Classification of Probabilistic Systems

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Outline

- probabilistic systems as coalgebras
- two strong semantics
  - * bisimilarity
  - * behaviour equivalence
- expressiveness comparison
- a hierarchy
- beyond discrete probabilities, beyond $\mathbf{Sets}$

Formal methods

In general:

- models - transition systems, automata, terms,...
  with a clear semantics
- analysis - model checking
  process algebra
  theorem proving...

In this talk

- models - probabilistic transition systems
- semantics - bisimilarity/behavior equivalence

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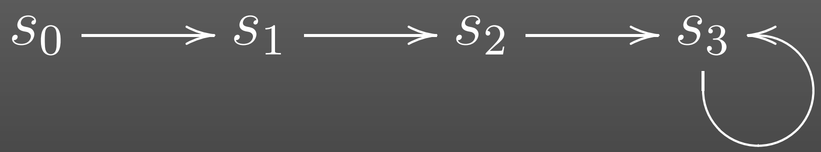
Aim: expressiveness of many models

in a single framework

Example models

deterministic systems



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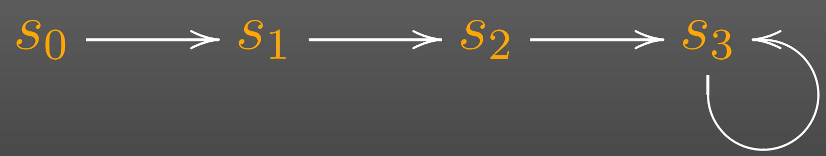

states $+$ transitions $\alpha : S \to S$

$$\alpha(s_0) = s_1,\ \alpha(s_1) = s_2,\ \ldots$$

Example models

labelled deterministic systems A - labels

$$s_0 \xrightarrow{a} s_1 \xrightarrow{b} s_2 \xrightarrow{b} s_3 \circlearrowleft a$$

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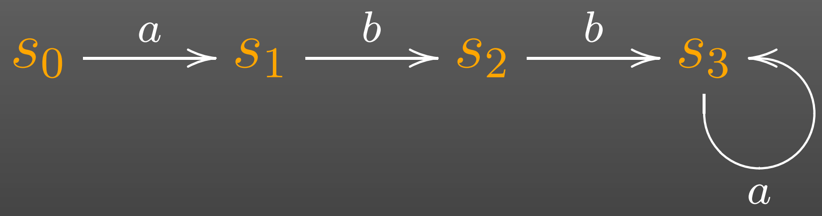
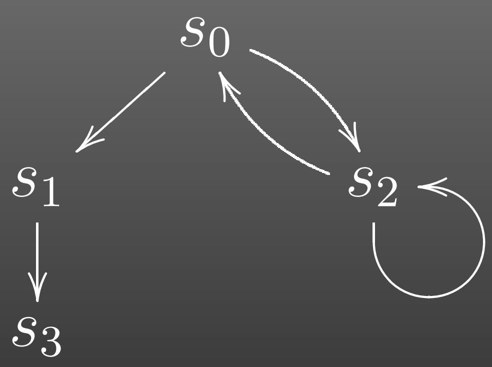


states $+$ transitions $\alpha : S \to A \times S$

$$\alpha(s_0) = \langle a, s_1 \rangle,\ \alpha(s_1) = \langle b, s_2 \rangle,\ \ldots$$

Example models

transition systems



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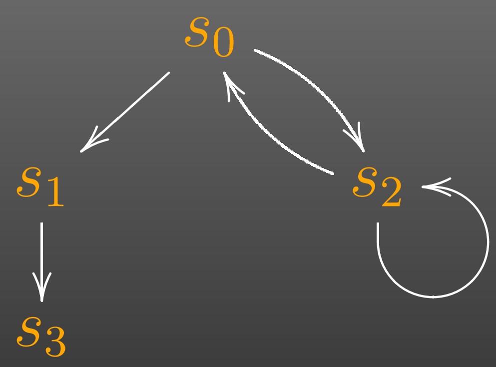

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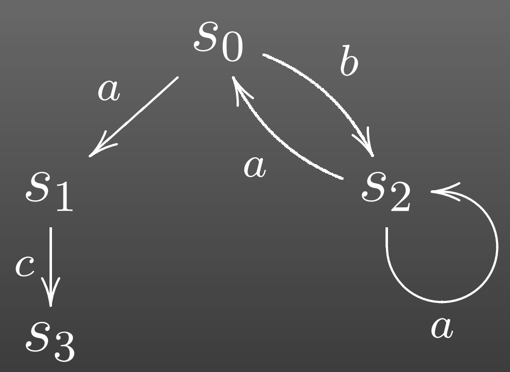

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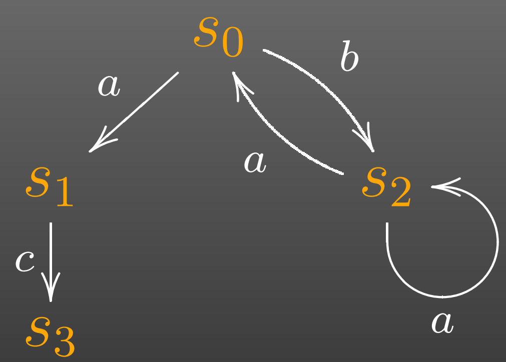

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- rich mathematical structure
- a uniform way for treating transition systems
- general notions and results, generic notion of bisimulation

Probabilistic systems

arise by enriching transition systems with (discrete) probabilities as labels on the transitions.

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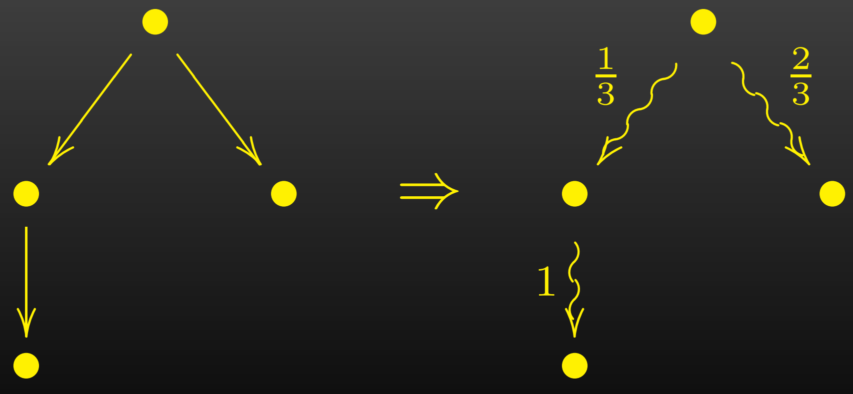

