Sound and Complete Axiomatization of Trace Semantics for Probabilistic Transition Systems

Alexandra Silva and Ana Sokolova
Radboud University Nijmegen and University of Salzburg

QAIS seminar 2011, Minho University, 17.10.2011

We will discuss

- history
- probabilistic transition systems
- (finite) trace semantics
- the sound and complete axiomatization
- in a coalgebraic setting

generative PTS,
fully probabilistic,
with labels

History

- For LTS Milner '84, JCSS
- expressions for LTS
- Kleene style theorem
- axiomatization
- sound and complete for bisimilarity

History

- For LTS Milner '84, JCSS
- expressions for LTS — $P ::= 0 \mid a.P \mid P + P \mid \mu x.P^g$
- Kleene style theorem
- axiomatization
- sound and complete for bisimilarity

History

behaviors of processes are characterized by expressions and vice-versa

- For LTS Milner '84, JCSS
- expressions for LTS

$$P ::= 0 \mid a.P \mid P + P \mid \mu x.P^g$$

- Kleene style theorem
- axiomatization
- sound and complete for bisimilarity

History

behaviors of processes are characterized by expressions and vice-versa

- For LTS Milner '84, JCSS
- expressions for LTS
  $P ::= 0 \mid a.P \mid P + P \mid \mu x.P^g$
- Kleene style theorem
- axiomatization
  $P + Q \equiv Q + P,\ P + 0 \equiv P,\ \mu x.P \equiv P[\mu x.P/x],\ \ldots$
- sound and complete for bisimilarity

History

behaviors of processes are characterized by expressions and vice-versa

- For LTS Milner '84, JCSS
- expressions for LTS $P ::= 0 \mid a.P \mid P + P \mid \mu x.P^g$
- Kleene style theorem
- axiomatization $P + Q \equiv Q + P,\ P + 0 \equiv P, \mu x.P \equiv P[\mu x.P/x], \ldots$
- sound and complete for bisimilarity

$$P \equiv Q \iff P \sim Q$$

Some years later

- Milner's result was extended by Rabinovich '93 MFPS for trace semantics
- expressions for LTS
- Kleene style theorem
- axiomatization
- sound and complete for trace semantics

Some years later

- Milner's result was extended by Rabinovich '93 MFPS for trace semantics
- expressions for LTS — Milner's expressions
- Kleene style theorem
- axiomatization
- sound and complete for trace semantics

Some years later

remains the same (bisimilarity implies trace equivalence)

- Milner's result was extended by Rabinovich '93 MFPS for trace semantics
- expressions for LTS

Milner's expressions

- Kleene style theorem
- axiomatization
- sound and complete for trace semantics

Some years later

remains the same (bisimilarity implies trace equivalence)

- Milner's result was extended by Rabinovich '93 MFPS for trace semantics
- expressions for LTS

  Milner's expressions
- Kleene style theorem
- axiomatization

  Milner's axioms plus $a.(P+Q) \equiv a.P + a.Q$
- sound and complete for trace semantics

Some years later

remains the same (bisimilarity implies trace equivalence)

- Milner's result was extended by Rabinovich '93 MFPS for trace semantics
- expressions for LTS

  Milner's expressions

- Kleene style theorem
- axiomatization

  Milner's axioms plus $a.(P + Q) \equiv a.P + a.Q$

- sound and complete for trace semantics

$$P \equiv Q \iff \mathrm{tr}(P) = \mathrm{tr}(Q)$$

Now we do it for PTS

- Expressions/axioms for PTS come in many flavors (mainly for bisimilarity)
  we build on Silva, Bonchi, Bonsangue, Rutten '09/'10

- Trace semantics for PTS also exists in variants
  we build on Hasuo, Jacobs, S. '06/'07

Now we do it for PTS

expressions, Kleene style theorem, sound and complete axiomatization for bisimilarity
(generic approach applicable to a large class of systems)

- Expressions/axioms for PTS come in many flavors (mainly for bisimilarity)
  we build on Silva, Bonchi, Bonsangue, Rutten '09/'10

- Trace semantics for PTS also exists in variants
  we build on Hasuo, Jacobs, S. '06/'07

Now we do it for PTS

expressions, Kleene style theorem, sound and complete axiomatization for bisimilarity
(generic approach applicable to a large class of systems)

- Expressions/axioms for PTS come in many flavors (mainly for bisimilarity)
  we build on Silva, Bonchi, Bonsangue, Rutten '09/'10
- Trace semantics for PTS also exists in variants
  we build on Hasuo, Jacobs, S. '06/'07

generic coalgebraic approach applicable to PTS

Now we do it for PTS

expressions, Kleene style theorem, sound and complete axiomatization for bisimilarity
(generic approach applicable to a large class of systems)

- Expressions/axioms for PTS come in many flavors (mainly for bisimilarity)
  we build on Silva, Bonchi, Bonsangue, Rutten '09/'10
- Trace semantics for PTS also exists in variants
  we build on Hasuo, Jacobs, S. '06/'07

same expressions, one more axiom,
sound and complete for trace semantics

generic coalgebraic approach
applicable to PTS

Probabilistic transition systems

PTS here are generative, labelled, with explicit termination

$$X \to \mathcal{D}_\omega(1 + A \times X)$$

Probabilistic transition systems

PTS here are generative, labelled, with explicit termination

$$X \to \mathcal{D}_\omega(1 + A \times X)$$

coalgebras on $\mathbf{Sets}$

Probabilistic transition systems

PTS here are generative, labelled, with explicit termination

$$X \to \mathcal{D}_\omega(1 + A \times X)$$

coalgebras on $\mathbf{Sets}$

$$\mathcal{D}_\omega X = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) \leqslant 1, \mathrm{supp}(X) \text{ is finite}\}$$

Probabilistic transition systems

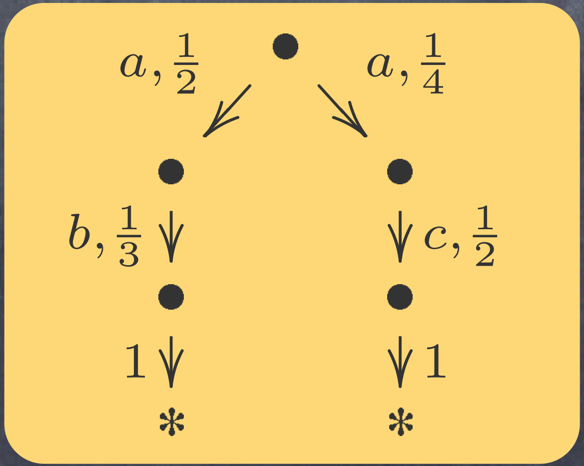
PTS here are generative, labelled, with explicit termination

$$X \to \mathcal{D}_\omega(1 + A \times X)$$

coalgebras on **Sets**

$$\mathcal{D}_\omega X = \{\mu{:}X \to [0,1] \mid \sum_{x \in X} \mu(x) \leqslant 1, \mathrm{supp}(X) \text{ is finite}\}$$

Example:



Coalgebra basics

Category $\mathbf{C}$, functor F, category of coalgebras:

$\mathbf{Coalg}_F$

Objects: $X \stackrel{c}{\to} FX$

Arrows:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ c\downarrow & & \downarrow d \\ FX & \xrightarrow{Fh} & FY \end{array}$$

Coalgebra basics

Category $\mathbf{C}$, functor F, category of coalgebras:

$\mathbf{Coalg}_F$

Objects: $X \xrightarrow{c} FX$

Arrows:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow{Fh} & FY \end{array}$$

Final coalgebra semantics:

$$\begin{array}{ccc} X & \xrightarrow{\exists!\,\mathrm{beh}} & \Omega_F \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \cong} \\ FX & \xrightarrow{F\,\mathrm{beh}} & F\Omega_F \end{array}$$

Coalgebra basics

Category $\mathbf{C}$, functor F, category of coalgebras:

$\mathbf{Coalg}_F$

Objects: $X \xrightarrow{c} FX$

Arrows:

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle d} \\ FX & \xrightarrow{Fh} & FY \end{array}$$

Final coalgebra semantics:

$$\begin{array}{ccc} X & \xrightarrow{\exists!\,\mathrm{beh}} & \Omega_F \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \cong} \\ FX & \xrightarrow{F\,\mathrm{beh}} & F\Omega_F \end{array}$$

bisimilarity in $\mathbf{Sets}$
(for wpp functors)

trace semantics in $\mathcal{K}\ell(T)$
(for TF-coalgebras)

Traces for PTS

Coalgebraic trace semantics [HJS'06/'07] applies to PTS

Traces for PTS

Coalgebraic trace semantics [HJS'06/'07] applies to PTS

trace-map as final coalgebra map in $\mathcal{K}\ell(\mathcal{D})$

$$\begin{array}{ccc} X & \xrightarrow{\ \mathrm{tr}\ } & A^* \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \cong} \\ 1 + A \times X & \longrightarrow & 1 + A \times A^* \end{array}$$

