correspondence result.

Lemma 4 *An equivalence relation R on a set S is a coalgebraic bisimulation on the generative system $\langle S, A, \alpha\rangle$ according to Definition 4 for the functor $\mathcal{G}_A$ if and only if it is a concrete bisimulation according to Definition 6.* $\square$

We end this section with a small discussion on the assumption of Lemma 1. Often we require a functor to weakly preserve pullbacks, so that it will be "well-behaved". For example, for bisimilarity being an equivalence. It can easily be seen that the milder condition of weakly preserving total pullbacks suffices for bisimilarity to be an equivalence. Moreover, we have relaxed the weak pullback preservation condition since in Section 5 we will need a bisimilarity characterization of a functor that transforms total pullbacks to weak pullbacks, but does not preserve weak pullbacks.

3   Weak bisimulation for action-type coalgebras

In this section we present a general definition of weak bisimulation for action-type systems. Our idea arises as a generalization of the notions of weak

bisimulation for concrete types of systems. In our opinion, a weak bisimulation on a given system is a strong bisimulation on a suitably transformed system obtained from the original one.

Weak bisimulation in concrete cases deals with hiding actions. Therefore we focus on weak bisimulation for action-type coalgebras. Recall that we have defined action-type coalgebras in Definition 1 as triples $\langle S, A, \alpha \rangle$ such that $\langle S, \alpha : S \to \mathcal{F}_A S \rangle$ is a coalgebra for the functor $\mathcal{F}_A$ induced by a bifunctor $\mathcal{F}$, as in Equation (1).

We proceed with the definition of weak bisimulation for action-type coalgebras. The definition consists of two phases. First we define the notion of a *-extended system, that captures the behavior of the original system when extending from the given set of actions A to A^*, the set of finite words over A. The *-extension should emerge from the original system in a faithful way (which will be made precise below). The second phase considers invisibility. Given a subset $\tau \subseteq A$ of invisible actions, we restrict the *-extension to visible behavior only, by defining its *weak-τ-extended system*. Then a weak bisimulation relation on the original system is obtained as a bisimulation relation on the weak-τ-extension.

Definition 8 *Let $\mathcal{F}$ and $\mathcal{G}$ be two bifunctors. Let Φ be a map assigning to every $\mathcal{F}_A$-coalgebra $\langle S, A, \alpha \rangle$, a $\mathcal{G}_{A^*}$ system $\langle S, A^*, \alpha' \rangle$, on the same set of states S, such that the following conditions are met*

(1) Φ is injective, i.e. $\Phi(\langle S, A, \alpha \rangle) = \Phi(\langle S, A, \beta \rangle) \Rightarrow \alpha = \beta$;

(2) Φ preserves and reflects bisimilarity, i.e. $s \sim t$ in the system $\langle S, A, \alpha \rangle$ if and only if $s \sim t$ in the transformed system $\Phi(\langle S, A, \alpha \rangle)$.

*Then Φ is called a *-translation, notation $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$. The $\mathcal{G}_{A^*}$-coalgebra $\Phi(\langle S, A, \alpha \rangle)$ is said to be a *-extension of the $\mathcal{F}_A$-coalgebra $\langle S, A, \alpha \rangle$.*

From the conditions (1) and (2) in Definition 8 it follows that the original system is "embedded" in its *-extension, cf. [10, 11, 43]. The fact that a *-translation may lead to systems of a new type, viz. of the bifunctor $\mathcal{G}$, might seem counter intuitive at first sight. However, this extra freedom is exploited in Section 5 when the starting functor itself is not expressive enough to allow for a *-extension.

A way to obtain *-translations follows from a previous result. Namely, if $\lambda : \mathcal{F}_A \Rightarrow \mathcal{G}_{A^*}$ is a natural transformation with injective components and the functor $\mathcal{F}_A$ preserves weak pullbacks, then the induced functor (see Equation (3)) is a *-translation [10, 11]. However, we shall see later (cf.

Example 1 and the preceding discussion) that $*$-translations emerging from natural transformations do not suffice.

Having described how to extend an $\mathcal{F}_A$ system to its $*$-extension we show how to hide invisible actions. Fix a set of invisible actions $\tau \subseteq A$. Consider the function $h_\tau \colon A^* \to (A \setminus \tau)^*$ induced by $h_\tau(a) = a$ if $a \notin \tau$ and $h_\tau(a) = \varepsilon$ for $a \in \tau$ (where ε denotes the empty word). The function h_τ is deleting all the occurrences of elements of τ in a word of A^*. We put $A_\tau = (A \setminus \tau)^*$. By Proposition 1, we get the following.

Corollary 1 *The transformation $\eta^\tau \colon \mathcal{G}_{A^*} \Rightarrow \mathcal{G}_{A_\tau}$ given by $\eta^\tau_S = \mathcal{G}\langle h_\tau, id_S \rangle$ is natural.* $\qquad\square$

Let Ψ_τ be the functor from $\mathsf{Coalg}_{\mathcal{G}}^{A^*}$ to $\mathsf{Coalg}_{\mathcal{G}}^{A_\tau}$ induced by the natural transformation η^τ, i.e. $\Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \alpha'' \rangle$ for $\alpha'' = \eta^\tau_S \circ \alpha'$ and $\Psi_\tau f = f$ for any morphism $f : S \to T$. As mentioned above, the induced functor preserves bisimilarity. The composition of a $*$-translation Φ and the hiding functor Ψ_τ is denoted by $\Omega_\tau = \Psi_\tau \circ \Phi$ and is called a weak-τ-translation. The resulting system $\langle S, A_\tau, \eta^\tau_S \circ \alpha' \rangle$ is called a weak-τ-extension of $\langle S, A, \alpha \rangle$.

The transformation to a weak-τ-extension is presented in the following scheme.

$$
\begin{array}{ccccc}
S & & S & & S \\
\alpha \downarrow & \xrightarrow{\ \ \Phi\ \ } & \downarrow \alpha' & \xrightarrow{\ \ \Psi_\tau\ \ } & \downarrow \alpha''=\eta^\tau_S \circ \alpha' \\
\mathcal{F}_A S & & \mathcal{G}_{A^*} S & & \mathcal{G}_{A_\tau} S \\[2mm]
\mathcal{F}_A \text{ - coalgebra} & & \mathcal{G}_{A^*} \text{ - coalgebra} & & \mathcal{G}_{A_\tau} S
\end{array}
$$

$$\mathcal{G}_{A_\tau} \text{ - coalgebra}$$

A weak-τ-translation, or equivalently, the pair $\langle \Phi, \tau \rangle$, yields a notion of weak bisimulation with respect to Φ and τ.

Definition 9 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ a $*$-translation and $\tau \subseteq A$. Let $\langle S, A, \alpha \rangle$ and $\langle T, A, \beta \rangle$ be two $\mathcal{F}_A$ systems. A relation $R \subseteq S \times T$ is a weak bisimulation with respect to $\langle \Phi, \tau \rangle$ if and only if it is a bisimulation between $\Omega_\tau(\langle S, A, \alpha \rangle)$ and $\Omega_\tau(\langle T, A, \beta \rangle)$. Two states $s \in S$ and $t \in T$ are weakly bisimilar with respect to $\langle \Phi, \tau \rangle$, notation $s \approx_\tau t$, if they are related by some weak bisimulation with respect to $\langle \Phi, \tau \rangle$.*

Concrete examples of weak bisimulation will be discussed in Section 4 and Section 5. We continue with verifying that weak bisimulations $\approx_\tau$ posses the intuitively expected properties.

Proposition 3 *Let $\mathcal{F}$, $\mathcal{G}$ be two bifunctors, $\Phi : \mathcal{F} \xrightarrow{*} \mathcal{G}$ a $*$-translation, $\langle S, A, \alpha \rangle$ an $\mathcal{F}_A$-coalgebra, $\tau \subseteq A$ and let $\approx_\tau$ denote the weak bisimilarity on $\langle S, A, \alpha \rangle$ w.r.t. $\langle \Phi, \tau \rangle$. Then the following hold:*

(i) $\sim \subseteq \approx_\tau$ *for any $\tau \subseteq A$*
i.e. strong bisimilarity implies weak bisimilarity.

(ii) $\sim \; = \; \approx_\emptyset$
i.e. strong bisimilarity is weak bisimilarity in absence of invisible actions.

(iii) $\tau_1 \subseteq \tau_2 \Rightarrow \approx_{\tau_1} \subseteq \approx_{\tau_2}$ *for any $\tau_1, \tau_2 \subseteq A$,*
i.e. the more actions are invisible, the coarser the weak bisimilarity gets.

Proof: Let $\mathcal{F}, \mathcal{G}, \Phi, \langle S, A, \alpha \rangle$ and τ be as in the assumptions of the Lemma.

(i) Assume $s \sim t$ in $\langle S, A, \alpha \rangle$. Since Φ preserves bisimilarity (Definition 8) we have that $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle)$. Next, since Ψ_τ preserves bisimilarity we get $s \sim t$ in $\Psi_\tau \circ \Phi(\langle S, A, \alpha \rangle)$, which by Definition 9 means $s \approx_\tau t$ in $\langle S, A, \alpha \rangle$.

(ii) From (i) we get $\sim \subseteq \approx_\emptyset$. For the opposite inclusion, note that $h_\emptyset : A^* \to A^*$ is the identity map, hence the natural transformation $\eta^\emptyset$ from Corollary 1 is the identity natural transformation. Therefore the induced functor $\Psi_\emptyset$ is the identity functor on $\mathsf{Coalg}_\mathcal{G}^{A^*}$. Now assume $s \approx_\emptyset t$ in $\langle S, A, \alpha \rangle$. This means $s \sim t$ in $\Omega_\emptyset(\langle S, A, \alpha \rangle)$, i.e. $s \sim t$ in $\Psi_\emptyset \circ \Phi(\langle S, A, \alpha \rangle)$, i.e. $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle)$. Since, by Definition 8, every $*$-translation reflects bisimilarity we get $s \sim t$ in $\langle S, A, \alpha \rangle$.

(iii) Let $\tau_1 \subseteq \tau_2$. Consider the diagram

$$
\begin{array}{ccc}
A^* & \xrightarrow{\;h_{\tau_2}\;} & A_{\tau_2} \\
{\scriptstyle h_{\tau_1}}\big\downarrow & \nearrow {\scriptstyle h_{\tau_1,\tau_2}} & \\
A_{\tau_1} & &
\end{array}
$$

where h_{τ_1,τ_2} is the map deleting all occurrences of elements of τ_2 in a word of A_{τ_1}. The diagram commutes since first deleting all occurrences

of elements of τ_1 followed by deleting all occurrences of elements of τ_2, in a word of A^* is the same as just deleting all occurrences of elements of τ_2. Let η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2} be the natural transformations induced by h_{τ_1}, h_{τ_2}, h_{τ_1,τ_2}, respectively (see Proposition 1 and Corollary 1). Then the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{G}_{A^*} & \xrightarrow{\;\eta^{\tau_2}\;} & \mathcal{G}_{A_{\tau_2}} \\
{\scriptstyle \eta^{\tau_1}}\Big\Vert & \nearrow{\scriptstyle \eta^{\tau_1,\tau_2}} & \\
\mathcal{G}_{A_{\tau_1}} & &
\end{array}
$$

Let Ψ_{τ_1}, Ψ_{τ_2}, Ψ_{τ_1,τ_2} be the functors induced by the natural transformations η^{τ_1}, η^{τ_2}, η^{τ_1,τ_2}, respectively. By Equation (3) it holds that

$$\Psi_{\tau_2} = \Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \tag{7}$$

and they all preserve bisimilarity. Now assume $s \approx_{\tau_1} t$ in $\langle S, A, \alpha\rangle$. This means that $s \sim t$ in the system $\Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha\rangle)$. Then, since Ψ_{τ_1,τ_2} preserves bisimilarity we have $s \sim t$ in the system $\Psi_{\tau_1,\tau_2} \circ \Psi_{\tau_1} \circ \Phi(\langle S, A, \alpha\rangle)$ which by equation (7) is the system $\Psi_{\tau_2} \circ \Phi(\langle S, A, \alpha\rangle)$ and we find $s \approx_{\tau_2} t$ in $\langle S, A, \alpha\rangle$. $\qquad\square$

For further use, we introduce some more notation. For any $w \in A_\tau$, we put $B_w = h_\tau^{-1}(\{w\}) \subseteq A^*$. We refer to the sets B_w as blocks. Note that $B_w = \tau^* a_1 \tau^* \cdots \tau^* a_k \tau^*$ for $w = a_1 \dots a_k \in A_\tau = (A \setminus \tau)^*$.