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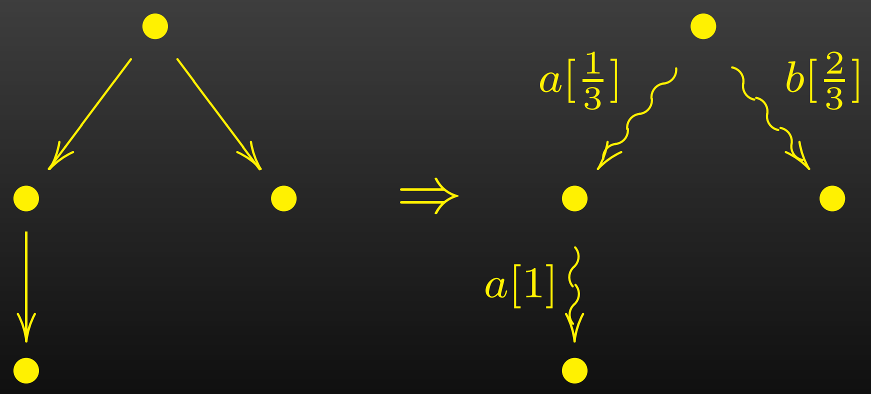

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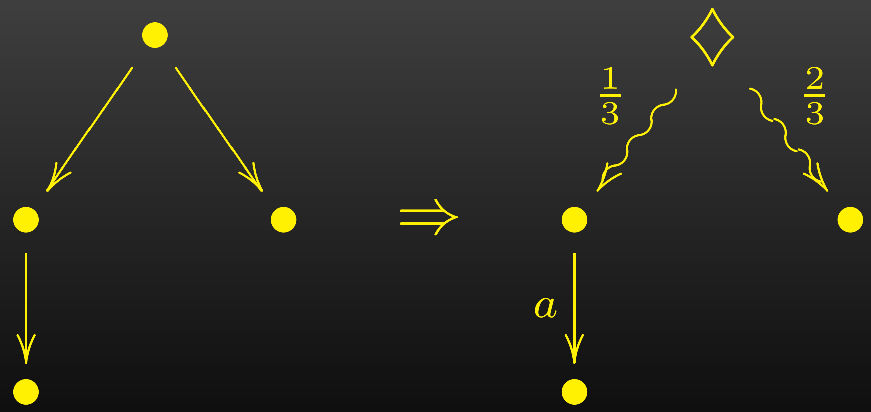

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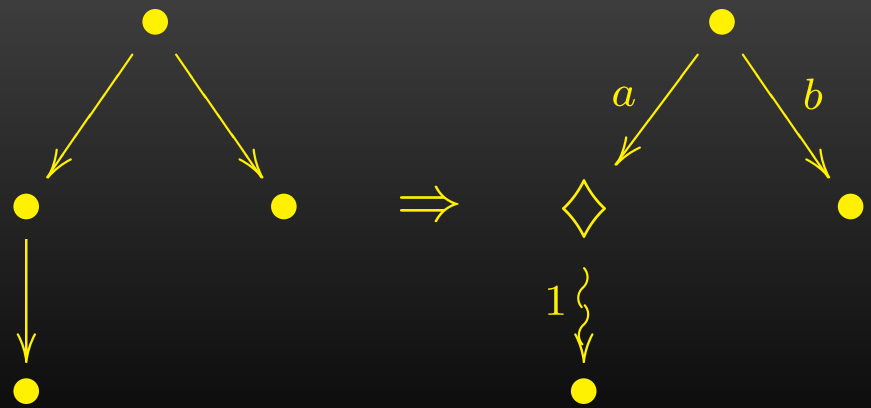

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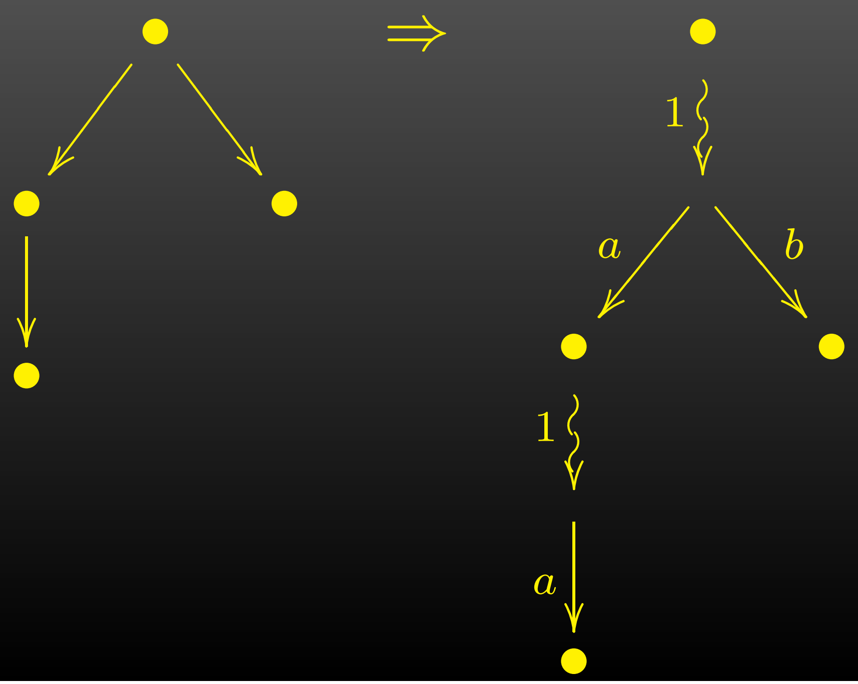

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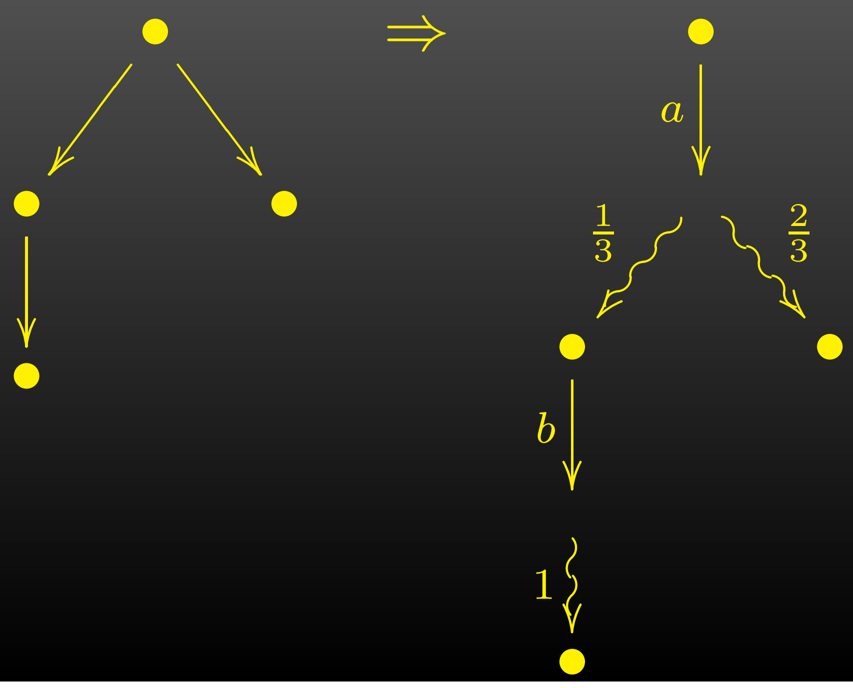