$\mathrm{tr}: X \to \mathcal{D}(A^*)$ in **Sets**

Traces for PTS

Coalgebraic trace semantics [HJS'06/'07] applies to PTS

trace-map as final coalgebra map in $\mathcal{K}\ell(\mathcal{D})$

$$\begin{array}{ccc} X & \xrightarrow{\ \mathrm{tr}\ } & A^* \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \cong} \\ 1 + A \times X & \longrightarrow & 1 + A \times A^* \end{array}$$

$\mathrm{tr} : X \to \mathcal{D}(A^*)$ in **Sets**

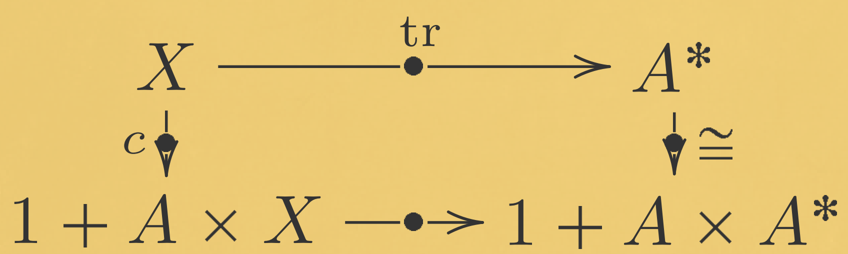
It instantiates to finite trace distribution:

$$\begin{aligned} \mathrm{tr}(x)(\varepsilon) &= c(x)(*) \\ \mathrm{tr}(x)(aw) &= \textstyle\sum_{x' \in X} c(x)(a, x') \cdot \mathrm{tr}(x)(w) \end{aligned}$$

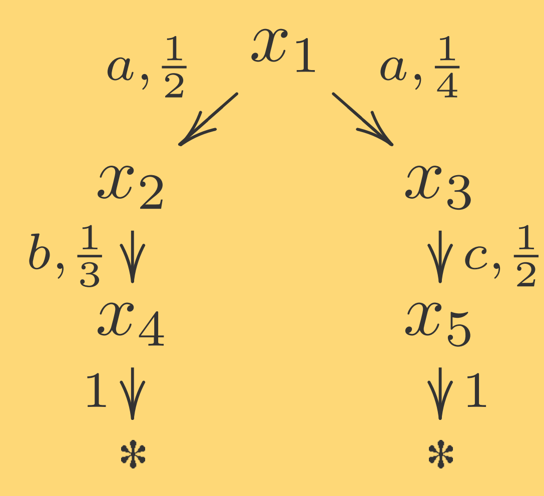
Traces for PTS

Coalgebraic trace semantics [HJS'06/'07] applies to PTS

trace-map as final coalgebra map in $\mathcal{K}\ell(\mathcal{D})$



$\text{tr}: X \to \mathcal{D}(A^*)$ in **Sets**



$$\text{tr}(x_1)(ab) = \frac{1}{6}$$

$$\text{tr}(x_1)(ac) = \frac{1}{8}$$

It instantiates to finite trace distribution:

$$\begin{aligned} \text{tr}(x)(\varepsilon) &= c(x)(*) \\ \text{tr}(x)(aw) &= \textstyle\sum_{x' \in X} c(x)(a, x') \cdot \text{tr}(x)(w) \end{aligned}$$

Expressions for PTS

Expressions for PTS

$$\mathsf{E} ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{E}^g ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{F}_i ::= * \mid a \cdot \mathsf{E}$$

Expressions for PTS

carry a scalar product

$$\mathsf{E} ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{E}^g ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{F}_i ::= * \mid a \cdot \mathsf{E}$$

Expressions for PTS

expressions behave! (Kleene-style theorem)

carry a scalar product

$$\mathsf{E} ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \mid x \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{E}^g ::= \bigoplus_{i\in I} p_i \cdot \mathsf{F}_i \mid \mu x.\mathsf{E}^g \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$\mathsf{F}_i ::= * \mid a \cdot \mathsf{E}$$

Expressions for PTS

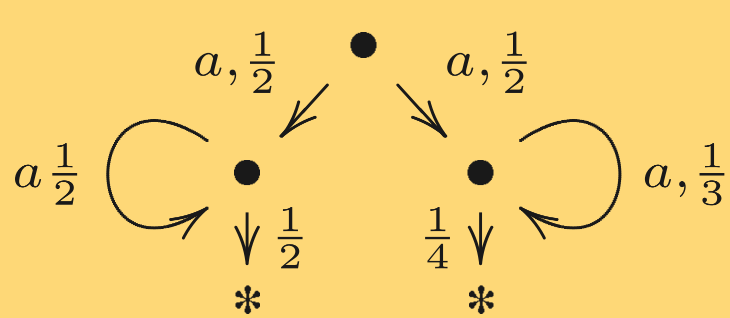
expressions behave! (Kleene-style theorem)

carry a scalar product

$$E ::= \bigoplus_{i\in I} p_i \cdot F_i \mid \mu x.E^g \mid x \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$E^g ::= \bigoplus_{i\in I} p_i \cdot F_i \mid \mu x.E^g \qquad (p_i \in [0,1], \sum_{i\in I} p_i \leqslant 1)$$

$$F_i ::= * \mid a \cdot E$$



$$\frac{1}{2}\cdot a\cdot \mu x.\left(\frac{1}{2}\cdot a\cdot x \oplus \frac{1}{2}\cdot *\right)$$
$$\oplus \frac{1}{2}\cdot a\cdot \mu x.\left(\frac{1}{3}\cdot a\cdot x \oplus \frac{1}{4}\cdot *\right)$$