4 Weak bisimulation for LTSs

In this section we show that in the case of LTSs there exists a $*$-translation according to the Definition 8, such that weak bisimulation in the concrete case [29] coincides with weak bisimulation induced by this $*$-translation. First we recall the standard definition of concrete weak bisimulation for LTSs.

Definition 10 *Let $\langle S, A, \rightarrow\rangle$ be an LTS. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, \alpha\rangle$ if and only if $\langle s, t\rangle \in R$ implies that*

$$
\begin{aligned}
&\text{if } s \xrightarrow{\;a\;} s', \text{ then there exists } t' \in S \text{ with} \\
&\qquad t \xrightarrow{\;\tau\;}{}^* \circ \xrightarrow{\;a\;} \circ \xrightarrow{\;\tau\;}{}^* t' \text{ and } \langle s', t'\rangle \in R
\end{aligned}
$$

for all $a \in A \setminus \{\tau\}$, and

if $s \xrightarrow{\tau} s'$, then there exists $t' \in S$ with $t \xrightarrow{\tau}{}^* t'$ and $\langle s', t' \rangle \in R$.

Two states s and t are called weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_\ell t$.

We now present a definition of a $*$-translation that will give rise to a notion of weak bisimulation that coincides with the standard one of Definition 10. Recall that $\mathcal{L}$, $\mathcal{L}_A$ are the functors for LTSs, as introduced in Section 2.

Definition 11 *Let Φ assign to every LTS, i.e. any $\mathcal{L}_A$-coalgebra $\langle S, A, \alpha \rangle$, the $\mathcal{L}_{A^*}$ coalgebra $\langle S, A^*, \alpha' \rangle$ where for $w = a_1 \ldots a_k \in A^*$, $k > 0$,*

$$\langle a_1 \ldots a_k, s' \rangle \in \alpha'(s) \quad \Longleftrightarrow \quad s \xrightarrow{a_1} \circ \xrightarrow{a_2} \circ \cdots \circ \xrightarrow{a_k} s'$$

and $\langle \varepsilon, s' \rangle \in \alpha'(s) \iff s = s'$. We use the notation $s \overset{w}{\Rightarrow} s'$ for $\langle w, s' \rangle \in \alpha'(s)$.

Hence, for $w = a_1 \ldots a_k$, we have $s \overset{w}{\Rightarrow} s'$ if and only if there exist states $s_1, \ldots, s_{k-1}$ such that

$$s \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots \xrightarrow{a_{k-1}} s_{k-1} \xrightarrow{a_k} s'.$$

Furthermore, note that for $a \in A$, since no hiding applies, it holds that

$$s \xrightarrow{a} s' \text{ in } \langle S, A, \alpha \rangle \quad \text{if and only if} \quad s \overset{a}{\Rightarrow} s' \text{ in } \langle S, A, \alpha' \rangle = \Phi(\langle S, A, \alpha \rangle)$$

i.e.,

$$\langle a, s' \rangle \in \alpha(s) \quad \Longleftrightarrow \quad \langle a, s' \rangle \in \alpha'(s).$$

Proposition 4 *The assignment Φ from Definition 11 is a $*$-translation.*

Proof: We need to prove that Φ is injective and reflects and preserves bisimilarity. Let $\Phi(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$, $\Phi(\langle S, A, \beta \rangle) = \langle S, A^*, \beta' \rangle$. Assume that $\alpha' = \beta'$. Then, for any state s,

$$\langle a, s' \rangle \in \alpha(s) \iff \langle a, s' \rangle \in \alpha'(s) \iff \langle a, s' \rangle \in \beta'(s) \iff \langle a, s' \rangle \in \beta(s).$$

Hence $\alpha(s) = \beta(s)$, i.e., $\alpha = \beta$.

For the reflection of bisimilarity, let $s \sim t$ in $\Phi(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$. Hence there exists an equivalence bisimulation relation R such that $\langle s, t \rangle \in R$ and (according to Definition 5) for all $w \in A^*$,

if $s \overset{w}{\Rightarrow} s'$ then there exists $t' \in S$ such that $t \overset{w}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$.

Assume $s \overset{a}{\longrightarrow} s'$ in $\langle S, A, \alpha \rangle$. Then $s \overset{a}{\Rightarrow} s'$ in $\langle S, A, \alpha' \rangle$ and therefore there exists $t' \in S$ with $\langle s', t' \rangle \in R$ and $t \overset{a}{\Rightarrow} t'$, i.e., $t \overset{a}{\longrightarrow} t'$. Hence, R is a bisimulation on $\langle S, A, \alpha \rangle$ i.e. $s \sim t$ in the original system.

For the preservation of bisimulation, let $s \sim t$ in $\langle S, A, \alpha \rangle$ and let R be an equivalence bisimulation relation such that $\langle s, t \rangle \in R$. Assume $s \overset{w}{\Rightarrow} s'$, for some word $w \in A^*$. We show by induction on the length of w that there exists t' with $t \overset{w}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$. If w has length 0, then $w = \varepsilon$, $s' = s$ and we take $t' = t$. Assume w has length $k+1$, i.e. $w = a \cdot w'$ for $a \in A, w' \in A^*$. Pick s'' such that $s \overset{a}{\longrightarrow} s'' \overset{w'}{\Rightarrow} s'$. Since $\langle s, t \rangle \in R$ we can pick t'' such that $t \overset{a}{\longrightarrow} t''$ and $\langle s'', t'' \rangle \in R$. By the inductive hypothesis, for w' we can choose t' such that $t'' \overset{w'}{\Rightarrow} t'$ and $\langle s', t' \rangle \in R$. Note that $t \overset{a}{\longrightarrow} t'' \overset{w'}{\Rightarrow} t'$, i.e., $t \overset{w}{\Rightarrow} t'$. Hence R is a bisimulation on $\langle S, A^*, \alpha' \rangle$ and $s \sim t$ holds in the $*$-extension. $\qquad \square$

Note that if $\mathcal{T}$ is a functor induced by a natural transformation η, in the context of Equation (3), and if $\langle S, A, \alpha \rangle, \langle S, A, \beta \rangle$ are two systems such that, for some $s \in S$, $\alpha(s) = \beta(s)$, then, clearly,

$$\alpha'(s) = \eta_S(\alpha(s)) = \eta_S(\beta(s)) = \beta'(s) \tag{8}$$

for $\langle S, A, \alpha' \rangle = \mathcal{T}(\langle S, A, \alpha \rangle), \langle S, A, \beta' \rangle = \mathcal{T}(\langle S, A, \beta \rangle)$.

Having $*$-translations induced by natural transformations is desirable, since such *-translations are functorial and also obtained by a categorical construct. However, the following simple example shows that the $*$-translation Φ from Definition 11 violates (8). Therefore it can not be induced by a natural transformation.

Example 1 *Let $S = \{s_1, s_2, s_3\}$ and $A = \{a, b, c\}$. Consider the LTSs:*

$$\langle S, A, \alpha \rangle : s_1 \overset{a}{\longrightarrow} s_2 \overset{b}{\longrightarrow} s_3 \quad and \quad \langle S, A, \beta \rangle : s_1 \overset{a}{\longrightarrow} s_2 \overset{c}{\longrightarrow} s_3.$$

Obviously $\alpha(s_1) = \beta(s_1)$. However, $\alpha'(s_1) = \{\langle \varepsilon, s_1 \rangle, \langle a, s_2 \rangle, \langle ab, s_3 \rangle\}$ while $\beta'(s_1) = \{\langle \varepsilon, s_1 \rangle, \langle a, s_2 \rangle, \langle ac, s_3 \rangle\}$.

We next show that the coalgebraic and the concrete definitions coincide in the case of LTS.

Theorem 1 *Let $\langle S, A, \alpha \rangle$ be an LTS. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ with respect to the pair $\langle \Phi, \{\tau\} \rangle$ if and only if $s \approx_\ell t$.*

Proof: Assume $s \approx_{\{\tau\}} t$ for $s, t \in S$ of an LTS $\langle S, A, \alpha \rangle$. This means that $s \sim t$ in the LTS $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha' \rangle$, i.e., there exists an equivalence bisimulation R on this system with $\langle s, t \rangle \in R$.

As usual, α' is such that $\langle S, A^*, \alpha' \rangle = \Phi(\langle S, A, \alpha \rangle)$. Here we have $\eta_S^{\{\tau\}} = \mathcal{L}(h_{\{\tau\}}, id_S) = \mathcal{P}(h_{\{\tau\}}, id_S)$ and

$$
\begin{aligned}
(\eta_S^{\{\tau\}} \circ \alpha')(s) &= \eta_S^{\{\tau\}}(\alpha'(s)) \\
&= \mathcal{P}(h_{\{\tau\}}, id_S)(\alpha'(s)) \\
&= \{\langle h_{\{\tau\}}(w), s' \rangle \mid \langle w, s' \rangle \in \alpha'(s)\} \\
&= \{\langle u, s' \rangle \mid \exists w \in B_u \colon s \overset{w}{\Rightarrow} s'\}
\end{aligned}
$$

We denote the transition relation of the weak-τ-system $\langle S, A_{\{\tau\}}, \eta_S^{\{\tau\}} \circ \alpha' \rangle$ by $\Rightarrow_\tau$, i.e., for $w \in A_\tau$,

$$
s \overset{w}{\Rightarrow}_\tau s' \iff \langle w, s' \rangle \in (\eta_S^{\{\tau\}} \circ \alpha')(s).
$$

The above shows that for a word $w = a_1 \ldots a_k \in A_\tau$

$$
s \overset{w}{\Rightarrow}_\tau s' \iff \exists v \in B_w = \tau^* a_1 \tau^* \ldots \tau^* a_k \tau^* \colon s \overset{v}{\Rightarrow} s'.
$$

We will show that the relation R is a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 10. Let $s \overset{a}{\longrightarrow} s'$ $(a \neq \tau)$. Then $s \overset{a}{\Rightarrow} s'$, implying $s \overset{a}{\Rightarrow}_\tau s'$. Since R is a bisimulation on the weak-τ-system, there exists t' such that $t \overset{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. We only need to note here that $\overset{a}{\Rightarrow}_\tau = \overset{\tau}{\longrightarrow}^* \circ \overset{a}{\longrightarrow} \circ \overset{\tau}{\longrightarrow}^*$. In case $s \overset{\tau}{\longrightarrow} s'$ we have $s \overset{\tau}{\Rightarrow} s'$ implying now $s \overset{\varepsilon}{\Rightarrow}_\tau s'$. Hence, there exists t' such that $t \overset{\varepsilon}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. Since $\overset{\varepsilon}{\Rightarrow}_\tau = \overset{\tau}{\longrightarrow}^*$, we have proved that R is a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 10.

For the opposite, let R be a weak bisimulation on $\langle S, A, \alpha \rangle$ according to Definition 10 such that $\langle s, t \rangle \in R$. It is easy to show that for any $a \in A$, if $s \overset{\tau}{\longrightarrow}^* \circ \overset{a}{\longrightarrow} \circ \overset{\tau}{\longrightarrow}^* s'$ then there exists t' such that $t \overset{\tau}{\longrightarrow}^* \circ \overset{a}{\longrightarrow} \circ \overset{\tau}{\longrightarrow}^* t'$ and $\langle s', t' \rangle \in R$. Hence, if $s \overset{a}{\Rightarrow}_\tau s'$ then there exists t' with $t \overset{a}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$. Based on this, a simple inductive argument on k leads to the conclusion that for any word $w = a_1 \ldots a_k \in A_\tau$, if $s \overset{w}{\Rightarrow}_\tau s'$ then there exists a t' such that $t \overset{w}{\Rightarrow}_\tau t'$ and $\langle s', t' \rangle \in R$, i.e. R is a bisimulation on the weak-τ-system and hence $s \approx_{\{\tau\}} t$. $\qquad \square$

5 Weak bisimulation for generative systems

In this section we deal with generative systems and their weak bisimilarity. We first focus on the concrete definition of weak bisimulation by Baier and

Hermanns [7, 6, 8]. Inspired by it, we provide a functor that suits for a definition of a $*$-translation for generative systems. This way we obtain a coalgebraic definition of weak bisimulation for this type of systems. We show that our definition, although at first sight much stronger, coincides with the definition of Baier and Hermanns for finite systems. Unlike in the case of LTSs, for generative systems the $*$-translation needs to leave its original class of systems, which justifies the generality of the definition.

This section is divided into several parts that lead to the correspondence result: First we introduce paths in a generative system and establish some notions and properties of paths. Next we define a measure on the set of paths, where we basically follow the lines of Baier and Hermanns [8, 6]. Furthermore, we present the definition of weak bisimulation by Baier and Hermanns, and we show some properties of weak bisimulation relations that will be used later on (without restricting to finite state systems as in [8, 6]). Then we define a translation and prove that it is a $*$-translation providing us with a notion of weak-τ-bisimulation. The final part of this section is devoted to the question of correspondence of the notion of weak-τ-bisimulation defined by means of the given $*$-translation and the concrete notion proposed by Baier and Hermanns.