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Thanks to the probability distribution functor $\mathcal{D}$

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$$\mathcal{D}S = \begin{array}{l}\text{the set of all discrete}\\ \text{probability distributions on } S\end{array}$$

Probabilistic systems

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$$\mathcal{D}S = \{\mu : S \to [0,1], \mu[S] = 1\}, \quad \mu[X] = \sum_{s \in X} \mu(x)$$

$$\mathcal{D}f : \mathcal{D}S \to \mathcal{D}T, \;\; \mathcal{D}f(\mu)(t) = \mu[f^{-1}(\{t\})]$$

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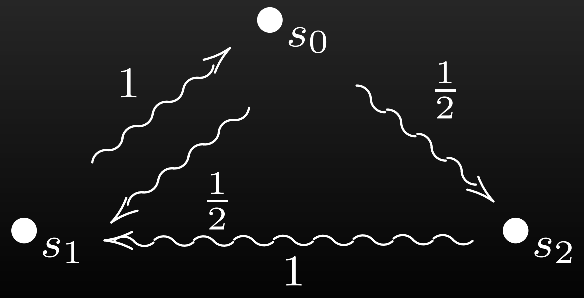
Example: $\alpha : S \to \mathcal{D}S$



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the probabilistic systems are also coalgebras
... of functors built by the following syntax

$$\mathcal{F} ::= \_ \mid A \mid \mathcal{P} \mid \mathcal{D} \mid \mathcal{G} + \mathcal{H} \mid \mathcal{G} \times \mathcal{H} \mid \mathcal{G}^A \mid \mathcal{G} \circ \mathcal{H}$$

reactive, generative

evolve from LTS - functor $\textcircled{\mathcal{P}}(A \times \_) \cong \textcircled{\mathcal{P}}^A$

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note: in the probabilistic case

$$(\mathcal{D} + 1)^A \ncong \mathcal{D}(A \times \_) + 1$$

Probabilistic system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
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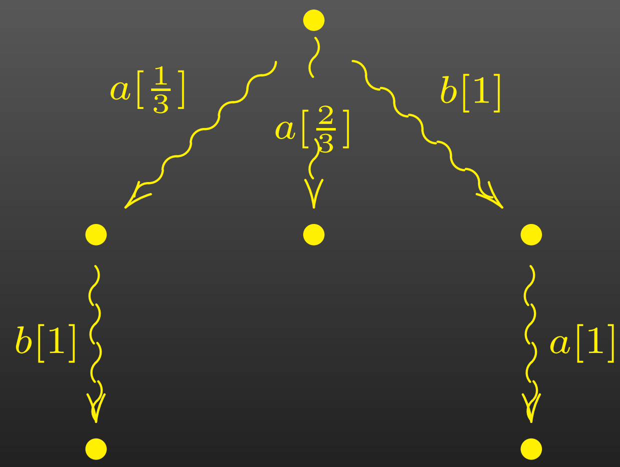

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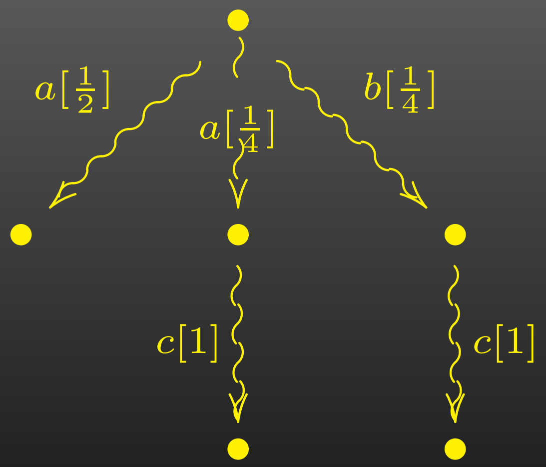

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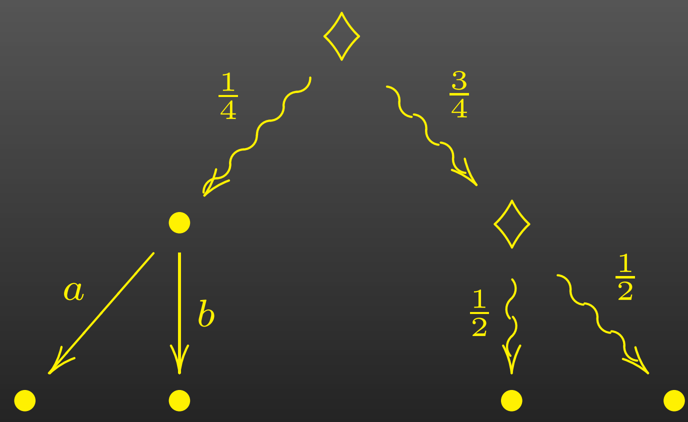

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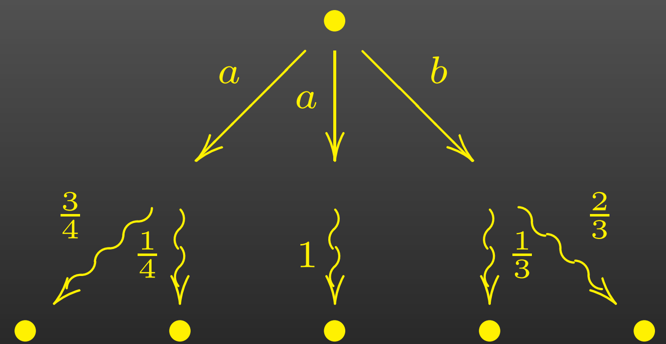

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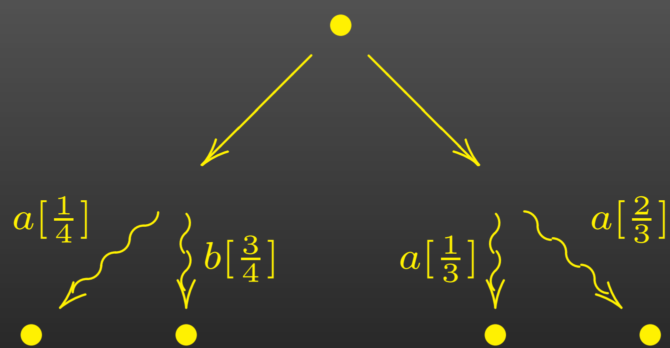

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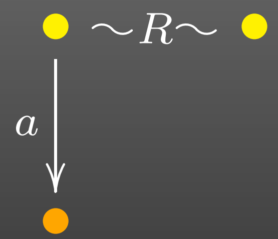