Axioms

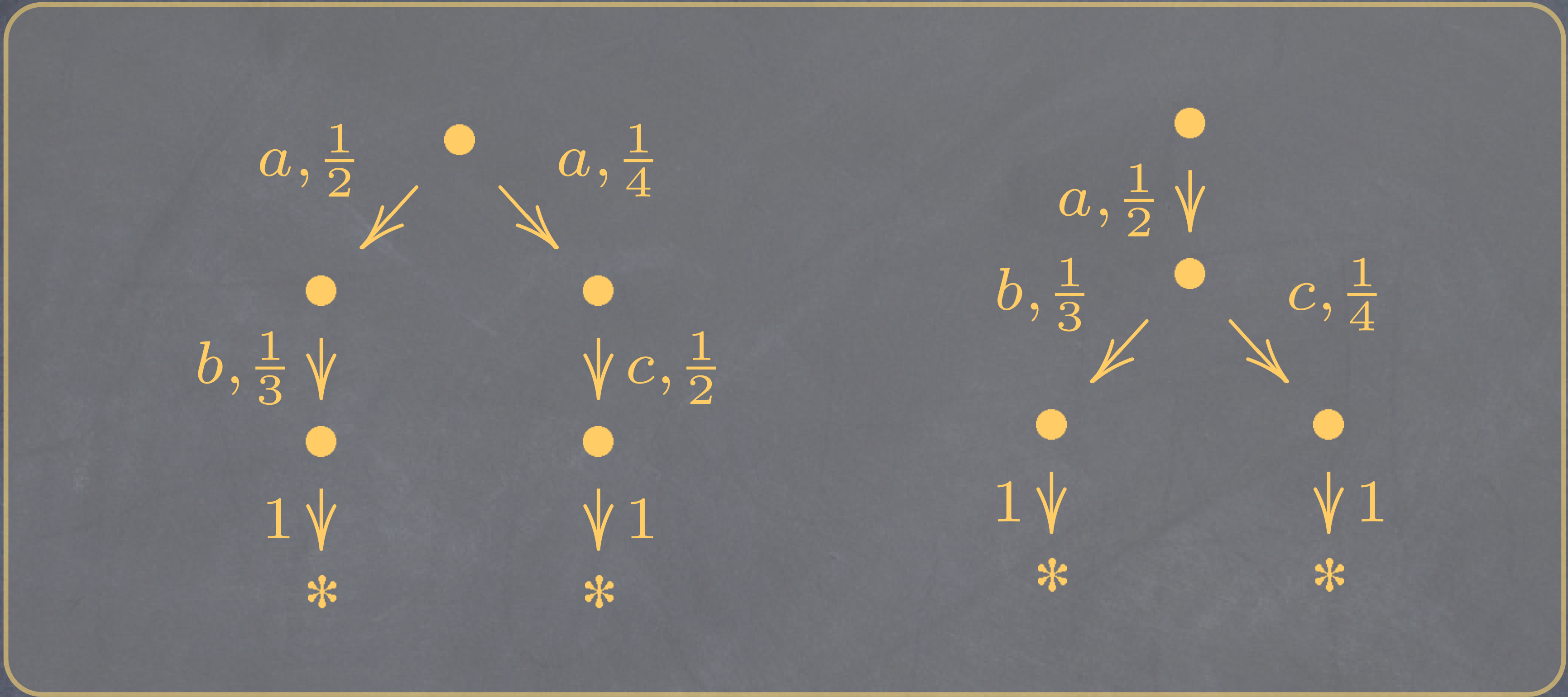
bisimilarity

$$\begin{array}{lcll}
E_1 \oplus (E_2 \oplus E_3) & \equiv & (E_1 \oplus E_2) \oplus E_3 & (A) \\
E_1 \oplus E_2 & \equiv & E_2 \oplus E_1 & (C) \\
E \oplus \underline{\emptyset} & \equiv & E & (E) \\
\mu x.E & \equiv & E[\mu x.E/x] & (FP) \\
\gamma[E/x] \equiv E & \Rightarrow & \mu x.\gamma \equiv E & (UFP) \\
\mu x.E & \equiv & \mu y.E[y/x] \ \text{ if } y \text{ is not free in } E & (\alpha - equiv) \\
E_1 \equiv E_2 & \Rightarrow & E[E_1/x] \equiv E[E_2/x] & (Cong) \\
0 \cdot E & \equiv & \underline{\emptyset} & (Z) \\
p \cdot E \oplus p' \cdot E & \equiv & (p + p') \cdot E & (S)
\end{array}$$

$$p \cdot a \cdot (p_1 E_1 \oplus p_2 E_2) \equiv p_1 \cdot a \cdot pE_1 \oplus p_2 \cdot a \cdot pE_2 \quad (D)$$

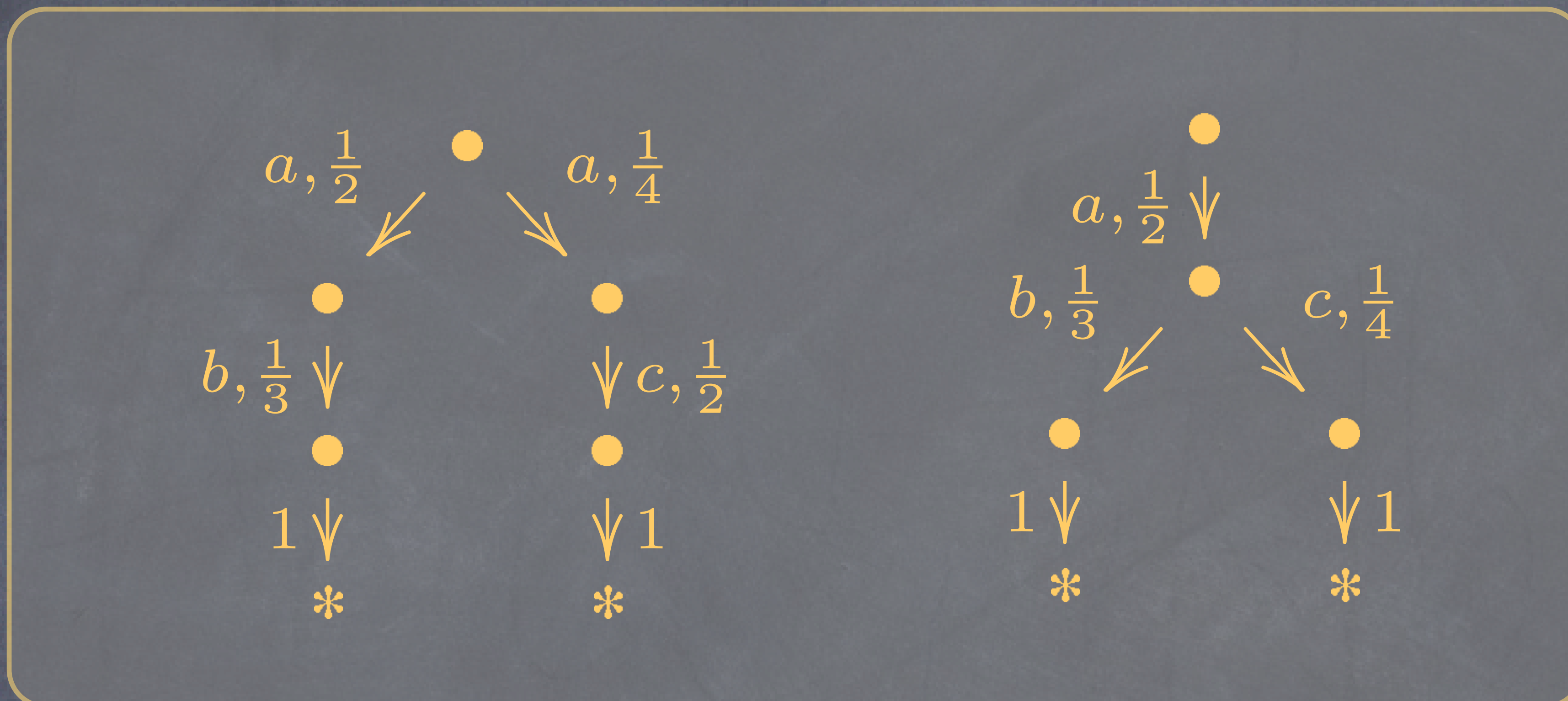
trace

Example



$a, \frac{1}{2}$
$a, \frac{1}{4}$
$b, \frac{1}{3}$
$c, \frac{1}{2}$
1
1
*
*
$a, \frac{1}{2}$
$b, \frac{1}{3}$
$c, \frac{1}{4}$
1
1
*
*

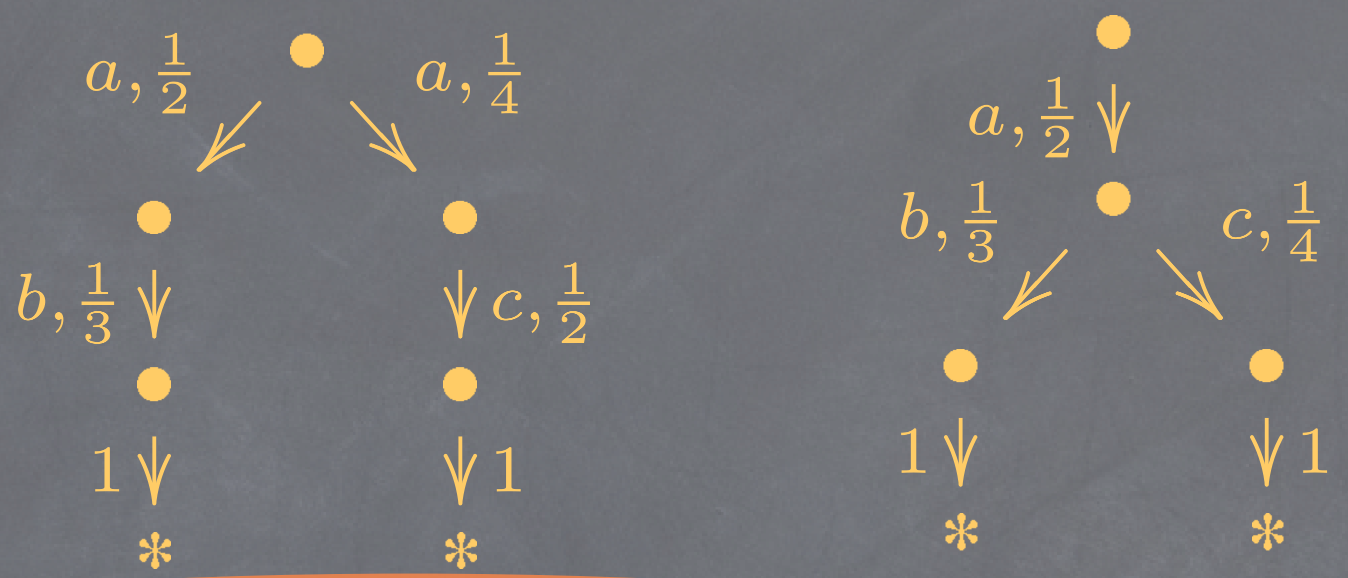
Example



$$\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) \overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right)\oplus\frac{1}{4}(1\cdot c\cdot 1\cdot *)\right)$$

$$= \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

Example



$$\frac{1}{3}\cdot b\cdot 1\cdot * = \frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right),\quad \frac{1}{2}\cdot c\cdot 1\cdot * = \frac{1}{2}(1\cdot c\cdot 1\cdot *)$$

$$\begin{aligned}\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot c\cdot 1\cdot *\right) &\overset{(D)}{\equiv} \frac{1}{2}\cdot a\cdot\left(\frac{1}{2}\left(\frac{2}{3}\cdot b\cdot 1\cdot *\right)\oplus\frac{1}{4}(1\cdot c\cdot 1\cdot *)\right)\\ &= \frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)\end{aligned}$$