The material presented in this section is to a large extent of technical nature. For readability, we provide a sketch-of-proof at a number of places. Full proofs can be found in [44].

5.1 Paths and cones in a generative system

Let $\langle S, A, P \rangle$ be a generative system. A finite path π of $\langle S, A, P \rangle$ is an alternating sequence $\langle s_0, a_1, s_1, a_2, \ldots, a_k, s_k \rangle$, where $k \in \mathbb{N}_0$, $s_i \in S$, $a_i \in A$, and $P(s_{i-1}, a_i, s_i) > 0$, $i = 1, \ldots, k$. We will denote a finite path $\pi = \langle s_0, a_1, s_1, a_2, \ldots, a_k, s_k \rangle$ more suggestively by

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots s_{k-1} \xrightarrow{a_k} s_k \, .$$

Moreover, in the situation above, we put

$$\text{length}(\pi) = k, \ \text{first}(\pi) = s_0, \ \text{last}(\pi) = s_k, \ \text{trace}(\pi) = a_1 a_2 \cdots a_k \, .$$

The path $\varepsilon_{s_0} = (s_0)$ will be understood as the empty path starting at s_0. We will often write just ε for an arbitrary empty path. Similar to the finite case, an infinite path π of $\langle S, A, P \rangle$ is an infinite sequence $\langle s_0, a_1, s_1, a_2, \ldots \rangle$, where $s_i \in S$, $a_i \in A$ and $P(s_{i-1}, a_i, s_i) > 0$, $i \in \mathbb{N}$, and will be written as

$$s_0 \xrightarrow{a_1} s_1 \xrightarrow{a_2} s_2 \cdots$$

Again we set $\mathrm{first}(\pi) = s_0$. A path π is called complete if it is either infinite or it is finite with $\mathrm{last}(\pi)$ a terminating state, i.e. $P(\mathrm{last}(\pi), \_, \_) = 0$.

The sets of all (finite or infinite) paths, of all finite paths and of all complete paths will be denoted by Paths, FPaths and CPaths, respectively. Moreover, if $s \in S$, we write

$$
\begin{aligned}
\mathrm{Paths}(s) &= \big\{ \pi \in \mathrm{Paths} \mid \mathrm{first}(\pi) = s \big\}, \\
\mathrm{FPaths}(s) &= \big\{ \pi \in \mathrm{FPaths} \mid \mathrm{first}(\pi) = s \big\}, \\
\mathrm{CPaths}(s) &= \big\{ \pi \in \mathrm{CPaths} \mid \mathrm{first}(\pi) = s \big\}.
\end{aligned}
$$

We next define sets of concatenated paths. If $\Pi_1, \Pi_2 \subseteq \mathrm{FPaths}$, we define

$$
\Pi_1 \cdot \Pi_2 = \big\{ \pi_1 \cdot \pi_2 \mid \pi_1 \in \Pi_1, \pi_2 \in \Pi_2, \mathrm{last}(\pi_1) = \mathrm{first}(\pi_2) \big\},
$$

where $\pi_1 \cdot \pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} \cdots \xrightarrow{a_n} s_n$ for $\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k$ and $\pi_2 \equiv s_k \xrightarrow{a_{k+1}} \cdots \xrightarrow{a_n} s_n$.

The set $\mathrm{Paths}(s)$ is partially ordered by the prefix relation. For $\pi, \pi' \in \mathrm{Paths}(s)$ we write $\pi \preceq \pi'$ if and only if the path π is a prefix of the path π'.

Note that if $\pi \prec \pi'$ then π is a finite path, and if $\pi_1 \preceq \pi$ and $\pi_2 \preceq \pi$, then either $\pi_1 \preceq \pi_2$ or $\pi_2 \preceq \pi_1$. The complete paths are exactly the maximal elements in this partial order. For every $\pi \in \mathrm{Paths}(s)$, there exists a $\pi' \in \mathrm{CPaths}(s)$ such that $\pi \preceq \pi'$.

The following statement will be used at several occasions throughout this section.

Lemma 5 *For any state $s \in S$, the set $\mathrm{FPaths}(s)$ is at most countable.*

Proof: Let $\mathrm{FPaths}_n(s)$ denote the set of finite paths starting from s with length n. Clearly, $\mathrm{FPaths}(s) = \cup_{n \in \mathbb{N}} \mathrm{FPaths}_n(s)$. The statement follows from the observation that for any state s and any $n \in \mathbb{N}$ the set $\mathrm{FPaths}_n(s)$ is at most countable. This observation can be proven by induction on n as follows. We have $\mathrm{FPaths}_0(s) = \{\epsilon\}$ and

$$
\mathrm{FPaths}_{n+1}(s) = \bigcup_{\langle a, s' \rangle : P(s,a,s') > 0} s \xrightarrow{a} s' \cdot \mathrm{FPaths}_n(s')
$$

which is at most countable by the inductive hypothesis and by the fact that $P(s, a, s') > 0$ for at most countably many a and s' (see Lemma 14 in Appendix B). $\square$

Definition 12 *For a finite path $\pi \in \mathrm{FPaths}(s)$, let $\pi\!\uparrow$ denote the set*

$$\pi\!\uparrow \; = \{\xi \in \mathrm{CPaths}(s) \mid \pi \preceq \xi\}$$

also called the cone of complete paths generated by the finite path π.

Note that always $\pi\!\uparrow \; \neq \emptyset$. Let

$$\mathrm{Cones}(s) = \big\{\pi\!\uparrow \; \mid \pi \in \mathrm{FPaths}(s)\big\} \subseteq \mathcal{P}(\mathrm{CPaths}(s))$$

denote the set of all cones starting in s. By Lemma 5 this set is at most countable. For the study of weak bisimulation for generative systems a thorough understanding of the geometry of cones is crucial. To begin with, we have the following elementary property:

Lemma 6 *Let $\pi_1, \pi_2 \in \mathrm{FPaths}(s)$. Then the cones $\pi_1\!\uparrow$ and $\pi_2\!\uparrow$ are either disjoint or one is a subset of the other. In fact,*

$$\pi_1\!\uparrow \cap \; \pi_2\!\uparrow \; = \begin{cases} \pi_2\!\uparrow & \text{if } \pi_1 \preceq \pi_2 \\ \pi_1\!\uparrow & \text{if } \pi_2 \preceq \pi_1 \\ \emptyset & \text{if } \pi_1 \npreceq \pi_2 \text{ and } \pi_2 \npreceq \pi_1 \end{cases}$$

Moreover, we have $\pi_1\!\uparrow \; = \pi_2\!\uparrow$ if and only if either

$$\pi_1 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k, \;\; \pi_2 \equiv s \xrightarrow{a_1} \cdots \xrightarrow{a_k} s_k \xrightarrow{a_{k+1}} s_{k+1} \cdots \xrightarrow{a_n} s_n \quad (9)$$

for $n \geq k \geq 0$, and

$$P(s_{i-1}, a_i, s_i) = 1, \;\; i = k+1, \ldots, n \quad (10)$$

or vice-versa. $\qquad\qquad\square$

Let $\Pi \subseteq \mathrm{FPaths}(s)$. We say that Π is *minimal* if for any two $\pi_1, \pi_2 \in \Pi$, $\pi_1 \neq \pi_2$, we have $\pi_1\!\uparrow \cap \; \pi_2\!\uparrow \; = \emptyset$. Hence in a minimal set of paths Π no path of Π is a proper prefix of another path of Π. We will express that Π is minimal by writing $\min(\Pi)$. As example note that every singleton set $\{\pi\}$, $\pi \in \mathrm{FPaths}(s)$, is minimal. Also every subset of $\mathrm{CPaths}(s)$ is minimal, too.

For $\Pi \subseteq \mathrm{FPaths}(s)$ we denote by $\Pi\!\uparrow$ the set

$$\Pi\!\uparrow \; = \bigcup_{\pi \in \Pi} \pi\!\uparrow \; .$$

Then the fact $\min(\Pi)$ just means that $\Pi{\uparrow}$ is actually the disjoint union of all $\pi{\uparrow}$, $\pi \in \Pi$, i.e.

$$\min(\Pi) \quad \Longleftrightarrow \quad \Pi{\uparrow} = \bigsqcup_{\pi \in \Pi} \pi{\uparrow},$$

where, here and in the sequel, the symbol $\sqcup$ denotes disjoint unions. It is an immediate consequence of the definition that,

$$\min(\Pi), \ \Pi' \subseteq \Pi \quad \Longrightarrow \quad \min(\Pi').$$

However, if Π_1 and Π_2 are minimal, their union need not necessarily be minimal, even if $\Pi_1 \cap \Pi_2 = \emptyset$. We will use the notation

$$\Pi = \biguplus_{i \in I} \Pi_i$$

to express that

$$\Pi_i \subseteq \mathrm{FPaths}(s), \ i \in I, \ \Pi = \bigsqcup_{i \in I} \Pi_i \ \text{and} \ \min(\Pi) \, .$$

Note that if $\Pi = \biguplus_{i \in I} \Pi_i$, also $\min(\Pi_i)$ for all $i \in I$. In particular this notation applies to minimal subsets Π written as the union of their one-element subsets:

$$\min(\Pi) \quad \Longrightarrow \quad \Pi = \biguplus_{\pi \in \Pi} \{\pi\}.$$

Observe that the following two properties hold, as can be readily checked.

- If $\Pi = \biguplus_{i \in I} \Pi_i$, then $\Pi{\uparrow} = \bigsqcup_{i \in I} \Pi_i{\uparrow} = \bigsqcup_{i \in I, \pi \in \Pi_i} \pi{\uparrow}$.

- We have $\Pi = \biguplus_{i \in I} \Pi_i$ if and only if

  - $\forall i \in I : \ \min(\Pi_i)$, and
  - $\forall i, j \in I : \ i \neq j \ \Longrightarrow \ \Pi_i \cap \Pi_j = \emptyset$, and
  - $\forall i, j \in I : \ i \neq j \ \Longrightarrow \ \forall \pi_i \in \Pi_i, \forall \pi_j \in \Pi_j : \ \pi_i \not\preceq \pi_j$ and $\pi_j \not\preceq \pi_i$.

Let $\Pi \subseteq \mathrm{FPaths}(s)$. Put $\Pi{\downarrow} = \{\pi \in \Pi \mid \forall \pi' \in \Pi : \pi' \not\prec \pi\}$.

Lemma 7 *For any subset $\Pi \subseteq \mathrm{FPaths}(s)$, it holds that $\Pi{\downarrow} \subseteq \Pi$, $\min(\Pi{\downarrow})$ and $\Pi{\uparrow} = (\Pi{\downarrow}){\uparrow}$.* $\qquad\qquad\square$

5.2 The measure Prob

We proceed with the construction of a probability measure Prob out of the distribution P of a generative system $\langle S, A, P \rangle$ on a certain σ-algebra on CPaths(s). This method was used in many papers, also in [8, 6], and before that in [39], where the setting is slightly different and/or only a part of the story is given. Here we give complete proofs for our setting. As a standard reference for measure theoretic notions and results we use the monograph [47]. An important measure theoretic result is the extension theorem which states that any pre-measure (σ-additive, monotone function with value zero for the empty set) on a semi-ring extends in a unique way to a measure on the σ-field generated by the semi-ring. Slightly different versions of this theorem apply to different definitions of the notion "semi-ring". For our purposes, the definition of a semi-ring from [47] fits best. Namely, a family of subsets of a given set S is a semi-ring if it contains the empty set, is closed under finite intersection and the set difference of any two of its elements is a disjoint union of at most countably many elements of the semi-ring.