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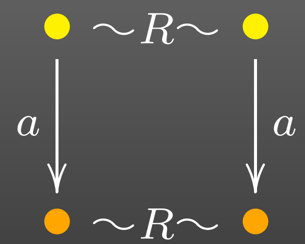

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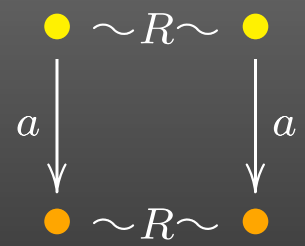
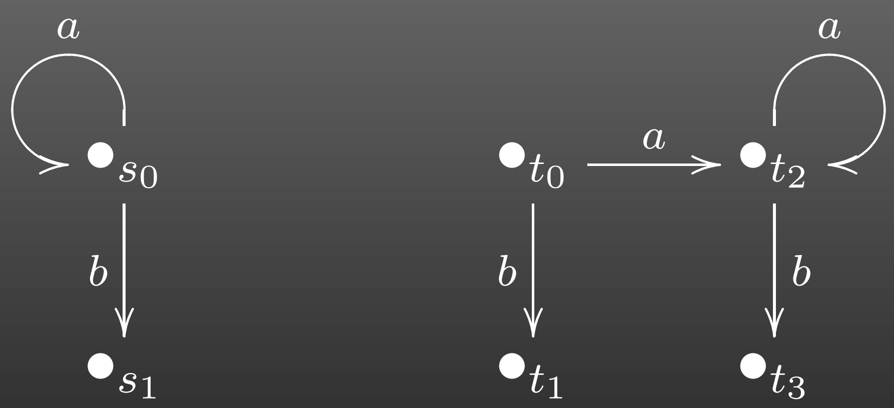


... two states are bisimilar if they are related by some bisimulation

Bisimulation - LTS

Example: Consider the LTS



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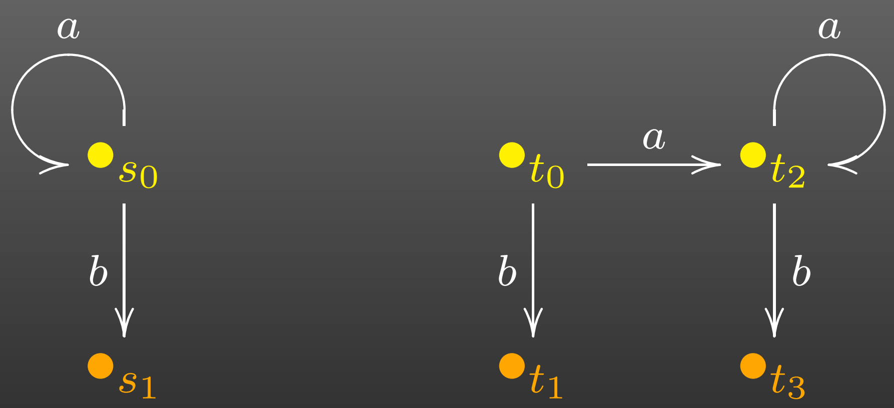

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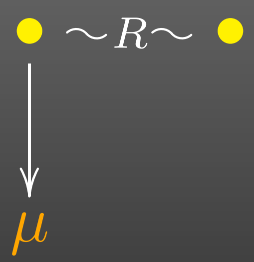

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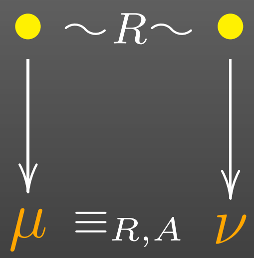

$\equiv_{R,A}$ relates distributions that assign the same probability to each label and each R-class

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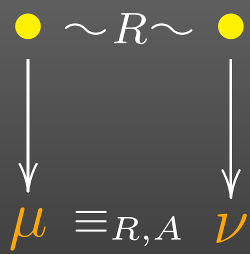

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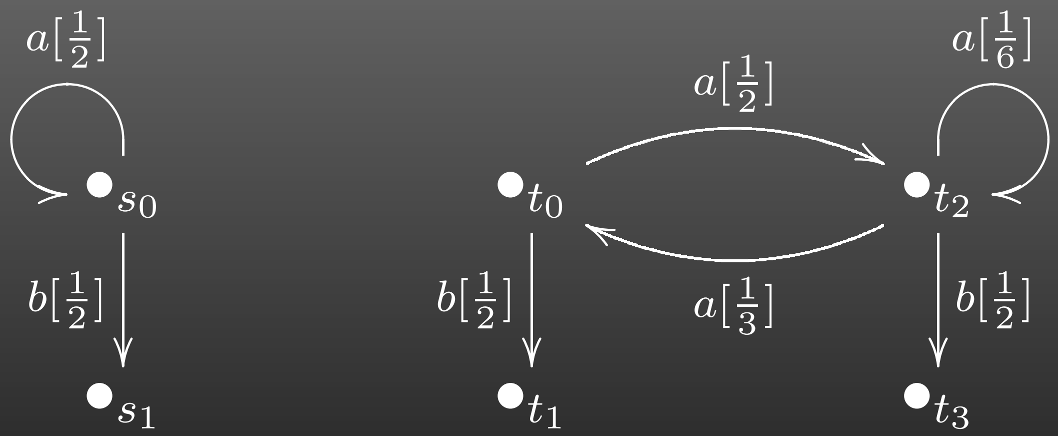

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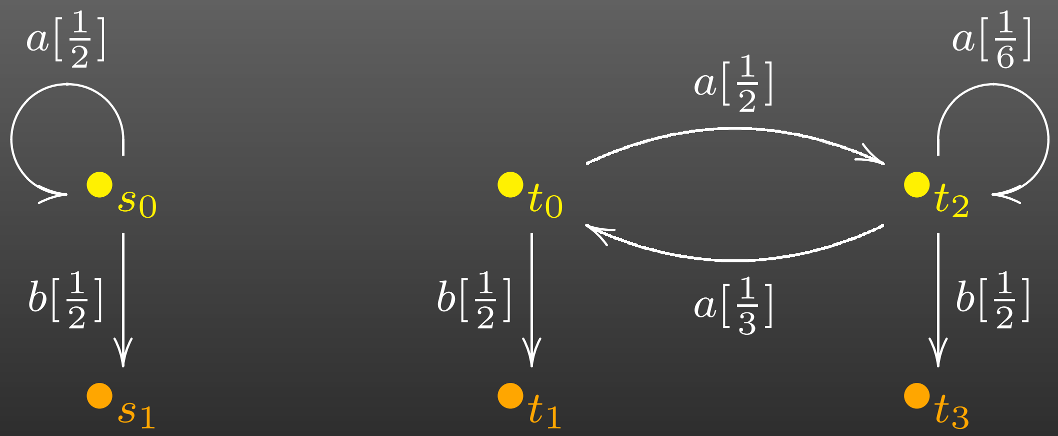