Soundness and Completeness

Find an injective map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness and Completeness

canonical map to $\equiv$ - classes

Find an injective map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness and Completeness

canonical map to $\equiv$ - classes

Find an **injective** map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness

$$\begin{aligned} & \mathsf{E}_1 \equiv \mathsf{E}_2 \\ \Leftrightarrow \quad & [\mathsf{E}_1] = [\mathsf{E}_2] \\ \overset{(*)}{\Rightarrow} \quad & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\ \overset{(\triangle)}{\Leftrightarrow} \quad & tr(\mathsf{E}_1) = tr(\mathsf{E}_2) \end{aligned}$$

Completeness

$$\begin{aligned} & tr(\mathsf{E}_1) = tr(\mathsf{E}_2) \\ \overset{(\triangle)}{\Leftrightarrow} \quad & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\ \overset{(\heartsuit)}{\Rightarrow} \quad & [\mathsf{E}_1] = [\mathsf{E}_2] \\ \Leftrightarrow \quad & \mathsf{E}_1 \equiv \mathsf{E}_2 \end{aligned}$$

Soundness and Completeness

canonical map to $\equiv$ - classes

Find an **injective** map $out_{\equiv}$ with $\mathrm{tr} = out_{\equiv} \circ [-]$

Soundness

$$\begin{array}{ll} & \mathsf{E}_1 \equiv \mathsf{E}_2 \\ \Leftrightarrow & [\mathsf{E}_1] = [\mathsf{E}_2] \\ \overset{(*)}{\Rightarrow} & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\ \overset{(\triangle)}{\Leftrightarrow} & tr(\mathsf{E}_1) = tr(\mathsf{E}_2) \end{array}$$

Completeness

$$\begin{array}{ll} & tr(\mathsf{E}_1) = tr(\mathsf{E}_2) \\ \overset{(\triangle)}{\Leftrightarrow} & out_{\equiv}([\mathsf{E}_1]) = out_{\equiv}([\mathsf{E}_2]) \\ \overset{(\heartsuit)}{\Rightarrow} & [\mathsf{E}_1] = [\mathsf{E}_2] \\ \Leftrightarrow & \mathsf{E}_1 \equiv \mathsf{E}_2 \end{array}$$

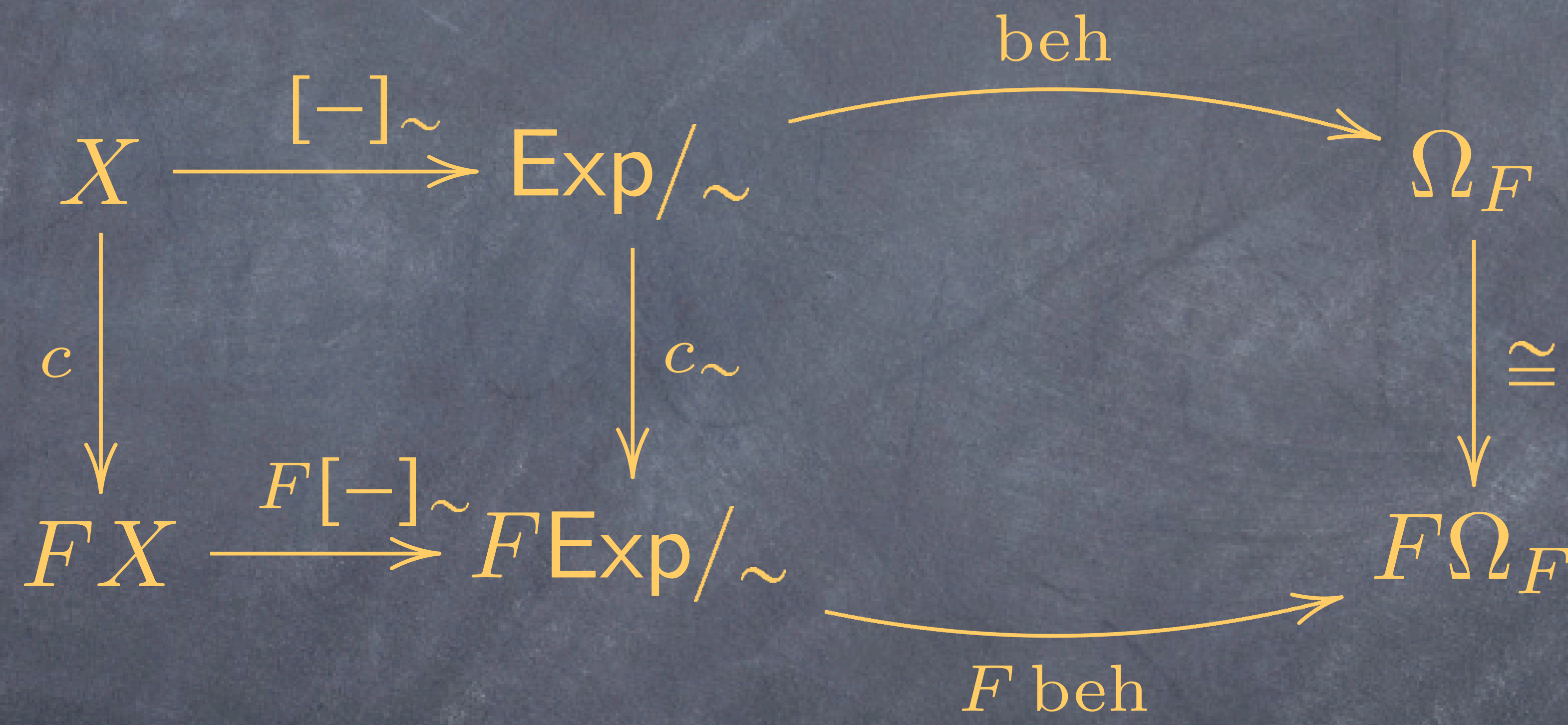
$(*)$ - existence of $out_{\equiv}$ $(\triangle)$ - $\mathrm{tr} = out_{\equiv} \circ [-]$ $(\heartsuit)$ - injectivity

How to get out?

Bisimilarity case, F-coalgebras

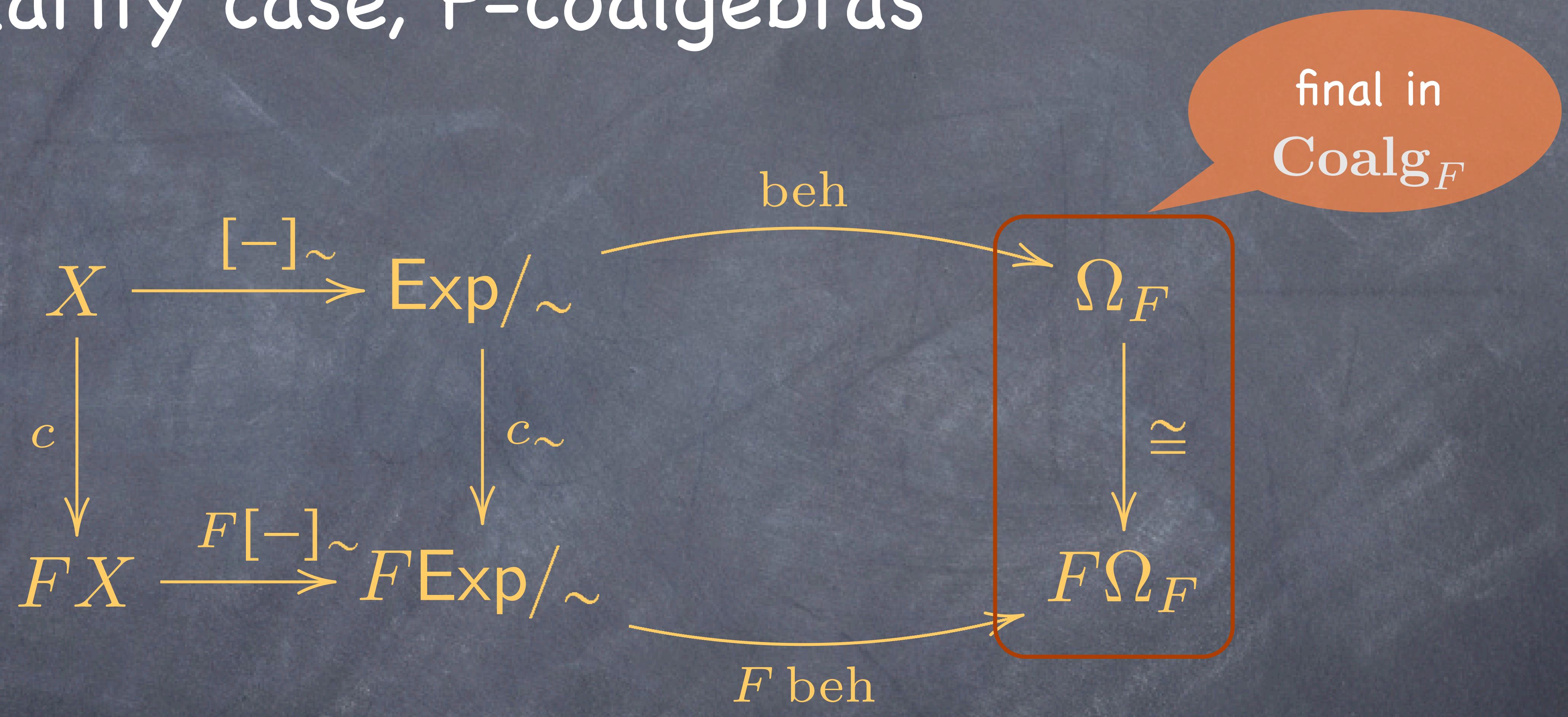
How to get out?