Lemma 8 *The set* Cones(s) $\cup \{\emptyset\}$ *is a semi-ring.*

Proof: Clearly, Cones(s) $\cup \{\emptyset\}$ contains the empty set and it is closed under intersection, by Lemma 6. We need to check that the set-difference of any two of its elements is a disjoint union of at most countably many elements of Cones(s) $\cup \{\emptyset\}$. Let $\pi_1 \uparrow, \pi_2 \uparrow \in$ Cones(s). We consider $\pi_1 \uparrow \setminus \pi_2 \uparrow$. Since $\pi_1 \uparrow \setminus \pi_2 \uparrow = \pi_1 \uparrow \setminus (\pi_1 \uparrow \cap \pi_2 \uparrow)$, by Lemma 6, the only interesting case is $\pi_1 \uparrow \cap \pi_2 \uparrow = \pi_2 \uparrow \neq \pi_1 \uparrow$ which implies $\pi_1 \prec \pi_2$. Let

$$\Pi = \{\pi \mid \pi = \pi' \cdot \text{last}(\pi') \xrightarrow{a} s', \pi_1 \preceq \pi' \prec \pi_2, \pi \npreceq \pi_2\}.$$

Then $\pi_1 \uparrow \setminus \pi_2 \uparrow = \Pi \uparrow = \sqcup_{\pi \in \Pi} \pi \uparrow$. This union is at most countable since the set Π is at most countable by Lemma 5. □

Now we are ready to introduce the desired extension of P to a measure. By Lemma 6, a function Prob : Cones(s) $\cup \{\emptyset\} \to [0, 1]$ is well-defined by Prob($\emptyset$) $= 0$, Prob($\varepsilon \uparrow$) $=$ Prob(CPaths(s)) $= 1$ and

$$\text{Prob}(C) = P(s, a, s') \cdot \text{Prob}(C'), \text{ for } C = \pi \uparrow, \ \pi = s \xrightarrow{a} s' \cdot \pi', C' = \pi' \uparrow$$

Lemma 9 *The function* Prob *is a pre-measure*[4] *on the semi-ring* Cones(s) $\cup \{\emptyset\}$.

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[4]In [47] pre-measures are also called measures.

Proof: By definition $\text{Prob}(\emptyset) = 0$. Further we need to check monotonicity and σ-additivity. To see that Prob is monotonic assume $\pi_1\uparrow \subseteq \pi_2\uparrow$. Then, by Lemma 6, we have two possibilities. The first one is $\pi_2 \prec \pi_1$ and since $P(s, a, t) \leq 1$ for all $s, t \in S, a \in A$, from the definition of Prob we get $\text{Prob}(\pi_1\uparrow) \leq \text{Prob}(\pi_2\uparrow)$. The second possibility is $\pi_1\uparrow = \pi_2\uparrow$, in which case $\text{Prob}(\pi_1\uparrow) = \text{Prob}(\pi_2\uparrow)$.

For the σ-additivity, assume

$$\pi\uparrow = \bigsqcup_{i \in I} \pi_i\uparrow \tag{11}$$

for some at most countable index set I. We need to show that $\text{Prob}(\pi\uparrow) = \sum_{i \in I} \text{Prob}(\pi_i\uparrow)$.

If $|I| = 1$, then the property is trivially satisfied. Therefore we assume that $|I| > 1$. In particular this means that π is not terminating.

There exists (via a Lemma of Zorn argument) a partial function depth[5] that assigns to some finite paths an ordinal number, satisfying:

1. If $\xi \in \text{FPaths}(s)$ is such that $\pi_i \preceq \xi$ for some $i \in I$, or if ξ terminates, then $\text{depth}(\xi) = 0$.

2. Otherwise, if ξ is a finite path such that all its one step successors $\{\xi' \mid \xi \preceq \xi', \text{length}(\xi') = \text{length}(\xi) + 1\}$ have assigned depth then also ξ belongs to the domain of depth and

   $$\text{depth}(\xi) = \sup\{\text{depth}(\xi') \mid \xi \preceq \xi', \text{length}(\xi') = \text{length}(\xi) + 1\} + 1. \tag{12}$$

Actually the function depth applied to a finite path ξ captures how deep in the cone generated by ξ one must go in order to be sure that all extensions of the path under consideration belong to some $\pi_i\uparrow$ for $i \in I$ or terminate. In other words, if $\text{depth}(\xi)$ is defined, and if Ξ is the set of paths that extend ξ in at least $\text{depth}(\xi)$ steps, then any path that extends any path in Ξ belongs to some of the cones $\pi_i\uparrow$ for $i \in I$ or terminates.

We first show, by reducing to contradiction, that our starting finite path π has been assigned a value for depth. Assume that π has not been assigned a value for depth. Let $\pi^0 = \pi$. For each $i > 0$ let π^i be a path such that $\text{length}(\pi^i) = \text{length}(\pi^{i-1}) + 1$, $\pi^{i-1} \preceq \pi^i$ and π^i has not been assigned a value for depth. Such a chain under the prefix ordering exists since if

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[5]The function depth has also been defined and used in a proof of a similar property by Segala [39].

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for some i all paths that extend π^i in one step would had been assigned depth, then π^i would also have been assigned a depth. Consider the infinite complete path π^∞ such that for all $i > 0$, $\pi^i \preceq \pi^\infty$. By definition $\pi^\infty \in \pi{\uparrow}$. By (11), there exists $i \in I$ such that $\pi^\infty \in \pi_i{\uparrow}$, implying that $\pi_i \preceq \pi^\infty$ and hence $\pi_i = \pi^n$ for some $n \geq 0$. However, then $\text{depth}(\pi^n) = \text{depth}(\pi_i) = 0$ contradicting that π^n has no value for depth assigned.

Let $\hat{\pi}$ be any non-terminating path and let $\{\pi_o \mid o \in O\}$ be the set of paths that extend $\hat{\pi}$ in one step, which means that

$$\forall o \in O: \quad \hat{\pi} \prec \pi_o, \text{length}(\pi_o) = \text{length}(\hat{\pi}) + 1. \tag{13}$$

Then

$$\hat{\pi}{\uparrow} = \bigsqcup_{o \in O} \pi_o{\uparrow} \tag{14}$$

and

$$
\begin{aligned}
\sum_{o \in O} \text{Prob}(\pi_o{\uparrow}) &= \sum_{a \in A, s' \in S} \text{Prob}(\hat{\pi}{\uparrow}) \cdot P(\text{last}(\hat{\pi}), a, s') \\
&= \text{Prob}(\hat{\pi}{\uparrow}) \cdot \sum_{a \in A, s' \in S} P(\text{last}(\hat{\pi}), a, s') \\
&= \text{Prob}(\hat{\pi}{\uparrow}) \tag{15}
\end{aligned}
$$

since $\hat{\pi}$ does not end in a terminating state, i.e. $\sum_{a \in A, s \in S} P(\text{last}(\hat{\pi}), a, s) = 1$.

We finally show, by induction on depth, that if $\hat{\pi}$ is a finite path which has been assigned a value for depth and if

$$\hat{\pi}{\uparrow} = \bigsqcup_{i \in I' \subseteq I} \pi_i{\uparrow}, \tag{16}$$

for some $I' \subseteq I$, then $\text{Prob}(\hat{\pi}{\uparrow}) = \sum_{i \in I' \subseteq I} \text{Prob}(\pi_i{\uparrow})$. Assume $\hat{\pi}$ is a path with $\text{depth}(\hat{\pi}) = 0$ satisfying the assumption above. Then either $\hat{\pi}$ terminates or $\hat{\pi}{\uparrow} = \pi_i{\uparrow}$ for some $i \in I'$ and therefore $|I'| = 1$ and the additivity holds trivially. Now assume $\text{depth}(\hat{\pi}) = \alpha$ and α is a successor ordinal (by definition α can not be a limit ordinal). This implies that $\hat{\pi}$ is not terminating. Moreover assume that the property holds for any path of the discussed form with depth smaller than α and let $\{\pi_o \mid o \in O\}$ be the set of paths that extend $\hat{\pi}$ in one step.

By (16) we have that

$$\forall i \in I': \quad \hat{\pi} \preceq \pi_i. \tag{17}$$

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Moreover, from (16) and (14), using Lemma 6 we easily conclude that

$$\forall i \in I', \exists! o \in O \colon \ \pi_o \preceq \pi_i \tag{18}$$

and

$$\forall o \in O, \exists i \in I' \colon \ \pi_o \preceq \pi_i. \tag{19}$$

Let

$$I'_o = \{ i \in I' \mid \pi_o \preceq \pi_i \}.$$

From (16), (18) and (19), we get that $I'_o \neq \emptyset$,

$$I' = \bigsqcup_{o \in O} I'_o \qquad \text{and} \qquad \pi_o{\uparrow} = \bigsqcup_{i \in I'_o} \pi_i{\uparrow} \text{ for } o \in O. \tag{20}$$

Then we get

$$
\begin{aligned}
\mathrm{Prob}(\hat{\pi}{\uparrow}) \ &\overset{(15)}{=}\ \sum_{o \in O} \mathrm{Prob}(\pi_o{\uparrow}) \\
&\overset{(I.H.)}{=}\ \sum_{o \in O} \sum_{i \in I'_o} \mathrm{Prob}(\pi_i{\uparrow}) \\
&\overset{(20)}{=}\ \sum_{i \in I'} \mathrm{Prob}(\pi_i{\uparrow}).
\end{aligned}
$$

where the inductive hypothesis is applicable since by (12) and (13), $\mathrm{depth}(\pi_o) < \alpha$ for all $o \in O$ and $I'_o \subseteq I' \subseteq I$. This completes the proof. □

Corollary 2 *The function* Prob *extends uniquely to a probability measure on the σ-algebra on* $\mathrm{CPaths}(s)$ *generated by* $\mathrm{Cones}(s) \cup \{\emptyset\}$. *We will denote this measure again by* Prob. □

Remark 2 *Note that, although paths are more or less just alternating sequences of elements of S and A, whether an alternating sequence of states and actions is a path depends on the distribution P. Therefore the function* Prob *itself, but also the σ-algebra where it is defined and in fact already the base set* $\mathrm{CPaths}(s)$ *depends heavily on P.*

The measure Prob induces a function on sets of finite paths, which we will also denote by Prob. We define $\mathrm{Prob} \colon \mathcal{P}(\mathrm{FPaths}(s)) \to [0, 1]$ by

$$\mathrm{Prob}(\Pi) = \mathrm{Prob}(\Pi{\uparrow}).$$

Note that $\Pi{\uparrow}$ is measurable since it is a countable union of cones. This notation is not in conflict with the already existing notation of the measure Prob. In fact, $\mathcal{P}(\text{FPaths}(s)) \cap \mathcal{P}(\text{CPaths}(s))$ consists entirely of Prob-measurable sets and on such sets both definitions coincide. To see this, note that if $\pi \in \text{FPaths}(s) \cap \text{CPaths}(s)$, then $\pi{\uparrow} = \{\pi\}$. Thus, if $\Pi \subseteq \text{FPaths}(s)$ and $\Pi \subseteq \text{CPaths}(s)$, we have

$$\Pi = \bigsqcup_{\pi \in \Pi} \{\pi\} = \bigsqcup_{\pi \in \Pi} \pi{\uparrow} = \Pi{\uparrow} \,,$$

and this union is at most countable.

It will always be clear from the context whether we mean the measure Prob or the just defined function Prob on sets of finite paths. Still, there is a word of caution in order: The function $\text{Prob} : \mathcal{P}(\text{FPaths}(s)) \to [0,1]$ is, in general, not additive. However, looking at the properties of $\uplus$ introduced above (on page 22), we find that

$$\Pi = \biguplus_{i \in I} \Pi_i \quad \implies \quad \text{Prob}(\Pi) = \sum_{i \in I} \text{Prob}(\Pi_i) \,.$$

For this reason, we will overload the notation $\uplus$ and use it also for sets of cones generated by sets of finite paths, i.e. from now on we will freely write

$$\Pi{\uparrow} = \biguplus_{i \in I} \Pi_i{\uparrow}$$

if and only if it holds that $\Pi = \biguplus_{i \in I} \Pi_i$ for $\Pi, \Pi_i \subseteq \text{FPaths}(s)$.

We obtain that $\text{Prob}(\Pi) = \sum_{\pi \in \Pi} \text{Prob}(\pi{\uparrow})$ for every minimal set Π. Moreover, by Lemma 7, we always have $\text{Prob}(\Pi) = \text{Prob}(\Pi{\downarrow})$.