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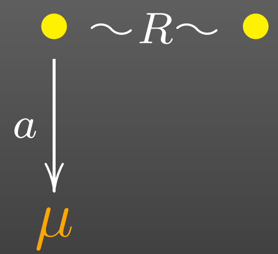

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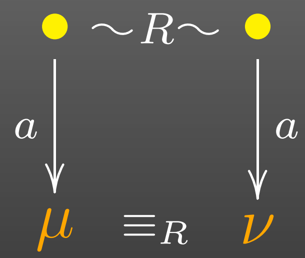

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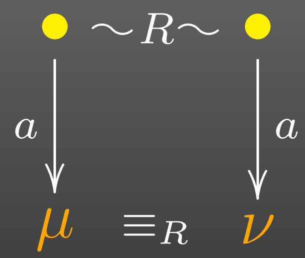

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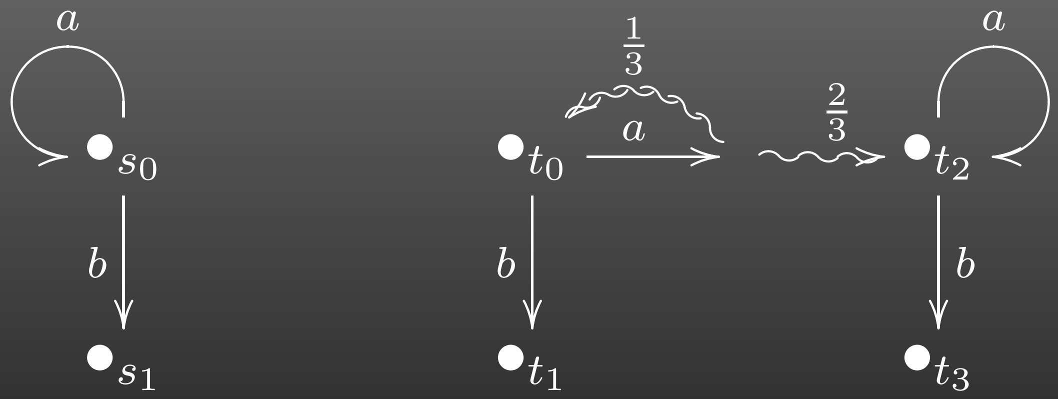

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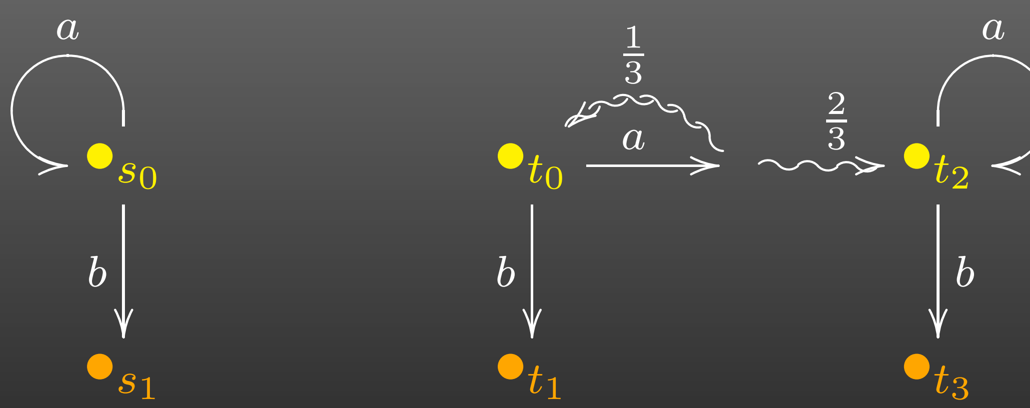

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Coalgebraic bisimulation

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$$\begin{array}{ccccc} S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & S \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{F}S & \xleftarrow[\mathcal{F}\pi_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}\pi_2]{} & \mathcal{F}S \end{array}$$

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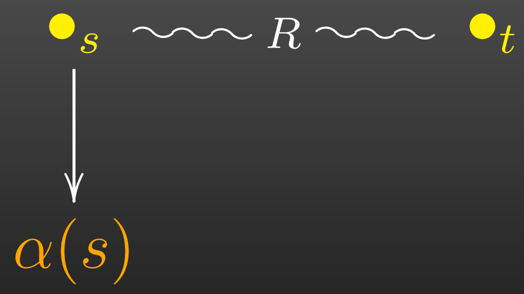

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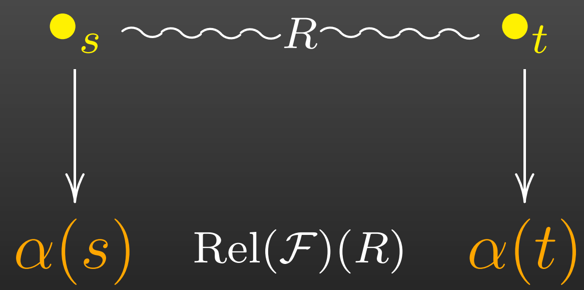

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$$\begin{array}{ccc} \bullet_s & \sim\!\sim\!\sim\!\sim R \sim\!\sim\!\sim\!\sim & \bullet_t \\ \downarrow & & \downarrow \\ \alpha(s) & \mathrm{Rel}(\mathcal{F})(R) & \alpha(t) \end{array}$$

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Theorem: Coalgebraic and concrete bisimilarity coincide for all probabilistic transition systems!

Expressiveness

When do we consider one type of system more expressive than another?

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Example:

LTS $\mathcal{P}(A \times \_)$

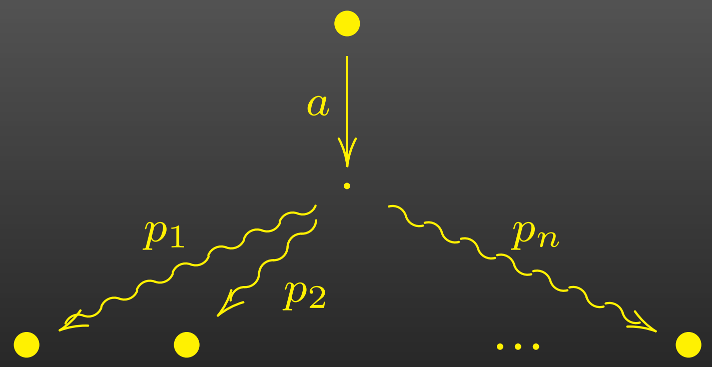
are clearly not more expressive than

Alternating Systems $\mathcal{D} + \mathcal{P}(A \times \_)$

Expressiveness

simple Segala system $\longrightarrow$ Segala system

$\mathcal{P}(A \times \mathcal{D})$ $\quad\quad$ $\mathcal{PD}(A \times \_)$



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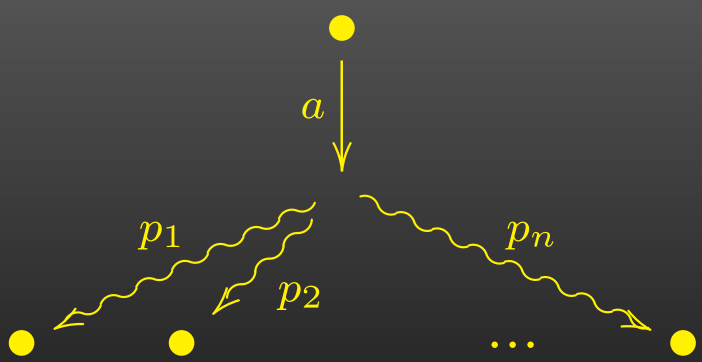