Bisimilarity case, F-coalgebras



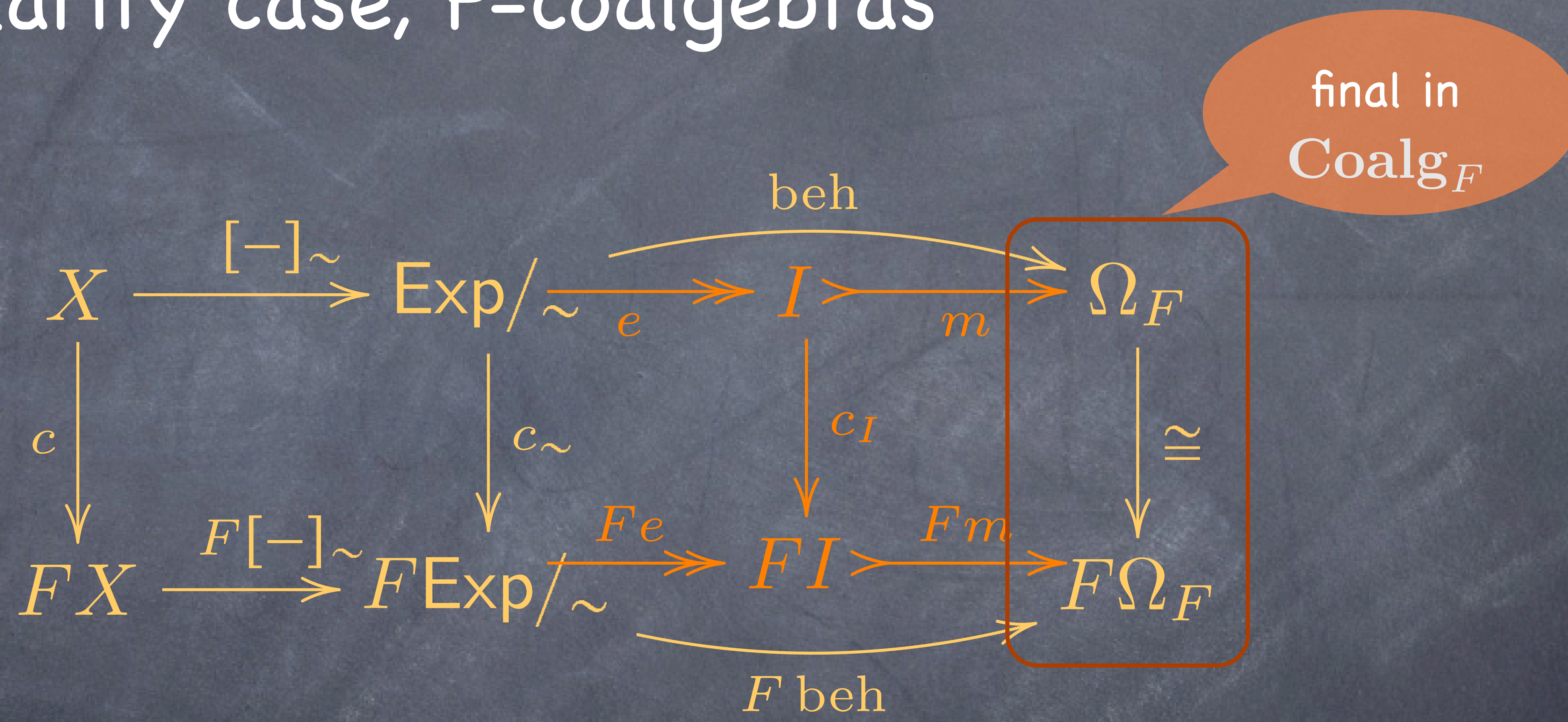
How to get out?

Bisimilarity case, F-coalgebras



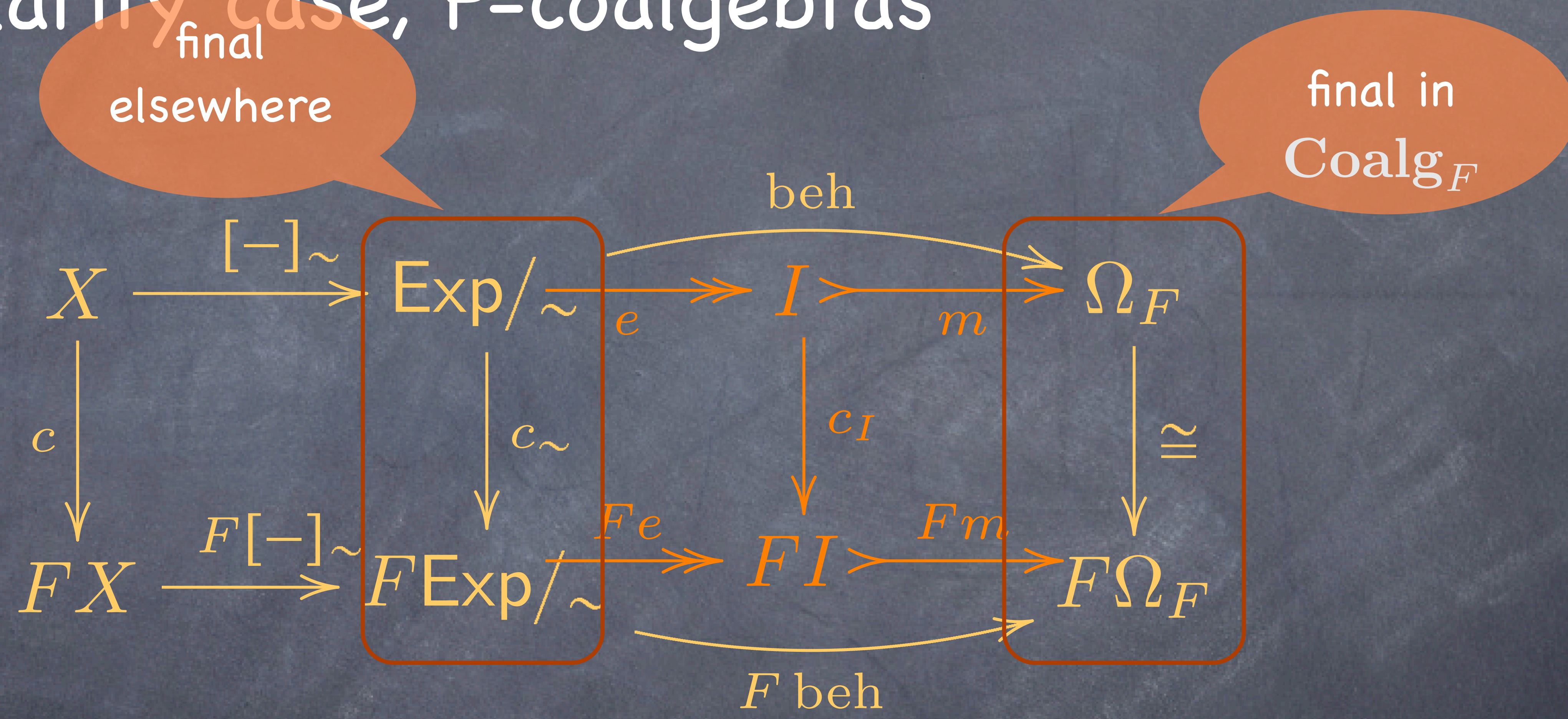
How to get out?