We next introduce some particular sets of paths. For $s \in S$, $S', S'' \subseteq S$ with $S' \subseteq S''$, and $W, W' \subseteq A^*$ with $W \subseteq W'$, by

$$s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S'$$

we denote the set of all finite paths that start in s, have a trace in W, end up in S', without passing a state in S'' having just performed a trace in the set W'. Formally,

$$s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' = \left\{ \pi \in \text{FPaths}(s) \mid \begin{array}{c} \text{last}(\pi) \in S', \text{trace}(\pi) \in W \\ \forall \xi \prec \pi : \text{trace}(\xi) \notin W' \vee \text{last}(\xi) \notin S'' \end{array} \right\}.$$

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We write $\mathrm{Prob}(s, W, \neg W, S', \neg S'') = \mathrm{Prob}(s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S')$. Since $S' \subseteq S''$ and $W \subseteq W'$ we always have $\min(s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S')$. For notational convenience we will drop redundant arguments whenever possible. Put

$$
\begin{aligned}
s \xrightarrow[\substack{\neg W' \\ \neg S'}]{W} S' &= s \xrightarrow[\substack{\neg W' \\ \neg S'}]{W} S', \\
s \xrightarrow[\substack{\neg W \\ \neg S''}]{W} S' &= s \xrightarrow[\substack{\neg W \\ \neg S''}]{W} S', \\
s \xrightarrow{W} S' &= s \xrightarrow[\substack{\neg W \\ \neg S'}]{W} S',
\end{aligned}
\tag{21}
$$

and, correspondingly,

$$
\begin{aligned}
\mathrm{Prob}(s, W, \neg W', S') &= \mathrm{Prob}(s, W, \neg W', S', \neg S'), \\
\mathrm{Prob}(s, W, S', \neg S'') &= \mathrm{Prob}(s, W, \neg W, S', \neg S''), \\
\mathrm{Prob}(s, W, S') &= \mathrm{Prob}(s, W, \neg W, S', \neg S').
\end{aligned}
\tag{22}
$$

Note that

$$
s \xrightarrow{W} S' = \bigl\{ \pi \in \mathrm{FPaths}(s) \mid \mathrm{trace}(\pi) \in W, \mathrm{last}(\pi) \in S' \bigr\}{\downarrow}
$$

and hence

$$
\begin{aligned}
\mathrm{Prob}(s, W, S') &= \mathrm{Prob}(s \xrightarrow{W} S') \\
&= \mathrm{Prob}(\{\pi \in \mathrm{FPaths}(s) \mid \mathrm{trace}(\pi) \in W, \mathrm{last}(\pi) \in S'\}).
\end{aligned}
\tag{23}
$$

Also, for $a \in A, t \in S$, we have

$$
\mathrm{Prob}(s, \{a\}, \{t\}) = \begin{cases} \mathrm{Prob}(s \xrightarrow{a} t) = P(s, a, t), & \text{if } s \xrightarrow{a} t \\ \mathrm{Prob}(\emptyset) = 0, & \text{otherwise} \end{cases}
\tag{24}
$$

Let S', S'', W, W' be as above. Suppose $F \subseteq S$. Then we put

$$
F \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' = \bigsqcup_{s \in F} s \xrightarrow[\substack{\neg W' \\ \neg S''}]{W} S' \subseteq \mathrm{FPaths}
$$

In case that for every $s \in F$ the value of $\mathrm{Prob}(s, W, \neg W', S', \neg S'')$ is the same, we speak of this value as $\mathrm{Prob}(F, W, \neg W', S', \neg S'')$. Also, in this context, we shall freely apply shorthand as in (21) and (22).

The next technical property concerning sets of concatenated paths will be used at several occasions in the paper. Note that, whenever a concatenation $\pi_1 \cdot \pi_2$ is defined, we have $\mathrm{Prob}(\{\pi_1 \cdot \pi_2\}) = \mathrm{Prob}(\{\pi_1\}) \cdot \mathrm{Prob}(\{\pi_2\})$. The proof is rather elementary and can be found in [44].

Proposition 5 *Let $\Pi_1 \subseteq \mathrm{FPaths}(s)$, $\Pi_2 \subseteq \mathrm{FPaths}$ and assume that the set of states S is represented as a disjoint union $S = \bigsqcup_{i \in I} S_i$. Denote $\Pi_{1,S_i} = \{\pi_1 \in \Pi_1 \mid \mathrm{last}(\pi_1) \in S_i\}$, $\Pi_{2,t} = \{\pi_2 \in \Pi_2 \mid \mathrm{first}(\pi_2) = t\}$. Assume that for every $i \in I$*

$$\mathrm{Prob}(\Pi_{2,t'}) = \mathrm{Prob}(\Pi_{2,t''}), \ t', t'' \in S_i .$$

Moreover, assume that Π_1, Π_2 and $\Pi_1 \cdot \Pi_2$ are minimal. Then, for every choice of $(t_i)_{i \in I} \in \prod_{i \in I} S_i$, we have

$$\mathrm{Prob}(\Pi_1 \cdot \Pi_2) = \sum_{i \in I} \mathrm{Prob}(\Pi_{1,S_i}) \cdot \mathrm{Prob}(\Pi_{2,t_i}) .$$

$\square$

It is worth to explicitly note the particular case of this proposition when $|I| = 1$.

Corollary 3 *Let $\Pi_1 \subseteq \mathrm{FPaths}(s)$, $\Pi_2 \subseteq \mathrm{FPaths}$. Let $\Pi_{2,t} = \{\pi_2 \in \Pi_2 \mid \mathrm{first}(\pi_2) = t\}$. Then, if $\min(\Pi_1)$, $\min(\Pi_2)$ and $\min(\Pi_1 \cdot \Pi_2)$, and if for any $t', t'' \in \mathrm{first}(\Pi_2)$, $\mathrm{Prob}(\Pi_{2,t'}) = \mathrm{Prob}(\Pi_{2,t''})$, we have that*

$$\mathrm{Prob}(\Pi_1 \cdot \Pi_2) = \mathrm{Prob}(\Pi_1) \cdot \mathrm{Prob}(\Pi_{2,t})$$

for arbitrary $t \in \mathrm{first}(\Pi_2)$.

$\square$

For further reference, we state the following simple property.

Proposition 6 *Consider a generative system $\langle S, A, P \rangle$. Let $s \in S$, $W \subseteq A^*$ and $S' \subseteq S$ such that it partitions as $S' = \sqcup_{i \in I} S_i$. Then*

$$\mathrm{Prob}(s, W, S') = \sum_{i \in I} \mathrm{Prob}(s, W, S_i, \neg S').$$

Proof: We have $s \xrightarrow{W} S' = \biguplus_{i \in I} s \xrightarrow{W}_{\neg S'} S_i$. $\square$

5.3   The concrete weak bisimulation

In this subsection we recall the original definition of weak bisimulation and branching bisimulation for generative systems proposed by Baier and Hermanns and we establish some properties of these relations that are essential for the correspondence result in Section 5.5 below.

Definition 13 *[7, 6, 8] Let $\langle S, A, P \rangle$ be a generative system. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a weak bisimulation on $\langle S, A, P \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for all actions $a \in A \setminus \{\tau\}$ and for all equivalence classes $C \in S/R$:*

$$\text{Prob}(s, \tau^* a \tau^*, C) = \text{Prob}(t, \tau^* a \tau^*, C) \tag{25}$$

and for all $C \in S/R$:

$$\text{Prob}(s, \tau^*, C) = \text{Prob}(t, \tau^*, C). \tag{26}$$

Two states s and t are weakly bisimilar if and only if they are related by some weak bisimulation relation. Notation $s \approx_g t$.

Note the analogy between the transfer conditions (25), (26) and (4). The definition of branching bisimulation for generative systems is given below.

Definition 14 *[7, 6, 8] Let $\langle S, A, P \rangle$ be a generative system. Let $\tau \in A$ be the invisible action. An equivalence relation $R \subseteq S \times S$ is a branching bisimulation on $\langle S, A, P \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for all actions $a \in A \setminus \{\tau\}$ and for all equivalence classes $C \in S/R$:*

$$\text{Prob}(s, \tau^* a, C) = \text{Prob}(t, \tau^* a, C) \tag{27}$$

and for all $C \in S/R$:

$$\text{Prob}(s, \tau^*, C) = \text{Prob}(t, \tau^*, C). \tag{28}$$

Two states s and t are branching bisimilar if and only if they are related by some branching bisimulation relation. Notation $s \approx_g^{br} t$.

Baier and Hermanns have shown [6, 8] the following correspondence result for finite systems, i.e. systems with finite set of states.

Proposition 7 *Any weak bisimulation on a finite generative system is a branching bisimulation and vice versa. Hence, branching bisimilarity and weak bisimilarity coincide on finite systems.* □

Also for arbitrary generative systems branching bisimilarity implies weak bisimilarity, i.e., the proof of this direction of Proposition 7 does not require finiteness, as shown below.

Proposition 8 *Any branching bisimulation on a generative system is a weak bisimulation as well.*

Proof: The property follows since we have $s \xrightarrow{\tau^* a \tau^*} C = \biguplus_{C' \in S/R} s \xrightarrow{\tau^* a} C' \cdot C' \xrightarrow{\tau^*} C$ given a branching bisimulation R, $s \in S$, $a \in A$ and $C \in S/R$. $\quad\square$

Whether a coincidence result as in Proposition 7 holds for arbitrary systems is an open question. The proof for finite systems can not be extended to arbitrary systems - in particular in Lemma 7.5.4 of [6] we can not obtain regularity for arbitrary matrices. On the other hand, up to now, an example showing the difference between weak and branching bisimilarity for arbitrary systems is not known to us. Therefore, we distinguish between the two notions.

Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Define a relation $\rightarrow$ on S/R by

$$C_1 \rightarrow C_2 \iff \operatorname{Prob}(C_1, \tau^*, C_2) = 1$$

and denote by $\leftrightarrow$ the equivalence closure of $\rightarrow$, i.e., $\leftrightarrow = (\rightarrow \cup \leftarrow)^*$.

A weak or branching bisimulation on $\langle S, A, P \rangle$ is called *complete*, if

$$\operatorname{Prob}(C_1, \tau^*, C_2) = 1 \iff C_1 = C_2$$

for all classes $C_1, C_2 \in S/R$. Hence, if R is a complete weak or branching bisimulation then for any two different classes $C_1, C_2 \in S/R$ it holds that $\operatorname{Prob}(C_1, \tau^*, C_2) < 1$.

The next proposition is essential for the correspondence result below. Its proof is long, involved, and includes a detailed study of the $\rightarrow$ relation. We only give a sketch, details can be found in [44]. A similar property is stated in [8, 6] without a proof.

Proposition 9 *Let $\langle S, A, P \rangle$ be a generative system and let $s \approx_g t$ or $s \approx_g^{br} t$. Then there exists a complete weak or a complete branching bisimulation R, respectively, relating s and t.*

Proof: (Sketch) The proof follows by a limit argument, using the Lemma of Zorn, from the following property:

Let R be a weak or branching bisimulation on $\langle S, A, P \rangle$. Let $C_0 \in S/R$ be a fixed class such that $U = [C_0]_{\leftrightarrow} \neq \{C_0\}$. Here $[C_0]_{\leftrightarrow}$ denotes the $\leftrightarrow$ -

equivalence class of C. Define an equivalence R' on S by

$$\langle s, t \rangle \in R' \iff \langle s, t \rangle \in R \vee \{s, t\} \subseteq \bigcup_{C \in U} C.$$

Then R' is a weak or branching bisimulation, respectively, and $R \subset R'$.

Hence, if R is not complete, then a larger weak or branching bisimulation can be derived from it (by joining some classes). $\quad\square$

5.4 Weak coalgebraic bisimulation for generative systems

In this subsection we provide a coalgebraic definition of weak bisimulation for generative systems, according to the approach from Section 3. For this we need a $*$-translation that will transform the generative systems with action set A into systems with action set A^*. Unlike for LTSs, the $*$-translation employed will yield coalgebras of a different type.

Let $\mathcal{G}^*$ be the bifunctor defined by

$$\mathcal{G}^*(A, S) = \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$$

on objects $\langle A, S \rangle$ and for morphisms $\langle f_1, f_2 \rangle \colon \langle A, S \rangle \to \langle B, T \rangle$ by

$$\mathcal{G}^*\langle f_1, f_2 \rangle = (\nu \mapsto \nu \circ (f_1^{-1} \times f_2^{-1}) \mid \nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]).$$

Consider the Set functor $\mathcal{G}_A^*$ corresponding to $\mathcal{G}^*$, so that

$$\mathcal{G}_A^*(S) = (\mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1])$$

and for a mapping $f \colon S \to T$,

$$\mathcal{G}_A^* f(\nu) = \nu \circ (id_A^{-1} \times f^{-1})$$

for $\nu \colon \mathcal{P}(A) \times \mathcal{P}(S) \to [0, 1]$.

We will use the functor $\mathcal{G}_A^*$ to model the $*$-translation of generative systems. Therefore we are interested in characterizing equivalence bisimulations for this functor. In order to apply the results from Section 2 we need the following proposition. We dedicate Appendix C to its proof.

Proposition 10 *The functor $\mathcal{G}_A^*$ weakly preserves total pullbacks, but it does not preserve weak pullbacks.* $\quad\square$

Let R be an equivalence relation on a set S. A subset $M \subseteq S$ is an *R-saturated set* if for all $s \in M$ the whole equivalence class of s is contained in M. We denote by $\mathrm{Sat}(R)$ the set of all R-saturated sets, $\mathrm{Sat}(R) \subseteq \mathcal{P}(S)$. Actually, M is a saturated set if and only if $M = \cup_{i \in I} C_i$ for $C_i \in S/R$. Hence there is a one-to-one correspondence between the R-saturated sets and the elements of $\mathcal{P}(S/R)$.