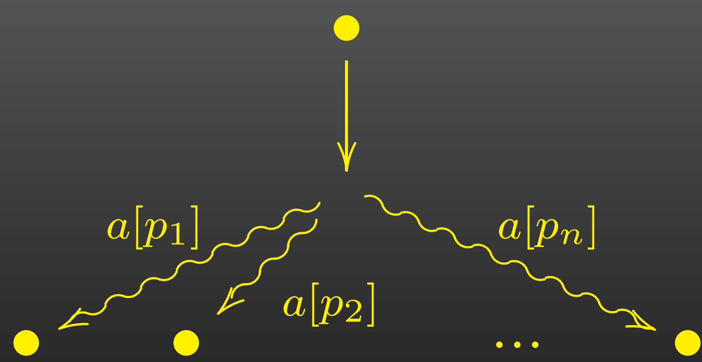

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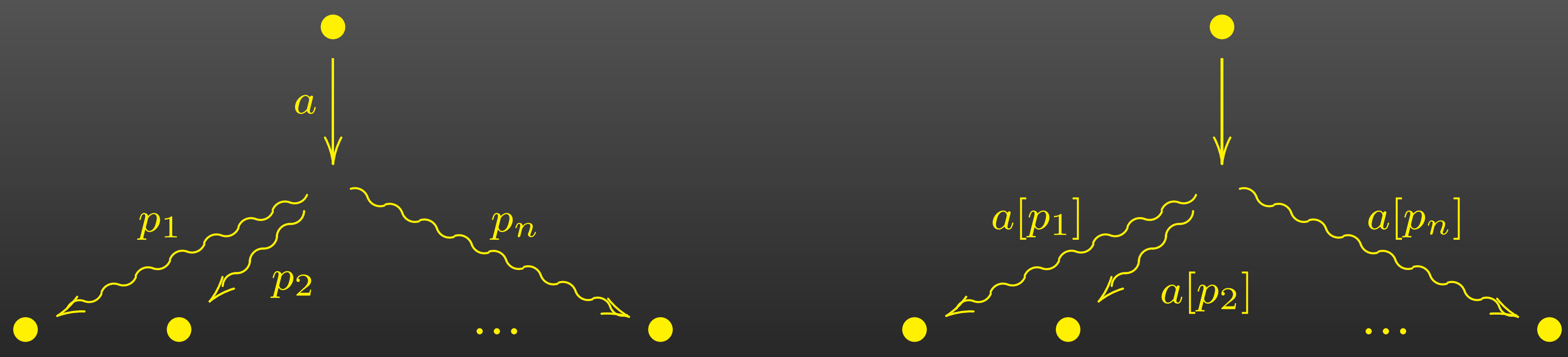

When do we consider one type of systems more expressive than another?

Our expressiveness criterion

$\mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$

if there is a mapping $\langle S, \alpha : S \to \mathcal{F}S \rangle \overset{\mathcal{T}}{\mapsto} \langle S, \tilde{\alpha} : S \to \mathcal{G}S \rangle$ that preserves and reflects bisimilarity

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Theorem: An injective natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ is sufficient for $\mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$

Proof idea

Generally, a natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ gives rise to a translation of coalgebras $\mathcal{T}_\tau : \mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$ as follows:

$$\begin{array}{ccc} S & & S \\ & & \downarrow{\scriptstyle \alpha} \\ {\scriptstyle \alpha}\Big\downarrow & \overset{\mathcal{T}_\tau}{\Longrightarrow} & \mathcal{F}S \\ & & \downarrow{\scriptstyle \tau_S} \\ \mathcal{F}S & & \mathcal{G}S \end{array}$$

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this translation we use in the proof!

Preservation - proof

The translation $\mathcal{T}_\tau$ preserves bisimulations:

A bisimulation $R \subseteq S \times T$ between $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$

$$\begin{array}{ccccc} S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & T \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ \mathcal{F}S & \xleftarrow{\mathcal{F}\pi_1} & \mathcal{F}R & \xrightarrow{\mathcal{F}\pi_2} & \mathcal{F}T \end{array}$$

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{\scriptstyle \tau_S}\downarrow & \text{nat. } \tau & \downarrow{\scriptstyle \tau_R} & \text{nat. } \tau & \downarrow{\scriptstyle \tau_T} \\
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\end{array}
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is a bisimulation between $\mathcal{T}_\tau \langle S, \alpha \rangle$ and $\mathcal{T}_\tau \langle T, \beta \rangle$

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But $\mathcal{T}_\tau$ need not reflect bisimilarity.

Example: The natural transformation

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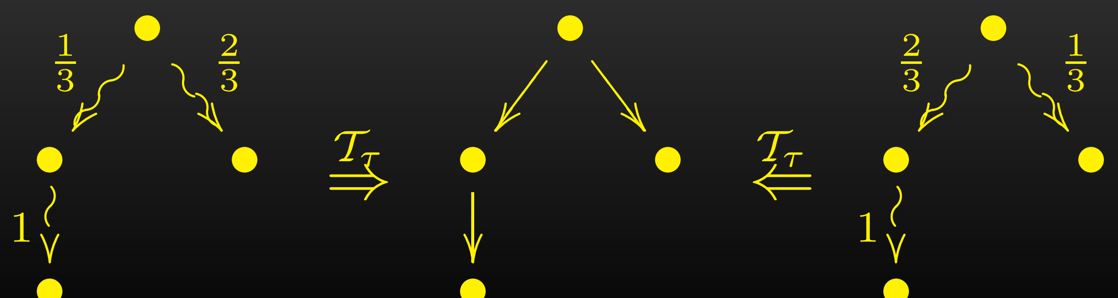

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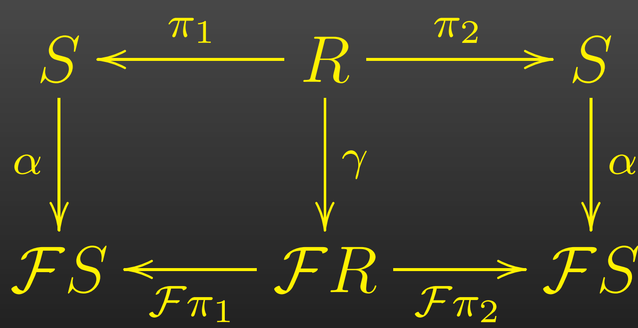
Injectivity turns out to be sufficient for reflection via cocongruences - behavioural equivalence

Recall bisimulation

A bisimulation on

$$\langle S, \alpha : S \to \mathcal{F}S \rangle$$

is $R \subseteq S \times S$ or (R, π_1, π_2) such that γ exists:



... two states are bisimilar if they are related by some bisimulation

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... two states are behaviourly equivalent if they are related by some cocongruence $s \approx t \iff q_1(s) = q_2(t)$

Results

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* The proof uses that $\mathbf{Sets}$ has a factorization system with a diagonal fill-in

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All our functors preserve w.p.