Bisimilarity case, F-coalgebras



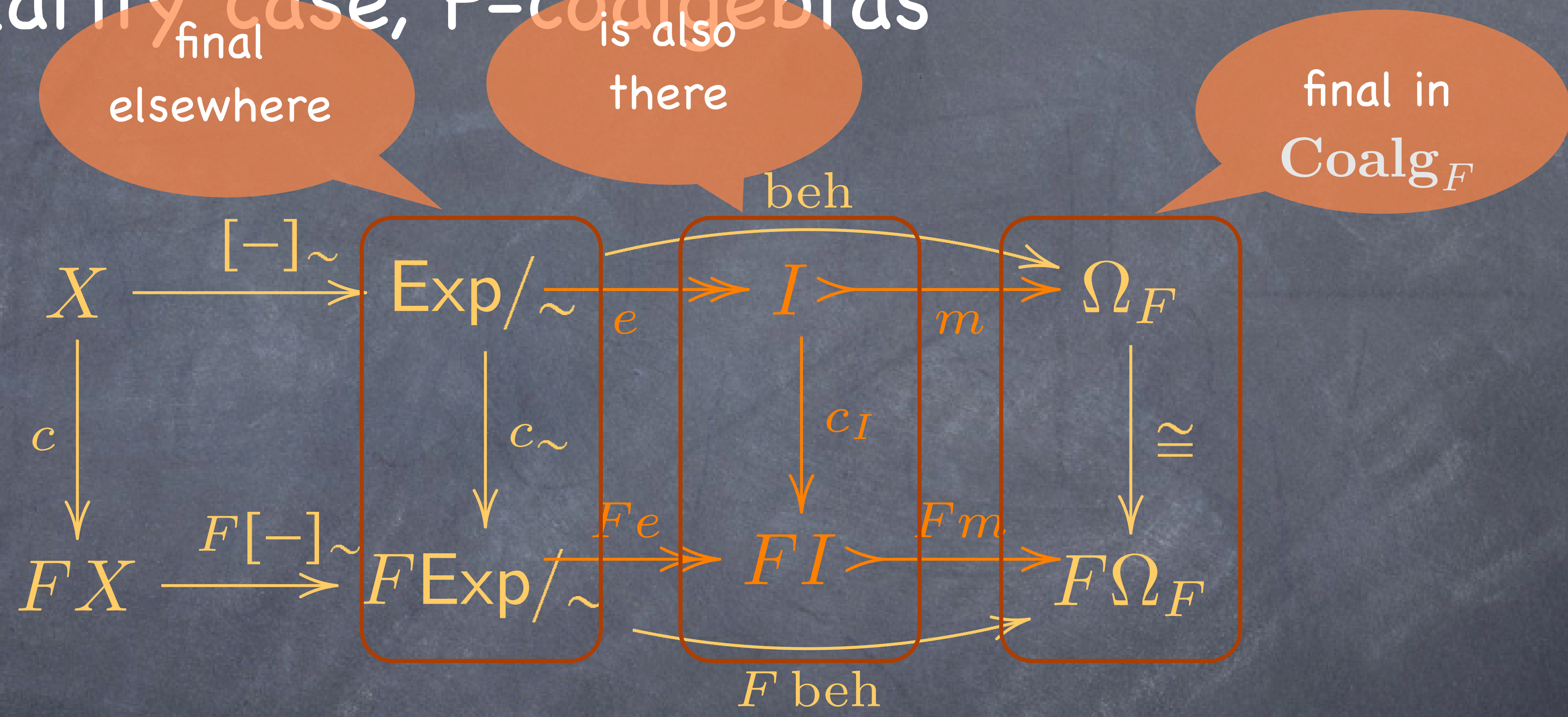
How to get out?

Bisimilarity case, F-coalgebras



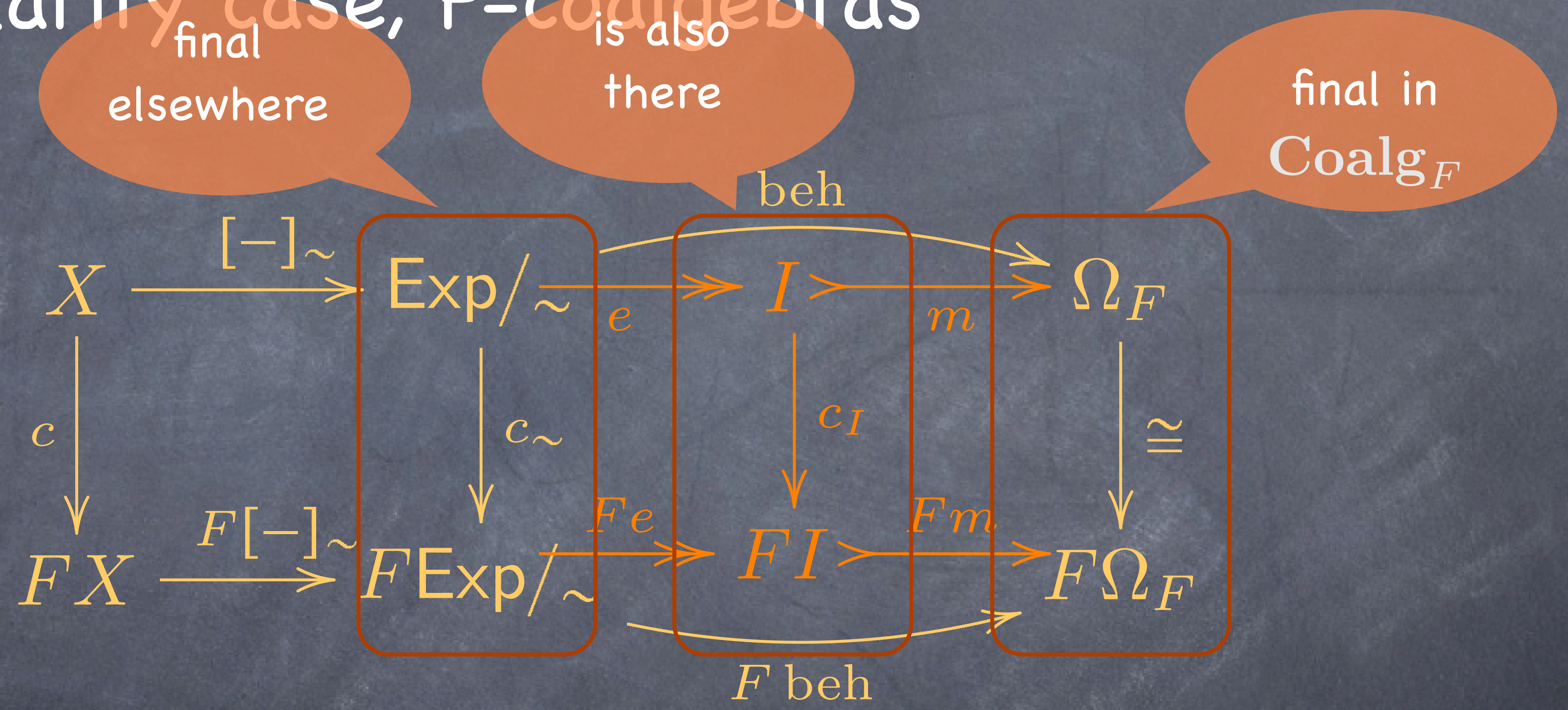
How to get out?

Bisimilarity case, F-coalgebras



How to get out?