The next lemma contains a transfer condition for equivalence bisimulations for systems of type $\mathcal{G}_A^*$. Its proof follows the approach discussed in Section 2 (see Lemma 2 and Lemma 3).

Lemma 10 *An equivalence relation R on a set S is a bisimulation on the $\mathcal{G}_A^*$ system $\langle S, A, \alpha \rangle$ if and only if*

$$\langle s, t \rangle \in R \implies \forall A' \subseteq A, \forall M \in \mathrm{Sat}(R) \colon \alpha(s)(A', M) = \alpha(t)(A', M).$$

Proof: Consider the pullback P of the cospan

$$\mathcal{G}_A^* S \xrightarrow{\mathcal{G}_A^* c} \mathcal{G}_A^*(S/R) \xleftarrow{\mathcal{G}_A^* c} \mathcal{G}_A^* S$$

where c is the canonical projection of S onto S/R. We have $\langle \mu, \nu \rangle \in P$ if and only if $\mathcal{G}_A^* c(\mu) = \mathcal{G}_A^* c(\nu)$, i.e. $\mu \circ (id_A^{-1} \times c^{-1}) = \nu \circ (id_A^{-1} \times c^{-1})$. This is equivalent to

$$\forall A' \subseteq A, \forall M \subseteq S/R \colon \mu(A', c^{-1}(M)) = \nu(A', c^{-1}(M))$$

and, since $c^{-1} \colon \mathcal{P}(S/R) \to \mathrm{Sat}(R)$ is a bijection, we get an equivalent condition

$$\forall A' \subseteq A, \forall M \in \mathrm{Sat}(R) \colon \mu(A', M) = \nu(A', M).$$

Now, using Lemma 2, and Proposition 10, we obtain the stated characterization. $\qquad\square$

We proceed by presenting a suitable $*$-translation for generative systems. The translation will yield a system of type $\mathcal{G}_{A^*}^*$. Recall that generative systems are coalgebras of the functor $\mathcal{G}_A = \mathcal{D}(A \times \mathcal{I}d) + 1$.

Definition 15 *Let Φ^g assign to every generative system $\langle S, A, P \rangle$, i.e. any $\mathcal{G}_A$-coalgebra $\langle S, A, \alpha \rangle$, the $\mathcal{G}_{A^*}^*$-coalgebra $\langle S, A^*, \alpha' \rangle$, where for $W \subseteq A^*$ and $S' \subseteq S$, $\alpha'(s)(W, S') = \mathrm{Prob}(s, W, S')$.*

In order to show that the translation defined above is indeed a $*$-translation we need the property below. Its proof is straightforward and can be found in [44].

Lemma 11 *Let $\langle S, A, \alpha \rangle$, i.e. $\langle S, A, P \rangle$, be a $\mathcal{G}_A$ system, R a bisimulation equivalence on $\langle S, A, \alpha \rangle$ and $\langle s, t \rangle \in R$. For $k \in \mathbb{N}, C_i \in S/R$ and $a_i \in A$, $i \in \{1, \ldots, k\}$, let $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ denote the set of paths*

$$s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k = \{ s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \mid s_i \in C_i, i = 1, \ldots, k \}.$$

Then $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ is minimal and

$$\mathrm{Prob}(s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) = \mathrm{Prob}(t \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k) \qquad (29)$$

$\square$

We can now show that the defined map is a $*$-translation.

Proposition 11 *The assignment Φ^g from Definition 15 is a $*$-translation.*

Proof: We need to check that Φ^g is injective and preserves and reflects bisimilarity. For injectivity, assume $\Phi^g(\langle S, A, \alpha \rangle) = \Phi^g(\langle S, A, \beta \rangle) = \langle S, A^*, \alpha' \rangle$. Then, by the definition of Prob, cf. (24), we get that for any $s, t \in S$ and any $a \in A$, $\alpha(s)(\langle a, t \rangle) = P(s, a, t) = \mathrm{Prob}(s, \{a\}, \{t\}) = \alpha'(s)(\{a\}, \{t\}) = \beta(s)(\langle a, t \rangle)$.

Reflection of bisimilarity is direct from Lemma 10: Assume $s \sim t$ in $\Phi^g(\langle S, A, \alpha \rangle) = \langle S, A^*, \alpha' \rangle$ and assume that R is an equivalence bisimulation on $\langle S, A^*, \alpha' \rangle$ such that $\langle s, t \rangle \in R$. By Lemma 10, we get that for $W \subseteq A^*$ and for $M \in \mathrm{Sat}(R)$,

$$\alpha'(s)(W, M) = \alpha'(t)(W, M). \qquad (30)$$

In particular, for all $a \in A$ and all $C \in S/R$, we have

$$\alpha'(s)(\{a\}, C) = \alpha'(t)(\{a\}, C). \qquad (31)$$

By the definition of α' and Prob we have

$$\alpha'(s)(\{a\}, C) = \mathrm{Prob}(s, \{a\}, C) = \sum_{s' \in C} P(s, a, s') = \sum_{s' \in C} \alpha(s)(\langle a, s' \rangle)$$

and therefore, for all $a \in A$ and all $C \in S/R$,

$$\sum_{s' \in C} \alpha(s)(\langle a, s' \rangle) = \sum_{s' \in C} \alpha(t)(\langle a, s' \rangle) \qquad (32)$$

which means that R is a bisimulation equivalence on the generative system $\langle S, A, \alpha \rangle$, i.e. $s \sim t$ in the original system.

The proof of preservation of bisimilarity uses Lemma 11. Let $s \sim t$ in the generative system $\langle S, A, \alpha \rangle$. Then there exists an equivalence bisimulation R with $\langle s, t \rangle \in R$. The relation R induces an equivalence R_s on FPaths(s) defined by

$$\langle s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \ , \ s \xrightarrow{a_1'} s_1' \cdots \xrightarrow{a_{k'}'} s_{k'}' \rangle \in R_s$$

if and only if $k = k'$, $a_i = a_i'$ and $\langle s_i, s_i' \rangle \in R$ for $i = 1, \ldots, k$. The classes of R_s are exactly the sets $s \xrightarrow{a_1} C_1 \cdots \xrightarrow{a_k} C_k$ for $C_i \in S/R$ and $a_i \in A$.

Assume $M \in \mathrm{Sat}(R)$ and $W \subseteq A^*$. We show that the set $s \xrightarrow{W} M$ is saturated with respect to R_s. Namely, let $\pi \equiv s \xrightarrow{a_1} s_1 \cdots \xrightarrow{a_k} s_k \in s \xrightarrow{W} M$ and let $\pi' \equiv s \xrightarrow{a_1} s_1' \cdots \xrightarrow{a_k} s_k'$ be a path such that $\langle \pi, \pi' \rangle \in R_s$. Then trace($\pi$) = trace($\pi'$), first($\pi$) = first($\pi'$) and $\langle \mathrm{last}(\pi), \mathrm{last}(\pi') \rangle \in R$. Since M is saturated, $\mathrm{last}(\pi') \in M$ for $\mathrm{last}(\pi) \in M$. Furthermore, π' does not have a proper prefix with trace in W and last in M, since this would imply that π has such a prefix, contradicting $\pi \in s \xrightarrow{W} M$. Hence, $\pi' \in s \xrightarrow{W} M$.

Therefore, the set $s \xrightarrow{W} M$ is a disjoint union of some R_s classes and, since $s \xrightarrow{W} M$ is minimal, we can write

$$s \xrightarrow{W} M = \biguplus_{i \in I} s \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik_i}} C_{ik_i} \ ,$$

and it follows that $\mathrm{Prob}(s, W, M) = \sum_{i \in I} \mathrm{Prob}(s \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik}} C_{ik})$. Similarly, $t \xrightarrow{W} M$ is a disjoint union of some R_t classes, for R_t being an equivalence on FPaths(t), defined as R_s with t instead of s. Using that R is a bisimulation and $\langle s, t \rangle \in R$, it is not difficult to see that actually

$$t \xrightarrow{W} M = \biguplus_{i \in I} t \xrightarrow{a_{i1}} C_{i1} \cdots \xrightarrow{a_{ik_i}} C_{ik_i} \ .$$

By Lemma 11, we get that $\mathrm{Prob}(s, W, M) = \mathrm{Prob}(t, W, M)$, i.e. $\alpha'(s)(W, M) = \alpha'(t)(W, M)$ proving that R is a bisimulation on $\langle S, A^*, \alpha' \rangle$ and $s \sim t$ in the $*$-extension $\langle S, A^*, \alpha' \rangle$. □

The $*$-translation Φ^g is also not induced by a natural transformation, as the systems of Example 1 in Section 4 show, interpreting each transition as probabilistic with probability 1.

Remark 3 *The $*$-translation Φ^g together with a subset $\tau \subseteq A$ determines a weak-τ-bisimulation. For a generative system $\langle S, A, \alpha \rangle$, the weak-$\tau$-system is*

$$\Psi_\tau \circ \Phi^g(\langle S, A, \alpha \rangle) = \Psi_\tau(\langle S, A^*, \alpha' \rangle) = \langle S, A_\tau, \alpha'' \rangle$$

where $\alpha''(s) : \mathcal{P}(A_\tau) \times \mathcal{P}(S) \to [0,1]$ is given by

$$\alpha''(s) = \eta_S^\tau(\alpha'(s)) = \mathcal{G}^*\langle h_\tau, id_S\rangle(\alpha'(s)) = \alpha'(s) \circ (h_\tau^{-1} \times id_S^{-1}).$$

Hence for $X \subseteq A_\tau$ and $S' \subseteq S$,

$$\alpha''(s)(X, S') = \alpha'(s)(h_\tau^{-1}(X), S') = \alpha'(s)(\bigcup_{w \in X} B_w, S') = \mathrm{Prob}(s, \bigcup_{w \in X} B_w, S'),$$

where, B_w is the block $B_w = \tau^* a_1 \tau^* \dots \tau^* a_k \tau^* = h_\tau^{-1}(\{w\})$, for a word $w = a_1 \dots a_k \in A_\tau$.

Therefore, from Lemma 10 we get that an equivalence relation R is a weak-τ-bisimulation w.r.t. $\langle \Phi^g, \tau \rangle$ on the generative system $\langle S, A, \alpha \rangle$ if and only if $\langle s, t \rangle \in R$ implies that for any collection $(B_i)_{i \in I}$ of blocks writing B_i as a shorthand for B_{w_i} for some word $w_i \in A^*$, and any collection $(C_j)_{j \in J}$ of classes $C_j \in S/R$,

$$\mathrm{Prob}(s, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j) = \mathrm{Prob}(t, \bigcup_{i \in I} B_i, \bigcup_{j \in J} C_j). \tag{33}$$

Sets of the form $\cup_{i \in I} B_i$ will be called saturated blocks.

5.5 Correspondence results

We are now able to state and prove the correspondence results for generative systems. The first statement is obvious from the definitions.

Theorem 2 *Let $\langle S, A, \alpha \rangle$ be a generative system. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_{\{\tau\}} t$ according to Definition 9 with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$ implies $s \approx_g t$ according to Definition 13.*

Proof: The statement holds trivially, having in mind Definition 13 and Remark 3, equation (33), since τ^* as well as $\tau^* a \tau^*$, for any $a \in A \setminus \{\tau\}$ is a saturated block and also each R-equivalence class is an R saturated set. Hence $\approx_{\{\tau\}}$ is at least as strong as $\approx_g$, $\approx_{\{\tau\}} \subseteq \approx_g$. □

In the opposite direction we have that coalgebraic weak bisimilarity is implied by branching bisimilarity.

Theorem 3 *Let $\langle S, A, \alpha \rangle$ be a generative system. Let $\tau \in A$ be the invisible action and $s, t \in S$ any two states. Then $s \approx_g^{br} t$ according to Definition 14 implies $s \approx_{\{\tau\}} t$ according to Definition 9 with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$.*

Proof: (Sketch) The proof of this theorem is rather technical, long, and divided into several steps. Here we discuss its outline. Details can be found in [44].

Let $\langle S, A, P \rangle$ be a generative system, and $s, t \in S$. For ease of presentation we write $\text{Holds}[s, t, W, S']$ for the statement $\text{Prob}(s, W, S') = \text{Prob}(t, W, S')$, where $W \subseteq A^*$ and $S' \subseteq S$. We also write $\text{Holds}[s, t, W, S', \neg S'']$ for the statement $\text{Prob}(s, W, S', \neg S'') = \text{Prob}(t, W, S', \neg S'')$ if needed.

Now assume $s \approx_g^{br} t$ for two states s and t. This means that there exists a branching bisimulation R, according to Definition 14, with $\langle s, t \rangle \in R$. Hence, R is an equivalence relation such that for all visible actions a and all equivalence classes $C \in S/R$ we have $\text{Holds}[s, t, \tau^*a, C]$. Additionally $\text{Holds}[s, t, \tau^*, C]$. By Proposition 9, we can assume that R is complete (this is essential for the proof).