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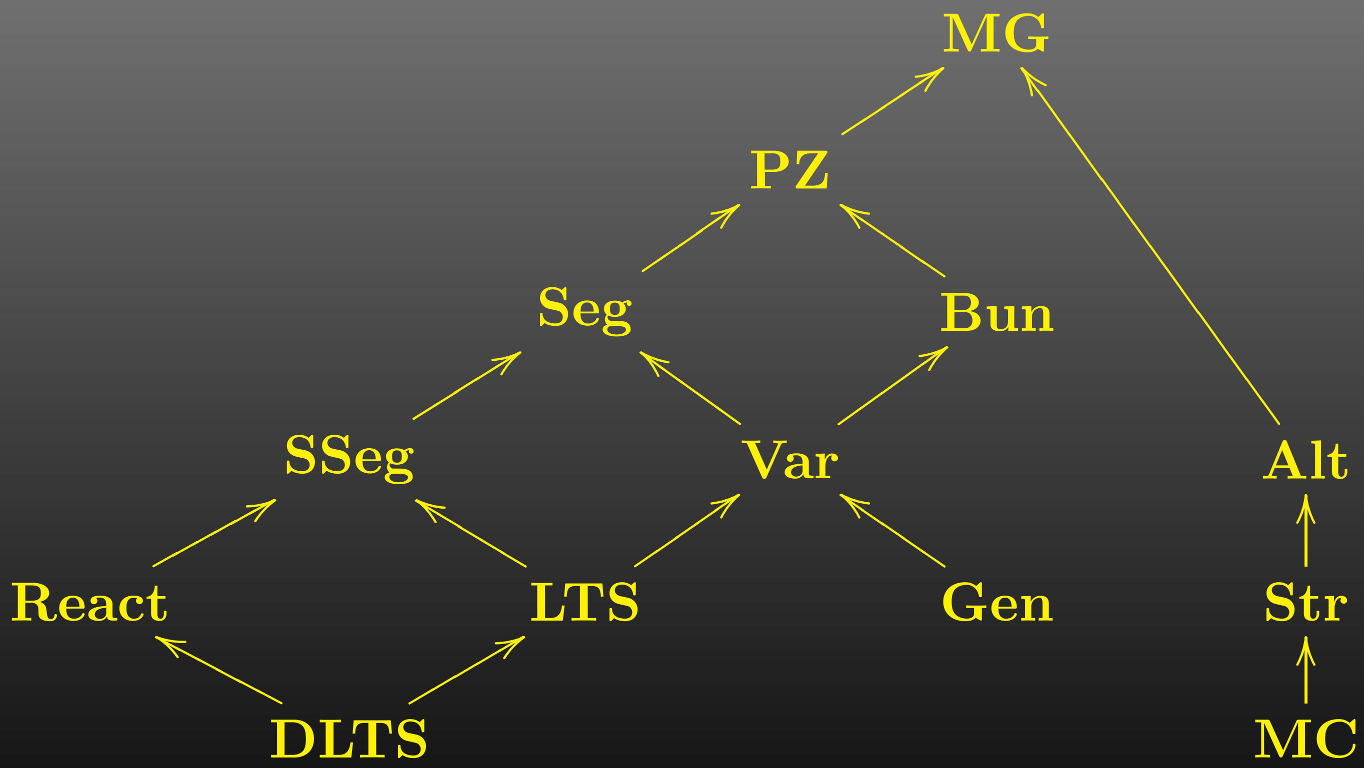
The hierarchy



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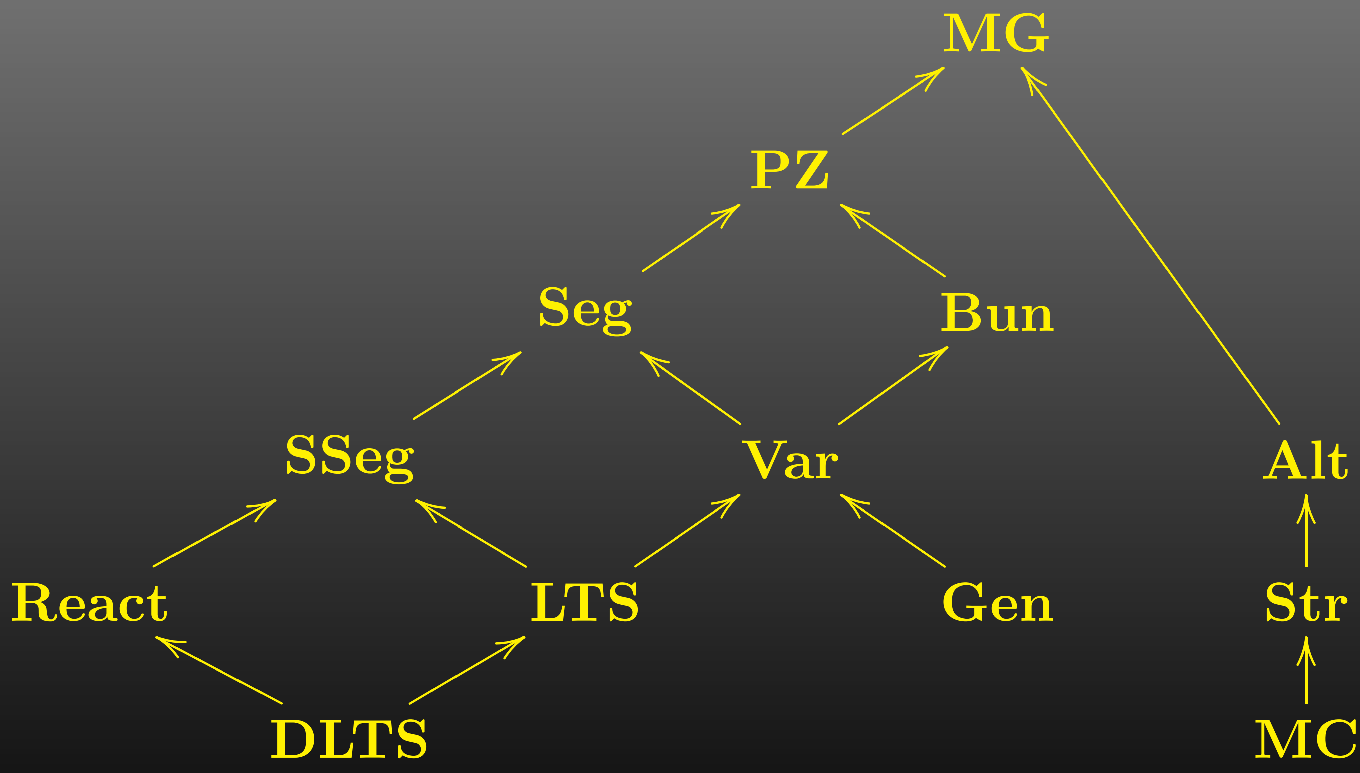

* Falk Bartels, Ana Sokolova, Erik de Vink, TCS 327

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Markov processes are Giry-coalgebras in Meas!