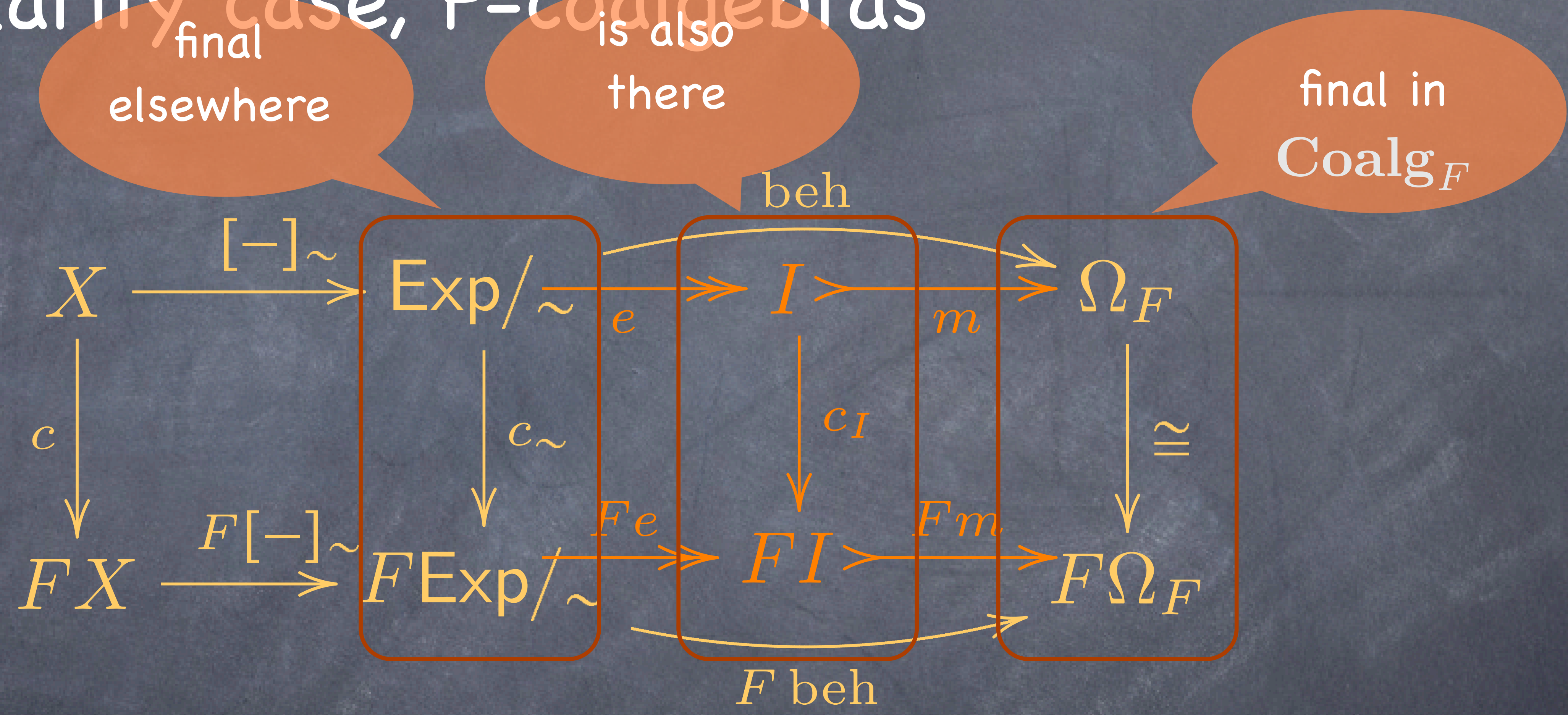
Bisimilarity case, F-coalgebras



Hence, e is iso, and out = beh is injective

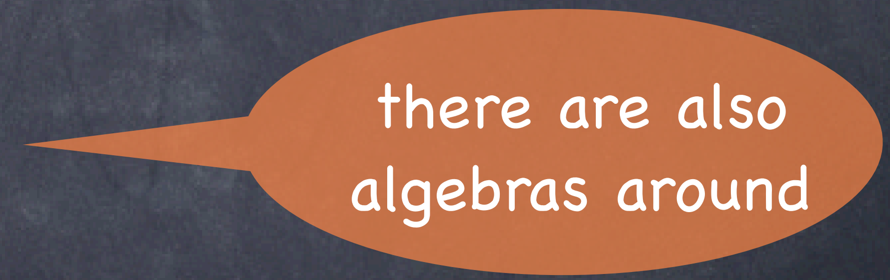
How to get out?

Bisimilarity case, F-coalgebras



Hence, e is iso, and out = beh is injective

Silva et al. '08/'09/'10, Jacobs'06

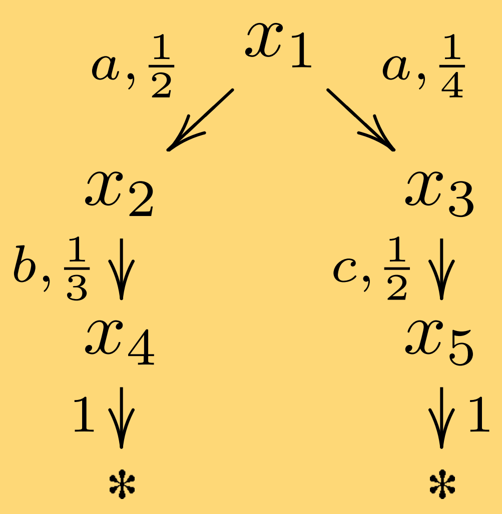


A way out for traces?

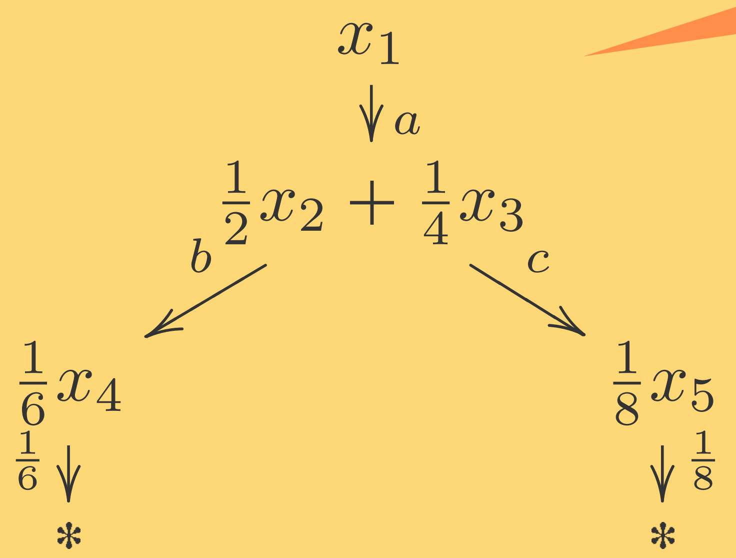
- It is tough to work in Kleisli categories
- Factorization ?
- So we find a way to stay in $\mathbf{Sets}$ or rather in $\mathbf{Sets}^{\mathcal{D}_\omega}$
- A way out - determinization