We want to show that the transfer condition (33) of Remark 3 holds, and hence R is a coalgebraic weak bisimulation witnessing that $s \approx_{\{\tau\}} t$. Therefore, we need to show that for any R-saturated set $M = \cup_j C_j$ and any union of blocks $W = \cup_i B_i$ it holds that

$$\text{Holds}[s, t, W, M].$$

We establish successively:

1. $\text{Holds}[s, t, B, C]$ where B is any block, $B = \tau^*a_1\tau^* \ldots \tau^*a_k\tau^*$ and C any class.

2. $\text{Holds}[s, t, B, C_i, \neg M]$ for any block B, any saturated set M, and $C_i \subseteq M$.

3. $\text{Holds}[s, t, B, M]$ for any block B and any saturated set M.

4. $\text{Holds}[s, t, W, M]$ for any saturated block (i.e. union of blocks) W and any saturated set M.

In each item, the main idea is to represent the set to be measured in terms of the sets for which the statement has already been proved. For example, for the simplest step 1., we observe that for any $B' = \tau^*a_1\tau^* \ldots \tau^*a_k$ we have

$$s \xrightarrow{B'} C = \biguplus_{C' \in S/R} s \xrightarrow{B} C' \cdot C' \xrightarrow{\tau^*a_{k+1}} C$$

and further

$$s \xrightarrow{B} C = \biguplus_{C' \in S/R} s \xrightarrow{B'} C' \cdot C' \xrightarrow{\tau^*} C.$$

Step 2. is most involved, but interesting in itself (cf. [44]). Step 3. is a direct consequence of Proposition 6. Showing 4. completes the proof. □

By Theorem 2, Theorem 3, and Proposition 7 we obtain the following corollary which gives us the correspondence result for finite systems.

Corollary 4 *For finite generative systems, coalgebraic weak bisimilarity $\approx_{\{\tau\}}$ according to Definition 9, with respect to the pair $\langle \Phi^g, \{\tau\} \rangle$, coincides with concrete weak bisimilarity $\approx_g$ according to Definition 13.* □

6 Concluding remarks

In this paper, we have proposed a coalgebraic definition of weak bisimulation for action-type systems. For its justification we have considered the case of familiar labelled transition systems and of generative probabilistic systems, and we have compared our notion to the concrete definitions. In particular, we have obtained that the coalgebraic definition of weak bisimulation (for a suitably chosen ∗-extension) for LTSs coincides with the standard definition of weak bisimulation.

For generative probabilistic systems, the situation is more complex. Most of the work and technical difficulties of this paper are related to the correspondence results for generative probabilistic systems. As the standard notion of concrete weak bisimulation we have adopted from a number of choices the one proposed by Baier and Hermanns. However, their investigations and results are limited to finite systems. As our set-up does not restrict to the finite case, the coalgebraic framework exploited in the present paper extends the concrete definition and provides a coalgebraic definition of weak bisimulation for generative systems also covering the infinite case. Moreover, the correspondence results of our Section 5 also position the definition of Baier and Hermanns as a natural one as it is canonically induced from the underlying generative transition systems, once captured coalgebraically.

Baier and Hermanns also propose a notion of branching bisimulation. They prove their concrete notions of weak and branching bisimulation to coincide for finite generative systems. For the coalgebraic definition of weak bisimulation for finite and infinite generative systems the situation is as

follows:

$$\text{concrete branching} \subseteq \text{coalgebraic weak} \subseteq \text{concrete weak}.$$

As mentioned before, in case of finite systems, we have

$$\text{concrete branching} = \text{concrete weak}.$$

So, in the finite case, that was considered for the concrete notions, all three notions –concrete branching, coalgebraic weak, and concrete weak bisimulation– coincide. The precise situation of strict inclusion and/or equality for the general case remains to be unraveled, although it seems that the coincidence of concrete branching and concrete weak bisimulation will carry over to a wide class of well-behaved infinite systems.

Various issues remain untackled by the present approach to the weak bisimulation problem for coalgebras. In particular, the main issue here is that one has to come up with a suitable definition of a $*$-translation oneself, in order to obtain a weak bisimulation for a class of coalgebras of a given type. Ideally, a coalgebraic construction would automatically induce the $*$-translation. A method for systematically obtaining $*$-translations is a topic for further research.

Also, other examples that fit in our framework are to be studied. For instance, while programs, modeled by automata with outputs for the functor $\mathcal{I}d + O$ with O being the set of outputs, allow for a coalgebraic definition of weak bisimulation along the lines described above quite naturally. The resulting definition coincides with the definition described in [37]. Namely, there the action set is a singleton $A = 1$ using that $\mathcal{I}d + O \cong 1 \times \mathcal{I}d + O$.

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A (Weak) Pullbacks and their preservation

A *span* $\langle S, s_1, s_2 \rangle$, between X and Y, is a diagram of the form $X \xleftarrow{s_1} S \xrightarrow{s_2} Y$. It is *jointly injective* if the mapping $\langle s_1, s_2 \rangle : S \to X \times Y$, defined by $\langle s_1, s_2 \rangle(s) = \langle s_1(s), s_2(s) \rangle$ is injective. A relation $R \subseteq X \times Y$ gives rise to the jointly injective span $\langle R, \pi_1, \pi_2 \rangle$ between X and Y. Dually, a cospan $\langle C, c_1, c_2 \rangle$ is a diagram of the form $X \xrightarrow{c_1} C \xleftarrow{c_2} Y$.

A pullback, of a cospan $\langle C, c_1, c_2 \rangle$, is a span $\langle P, p_1, p_2 \rangle$ as in the diagram below satisfying $c_1 \circ p_1 = c_2 \circ p_2$ and such that for every span $\langle S, s_1, s_2 \rangle$ with $c_1 \circ s_1 = c_2 \circ s_2$ there exists a unique mediating map $m : S \to P$ satisfying $s_1 = p_1 \circ m$ and $s_2 = p_2 \circ m$. A *weak pullback* is a pullback for which the mediating arrow m need not be unique.
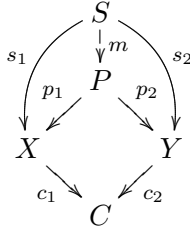
A pullback of a cospan $\langle C, c_1, c_2 \rangle$ between sets X and Y is the span arising from the relation

$$Q := \{ \langle x, y \rangle \in X \times Y \mid c_1(x) = c_2(y) \}.$$

A weak pullback arising from a relation $R \subseteq X \times Y$ is also an ordinary pullback, as one can derive from the joint injectivity of the two projections.

A functor $\mathcal{F}$ is said to *preserve* a (weak) pullback $\langle P, p_1, p_2 \rangle$ of a cospan $\langle C, c_1, c_2 \rangle$, if $\langle \mathcal{F}P, \mathcal{F}p_1, \mathcal{F}p_2 \rangle$ is again a (weak) pullback of $\langle \mathcal{F}C, \mathcal{F}c_1, \mathcal{F}c_2 \rangle$, i.e. if it transforms a (weak) pullback of a cospan into a (weak) pullback of the transformed cospan. The functor $\mathcal{F}$ *weakly preserves* a pullback of a cospan if it transforms it into a weak pullback of the transformed cospan. We note the following two properties taken from [18, 19].

Lemma 12 *Let $\mathcal{F}$ be a* Set *endofunctor. Then*

(i) *$\mathcal{F}$ preserves weak pullbacks if and only if it weakly preserves pullbacks.*

(ii) $\mathcal{F}$ *preserves weak pullbacks if and only if for any cospan* $\langle C, c_1, c_2 \rangle$ *we have: Given* u *and* v *with* $\mathcal{F}c_1(u) = \mathcal{F}c_2(v)$ *then there exists a* $w \in \mathcal{F}\{\langle x, y \rangle \mid c_1(x) = c_2(y)\}$ *with* $\mathcal{F}\pi_1(w) = u$ *and* $\mathcal{F}\pi_2(w) = v$. $\quad\square$

We end this section by mentioning a special type of pullback. A (weak) pullback $\langle P, p_1, p_2 \rangle$ is said to be *total* if its canonical morphisms, or legs, p_1 and p_2 are epi. In Set a pullback of a cospan $\langle C, c_1, c_2 \rangle$ where $c_1 : X \to C$ and $c_2 : Y \to C$ are surjective, is a total pullback. Moreover, it is easy to see the following.

Lemma 13 *In* Set*, the pullback of a cospan* $\langle C, c_1 : X \to C, c_2 : Y \to C \rangle$ *is total if and only if the images of* X *and* Y *under* c_1 *and* c_2*, respectively, are equal, i.e.* $c_1(X) = c_2(Y)$. $\quad\square$

We say that a functor weakly preserves total pullbacks if it transforms any total pullback into a weak pullback. According to Lemma 13, weakly preserving total pullbacks is the same as weakly preserving pullbacks of cospans $\langle C, c_1, c_2 \rangle$ with $c_1(X) = c_2(Y)$. Clearly, if a functor preserves weak pullbacks, then it weakly preserves total pullbacks. We shall see in Appendix C that weak preservation of total pullbacks is a strictly weaker notion, i.e., there exists a functor that weakly preserves total pullbacks but does not preserve weak pullbacks.

B Weak pullback preservation of the distribution functor

Here we establish the weak pullback preservation of $\mathcal{G}_A$, the functor defining generative probabilistic systems. Actually, we show weak pullback preservation of the probability distribution functor $\mathcal{D}$. For the probability distribution functor with finite support weak pullback preservation was proven by De Vink and Rutten [46], using the graph-theoretic min cut - max flow theorem, and by Moss [31], using an elementary matrix fill-in property. Following Moss [31] we show that the needed matrix fill-in property can be used and holds for arbitrary, infinite, matrices as well.

We start with a simple auxiliary property, that is also needed for the proof of Lemma 5 (Section 5.1). This property also justifies the name "discrete" probability distributions.

Lemma 14 *Let* $f : S \to \mathbb{R}_{\geq 0}$ *be a function with the property* $\sum_{s \in S} f(s) < \infty$. *Then the support set of this function,* $\operatorname{supp}(f) = \{s \in S \mid f(s) > 0\}$ *is at most countable.*

Proof: Let $s \in \text{supp}(f)$. Then $f(s) > 0$ and therefore there exists a natural number n such that $f(s) > 1/n$. So we have, $\text{supp}(f) \subseteq \cup_{n \in \mathbb{N}} \text{supp}_n(\mu)$ where $\text{supp}_n(\mu) = \{s \in \text{supp}(\mu) \mid f(s) > 1/n\}$. Now, since $\sum_{s \in \text{supp}(f)} f(s) = r < \infty$, the set $\text{supp}_n(f)$ has less than n/r elements, i.e., it is finite , for all $n \in \mathbb{N}$. Therefore the set $\text{supp}(f)$ is at most countable, being a countable union of finite sets. $\qquad\square$

Next we present the matrix fill-in property for infinite matrices.

Lemma 15 *Let I and J be arbitrary sets. For any two sets $\{x_i \mid i \in I\}$ and $\{y_j \mid j \in J\}$ of non-negative real numbers such that*

$$\sum_{i \in I} x_i = \sum_{j \in J} y_j < \infty,$$

there exist non-negative real numbers $\{z_{i,j} \mid i \in I, j \in J\}$ such that

$$\sum_{j \in J} z_{i,j} = x_i \quad and \quad \sum_{i \in I} z_{i,j} = y_j$$

for all $i \in I, j \in J$.

The proof is rather technical, though interesting, and can be found in [44, Lemma B.2, Lemma B.3]. Let us discuss the idea, also used in [31] for finite matrices, on a finite example. Let two finite sequences x and y be given by $x_1 = 2, x_2 = 1, x_3 = 3$ and $y_1 = 1, y_2 = 3, y_3 = 0, y_4 = 2$. Since

$$x_1 + x_2 + x_3 = y_1 + y_2 + y_3 + y_4$$

the statement claims that there exists a matrix Z, in this case of order 3×4, such that x_i is the sum of the i-th row and y_j the sum of the j-th column. The matrix

$$Z = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix}$$

satisfies that property. We have constructed it in the following way. For $z_{1,1}$ we take the minimum $\min\{x_1, y_1\}$, hence $z_{1,1} = y_1 = 1$. Since the first column sum has already been achieved we fill-in $z_{2,1} = z_{3,1} = 0$ and the next element to be filled-in is $z_{1,2}$. We fill it with the value $\min\{x_1 - z_{1,1}, y_2\} = x_1 - z_{1,1} = 1$. Since the first row-sum has been achieved, we put $z_{1,3} = z_{1,4} = 0$, and continue with $z_{2,2}$. It gets the value $\min\{x_2 - z_{2,1}, y_2 - z_{1,2}\} =$

$x_2 - z_{2,1} = 1$. Hence, $z_{2,3} = z_{2,4} = 0$ and the next element to be filled-in is $z_{3,2}$. Its value is then $\min\{x_3 - z_{3,1}, y_2 - z_{1,2} - z_{2,2}\} = y_2 - z_{1,2} - z_{2,2} = 1$, which completes the second column. Next is $z_{3,3} = \min\{x_3 - z_{3,1} - z_{3,2}, y_3 - z_{1,3} - z_{2,3}\} = y_3 - z_{1,3} - z_{2,3} = 0$. We fill-in the last element $z_{3,4}$ with the remaining value $x_3 - z_{3,1} - z_{3,2} - z_{3,3} = y_4 - z_{1,4} - z_{2,4} - z_{3,4} = 2$.