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The Giry functor (monad) on $\mathbf{Meas}$ is given on objects by

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and on arrows

$$\mathcal{G}(f)\Big(S_X \xrightarrow{\varphi} [0,1]\Big) \quad = \quad \Big(S_Y \xrightarrow{f^{-1}} S_X \xrightarrow{\varphi} [0,1]\Big)$$

Chains vs. processes

The situation is

$$\mathcal{G} \circlearrowright \mathbf{Meas} \underset{D}{\overset{U}{\rightleftarrows}} \mathbf{Sets} \circlearrowleft \mathcal{D} \qquad \text{with} \quad D \dashv U$$

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and we can translate chains into processes

$$\begin{array}{ccccc}
 & & \overset{\mathcal{T}}{\frown} & & \\
\mathsf{Coalg}_{\mathcal{D}} & \xrightarrow[\rho D \circ -]{} & \mathsf{Coalg}_{U\mathcal{G}D} & \xrightarrow[(*)]{} & \mathsf{Coalg}_{\mathcal{G}} \\
\searrow & & \swarrow & & \downarrow \\
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\end{array}$$

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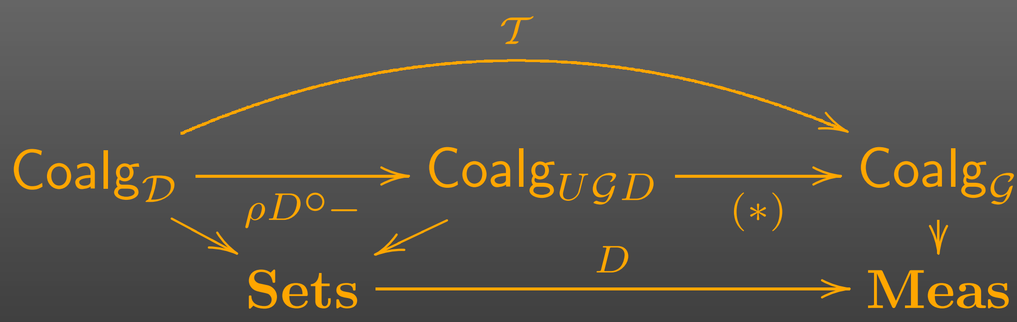

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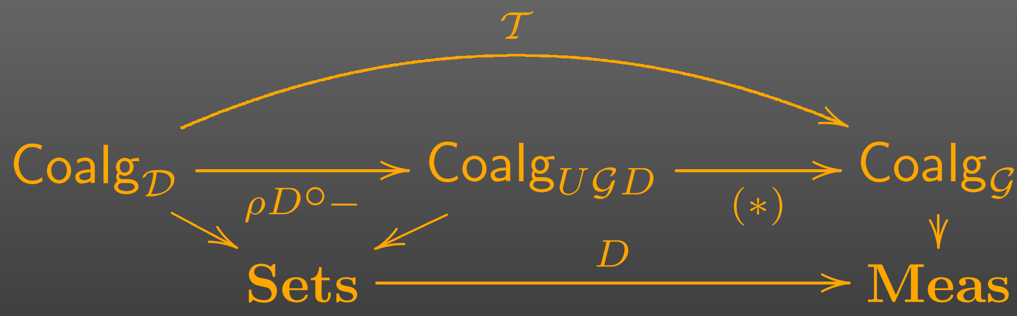

with $\left(X \xrightarrow{c} \mathcal{D}(X) = \mathcal{D}UD(X)\right) \longmapsto \left(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\right)$

and $(*)$ from

$$\frac{X \longrightarrow U\mathcal{G}D(X) \quad \text{in } \mathbf{Sets}}{D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}}$$

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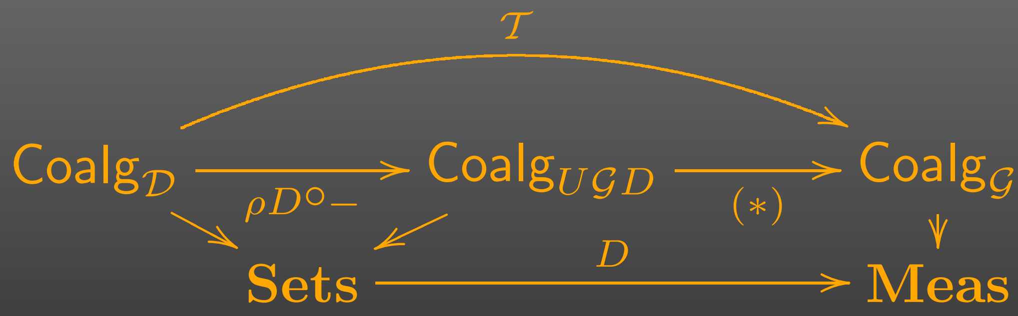

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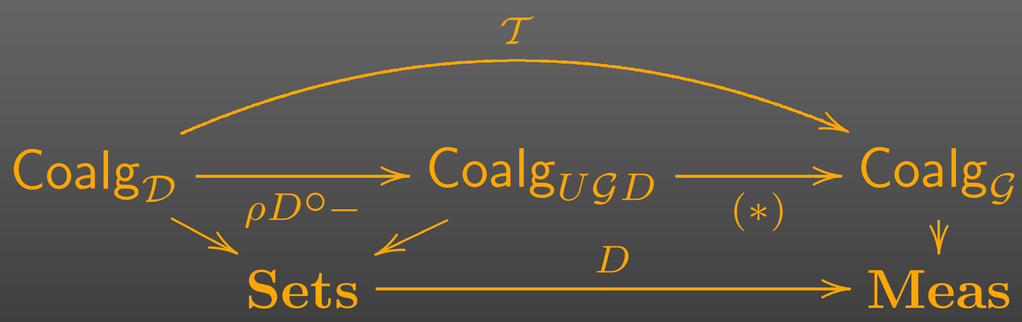

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Hence: $\mathbf{MC} = \mathsf{Coalg}_{\mathcal{D}}^{\mathbf{Sets}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}^{\mathbf{Meas}} = \mathbf{MP}$

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- future work: build another floor in $\mathbf{Meas}$