Lemma 16 *The functor $\mathcal{D}$ preserves weak pullbacks.*

Proof: It suffices to show that a pullback diagram

will be transformed to a weak pullback diagram (Lemma 12). Let P' be the pullback of the cospan $\mathcal{D}X \xrightarrow{\mathcal{D}f} \mathcal{D}Z \xleftarrow{\mathcal{D}g} \mathcal{D}Y$. Since $\mathcal{D}f \circ \mathcal{D}\pi_1 = \mathcal{D}g \circ \mathcal{D}\pi_2$, there exists $\gamma : \mathcal{D}P \to P'$ such that the next diagram commutes

and it is enough to show that γ is surjective in order to get a mediating morphism from P' to $\mathcal{D}P$. Let $\langle u, v \rangle \in P'$ be given. If $\mu \in \mathcal{D}P$ is such that

$$(\mathcal{D}\pi_1)(\mu) = u, \qquad (\mathcal{D}\pi_2)(\mu) = v \tag{34}$$

then $\gamma(\mu) = \langle u, v \rangle$ since π_1 and π_2 are jointly injective i.e. $\pi_1 \times \pi_2$ is injective. Hence the task is to find a function $\mu \in \mathcal{D}P$ which satisfies (34). More explicitely we have to find $\mu : P \to [0, 1]$ such that for all $x_0 \in X, y_0 \in Y$

$$\sum_{y \in Y : \langle x_0, y \rangle \in P} \mu(x_0, y) = u(x_0), \qquad \sum_{x \in X : \langle x, y_0 \rangle \in P} \mu(x, y_0) = v(y_0) \tag{35}$$

For if $\mu : P \to [0, 1]$ satisfies (35), then $\mu \in \mathcal{D}P$ and (34) holds.

The set P can be written as the union

$$P = \bigcup_{z \in Z} f^{-1}(\{z\}) \times g^{-1}(\{z\})$$

of disjoint rectangles, in fact rectangles with non-overlapping edges. Therefore, the existence of a map μ which satisfies condition (35) is equivalent to the condition that for all $z \in Z$ there exists a function $\mu_z : f^{-1}(\{z\}) \times g^{-1}(\{z\}) \to [0,1]$ such that for all $x_0 \in f^{-1}(\{z\})$, and all $y_0 \in g^{-1}(\{z\})$,

$$\sum_{y \in g^{-1}(\{z\})} \mu_z(x_0, y) = u(x_0), \qquad \sum_{x \in f^{-1}(\{z\})} \mu_z(x, y_0) = v(y_0). \qquad (36)$$

Since $\langle u, v \rangle \in P'$, we have

$$\sum_{x \in f^{-1}(\{z\})} u(x) = (\mathcal{D}f)(u)(z) = (\mathcal{D}g)(v)(z) = \sum_{y \in g^{-1}(\{z\})} v(y). \qquad (37)$$

Thus we may apply the matrix fill-in property, Lemma 15. $\qquad\square$

C   Weak pullback preservation of the functor $\mathcal{G}_A^*$

In this part we investigate the weak pullback preservation of the functor $\mathcal{G}_A^*$. We establish that the functor preserves total weak pullbacks, but does not preserve weak pullbacks, i.e. we give a proof of Proposition 10.

Lemma 17 *The functor $\mathcal{G}_A^*$ weakly preserves total pullbacks.*

Proof: Let $\langle P, \pi_1, \pi_2 \rangle$ be a total pullback in Set of the cospan $X \xrightarrow{f} Z \xleftarrow{g} Y$, i.e. $P = \{\langle x, y \rangle \mid f(x) = g(y)\}$ and π_1, π_2 surjective. Then the outer square of the diagram below commutes. Moreover, there exists a mediating morphism $\gamma : \mathcal{G}_A^* P \to P'$ from the candidate pullback $\langle \mathcal{G}_A^* P, \mathcal{G}_A^* \pi_1, \mathcal{G}_A^* \pi_2 \rangle$ to the pullback $\langle P', p_1, p_2 \rangle$ of the cospan $\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* Y$.

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It is enough to prove that γ is surjective (Lemma 12(ii)). So, we show that for every $\langle u, v \rangle \in P'$ there exists $w \in \mathcal{G}_A^* P$ with $\mathcal{G}_A^* \pi_1(w) = u$ and $\mathcal{G}_A^* \pi_2(w) = v$ which is equivalent to $w \circ (id_A^{-1} \times \pi_1^{-1}) = u$ and $w \circ (id_A^{-1} \times \pi_2^{-1}) = v$. Fix $\langle u, v \rangle \in P'$. We have

$$\langle u, v \rangle \in P' \implies \forall A' \subseteq A, \forall Z' \subseteq Z \colon u(A', f^{-1}(Z')) = v(A', g^{-1}(Z')). \quad (38)$$

Let $X' \subseteq X, Y' \subseteq Y$ and assume $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$. Then

(i) $f^{-1}(f(X')) = X'$:
  Clearly $X' \subseteq f^{-1}(f(X'))$. Let $x \in f^{-1}(f(X'))$ such that $f(x) = f(x')$ for some $x' \in X'$. Since π_1 is surjective, there exists $y \in Y$ with $\langle x, y \rangle \in P$ i.e. $f(x) = g(y)$, and hence also $f(x') = g(y)$, i.e. $\langle x', y \rangle \in P$. Thus $\langle x', y \rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ implying $y \in Y'$. Hence $\langle x, y \rangle \in \pi_2^{-1}(Y') = \pi_1^{-1}(X')$ i.e. $x \in X'$.

(ii) $g^{-1}(g(Y')) = Y'$: similar as (i).

(iii) $f(X') = g(Y')$:
  Let $z \in f(X')$, i.e. $z = f(x')$ for $x' \in X'$. Since π_1 is surjective there exists $y \in Y$ with $\langle x', y \rangle \in P$, i.e. $f(x') = g(y)$. Now, $\langle x', y \rangle \in \pi_1^{-1}(X') = \pi_2^{-1}(Y')$ and therefore $y \in Y'$, i.e. $z = f(x') = g(y) \in g(Y')$. We have shown $f(X') \subseteq g(Y')$. Similarly, $g(Y') \subseteq f(X')$.

Hence, if $\pi_1^{-1}(X') = \pi_2^{-1}(Y')$ for $X' \subseteq X, Y' \subseteq Y$ we get, for any $A' \subseteq A$,

$$u(A', X') \overset{(i)}{=} u(A', f^{-1}(f(X'))) \overset{(38)}{=} v(A', g^{-1}(f(X'))) \overset{(iii)}{=}$$

$$v(A', g^{-1}(g(Y'))) \overset{(ii)}{=} v(A', Y').$$

Since π_1 and π_2 are surjective,

$$\pi_1^{-1}(X') = \pi_1^{-1}(X'') \implies X' = X''$$

and

$$\pi_2^{-1}(Y') = \pi_2^{-1}(Y'') \implies Y' = Y''$$

for any $X', X'' \subseteq X$ and any $Y', Y'' \subseteq Y$. So the function $w \colon \mathcal{P}(A) \times \mathcal{P}(P) \to [0, 1]$ given by

$$w(A', Q) = \begin{cases} u(A', X') & Q = \pi_1^{-1}(X') \\ v(A', Y') & Q = \pi_2^{-1}(Y') \\ 0 & \text{otherwise} \end{cases}$$

is well defined. Clearly, $w \circ (id_A^{-1} \times \pi_1^{-1}) = u$ and $w \circ (id_A^{-1} \times \pi_2^{-1}) = v$. Thus the functor $\mathcal{G}_A^*$ weakly preserves total pullbacks. □

However, note that although $\mathcal{G}_A^*$ weakly preserves total pullbacks, it does not preserve weak pullbacks, as shown by the next example.

Example 2 $\mathcal{G}_A^*$ *does not preserve weak pullbacks.*

Choose X with $|X| \geq 3$. Fix $x_0 \in X$. Let $Z = \{1, 2, 3\}$ and consider the cospan $X \xrightarrow{f} Z \xleftarrow{g} X$ for the maps

$$f(x) = \begin{cases} 2 & x = x_0 \\ 1 & \text{otherwise} \end{cases} \qquad g(x) = \begin{cases} 2 & x = x_0 \\ 3 & \text{otherwise.} \end{cases}$$

The Set *pullback of this cospan is then $P = \{\langle x_0, x_0 \rangle\}$. On the other hand, let P' be the pullback of the cospan*

$$\mathcal{G}_A^* X \xrightarrow{\mathcal{G}_A^* f} \mathcal{G}_A^* Z \xleftarrow{\mathcal{G}_A^* g} \mathcal{G}_A^* X .$$

We have $\langle \mu, \nu \rangle \in P'$ if and only if

$$\mathcal{G}_A^* f(\mu) = \mathcal{G}_A^* g(\nu),$$

i.e.

$$\mu(A', f^{-1}(Z')) = \nu(A', g^{-1}(Z'))$$

for all $A' \subseteq A, Z' \subseteq Z$. Therefore, every pair $\langle \mu, \nu \rangle \in \mathcal{G}_A^* X \times \mathcal{G}_A^* X$ with the property

$$\mu(A', \emptyset) = \mu(A', \{x_0\}) = \mu(A', X \setminus \{x_0\}) = \mu(A', X) =$$

$$= \nu(A', \emptyset) = \nu(A', \{x_0\}) = \nu(A', X \setminus \{x_0\}) = \nu(A', X)$$

belongs to P', since $\emptyset, \{x_0\}, X \setminus \{x_0\}$ and X are the only subsets of X that are inverse images of subsets of Z under f and g.

Now we consider $\mathcal{G}_A^* P = \mathcal{P}(A) \times \mathcal{P}(P) \to [0, 1]$. If $\mu \in \mathcal{G}_A^* X$ is such that $\mu = (\mathcal{G}_A^* \pi_1)(\chi)$ for some $\chi \in \mathcal{G}_A^* P$, then $\mu = \chi \circ (id_A^{-1} \times \pi_1^{-1})$. Hence, for $A' \subseteq A, X' \subseteq X$ we have

$$\mu(A', X') = \begin{cases} \chi(A', \emptyset) & x_0 \notin X' \\ \chi(A', \{\langle x_0, x_0 \rangle\}) & x_0 \in X'. \end{cases}$$

Choose $x_1 \in X$, $x_1 \neq x_0$. Since $|X| \geq 3$ we have $\{x_0, x_1\} \notin \{\emptyset, \{x_0\}, X \setminus \{x_0\}, X\}$. Define $\xi \colon \mathcal{P}(A) \times \mathcal{P}(X) \to [0, 1]$ by

$$\xi(A', X') = \begin{cases} 1 & X' = \{x_0, x_1\} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\xi \in \mathcal{G}_A^*(X)$ and the pair $\langle \xi, \xi \rangle$ belongs to P', since for every $A' \subseteq A$,

$$\xi(A', \emptyset) = \xi(A', \{x_0\}) = \xi(A', X \setminus \{x_0\}) = \xi(A', X) = 0.$$

However, ξ can not be written as $(\mathcal{G}_A^* \pi_1)(\chi)$ for any $\chi \in \mathcal{G}_A^* P$, since

$$\xi(A', \{x_0, x_1\}) \neq \xi(A', \{x_0\}),$$

while, as noted above,

$$(\mathcal{G}_A^* \pi_1)(\chi)(A', \{x_0, x_1\}) = \chi(A', \{\langle x_0, x_0 \rangle\}) = (\mathcal{G}_A^* \pi_1)(\chi)(A', \{x_0\}).$$

Hence, for the pair $\langle \xi, \xi \rangle \in P'$ there does not exist an element $\chi \in \mathcal{G}_A^* P$ such that $\mathcal{G}_A^* \pi_1(\chi) = \xi$ and $\mathcal{G}_A^* \pi_2(\chi) = \xi$, which by Lemma 12 shows that $\mathcal{G}_A^*$ does not preserve weak pullbacks.