Determinization of PTS

PTS example



Its determinization

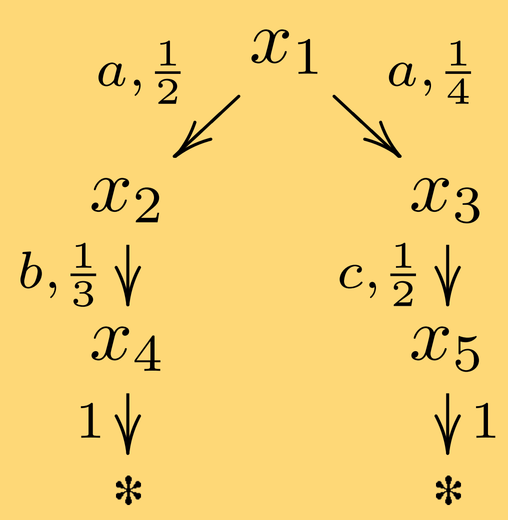


a G-coalgebra

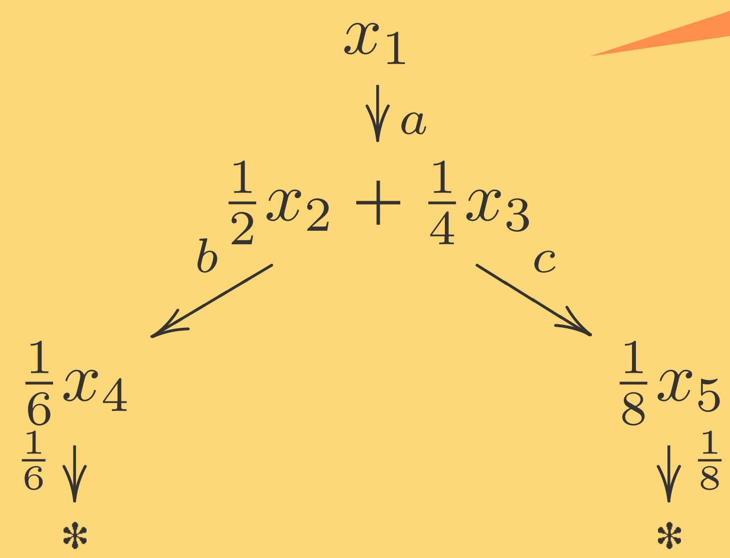
Monday, October 17, 2011

Determinization of PTS

PTS example

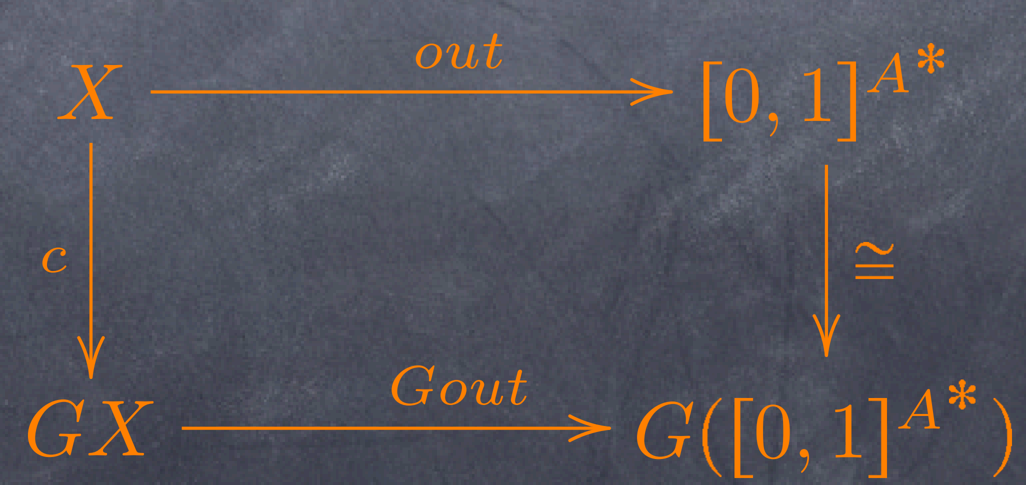


Its determinization



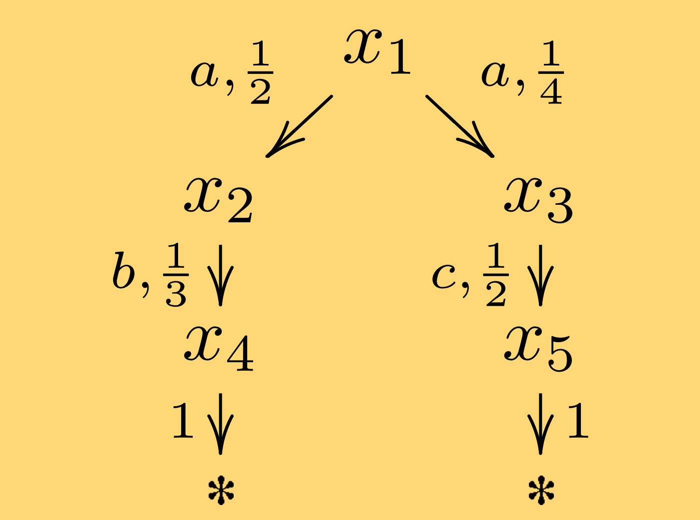
a G-coalgebra

$$GX = [0,1] \times X^A$$



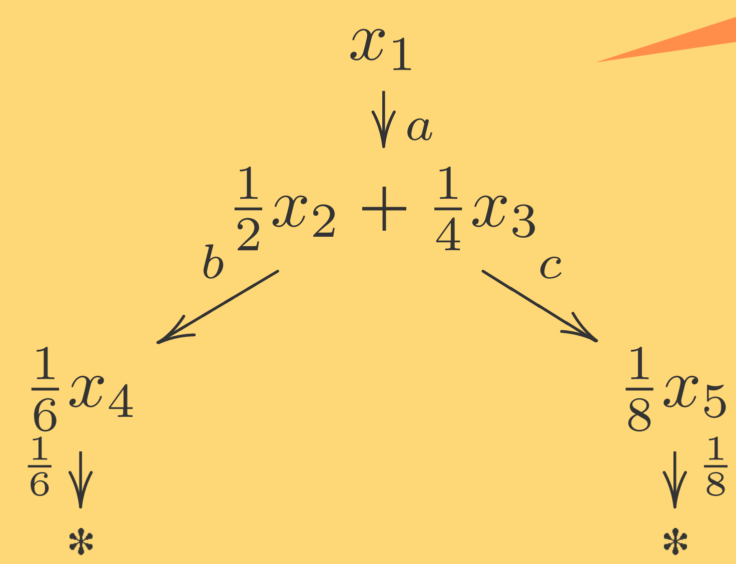
Determinization of PTS

PTS example



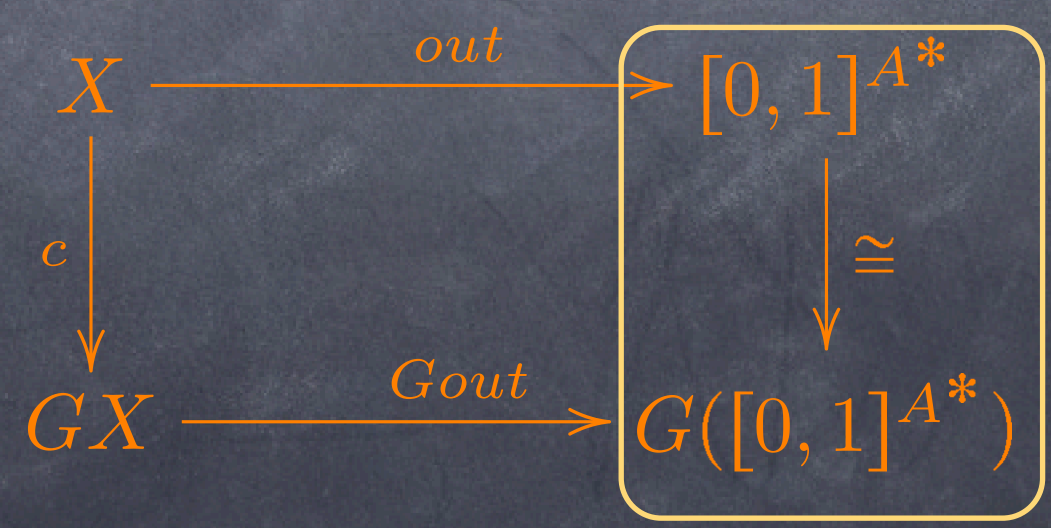
Its determinization

a G-coalgebra



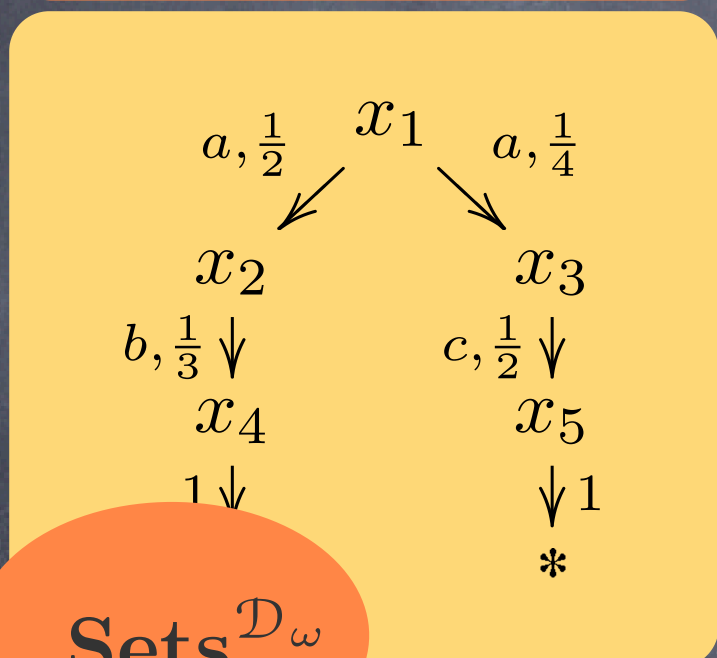
$$GX = [0,1] \times X^A$$

final G-coalgebra



Determinization of PTS

PTS example

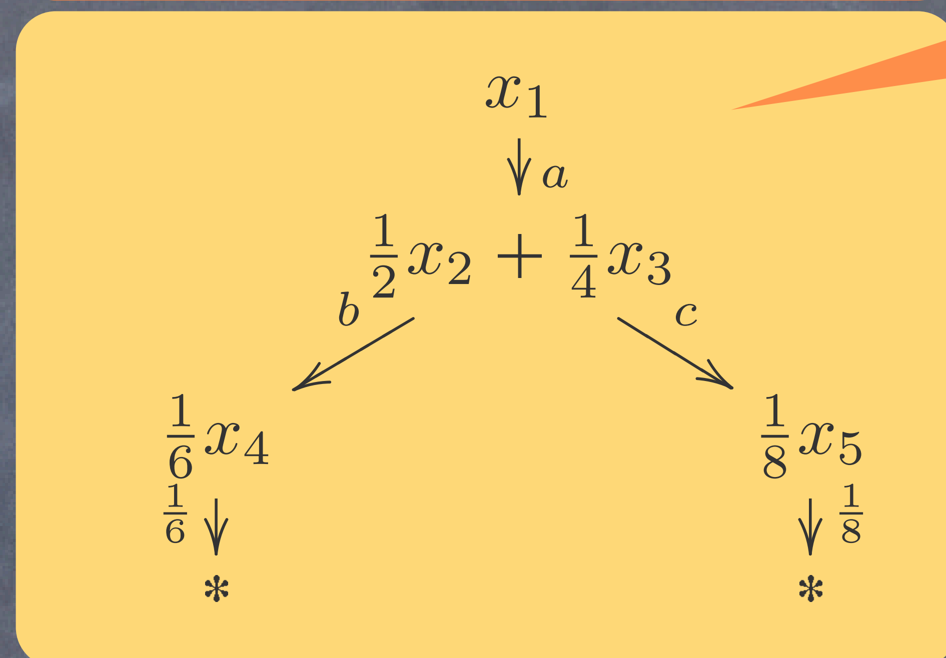


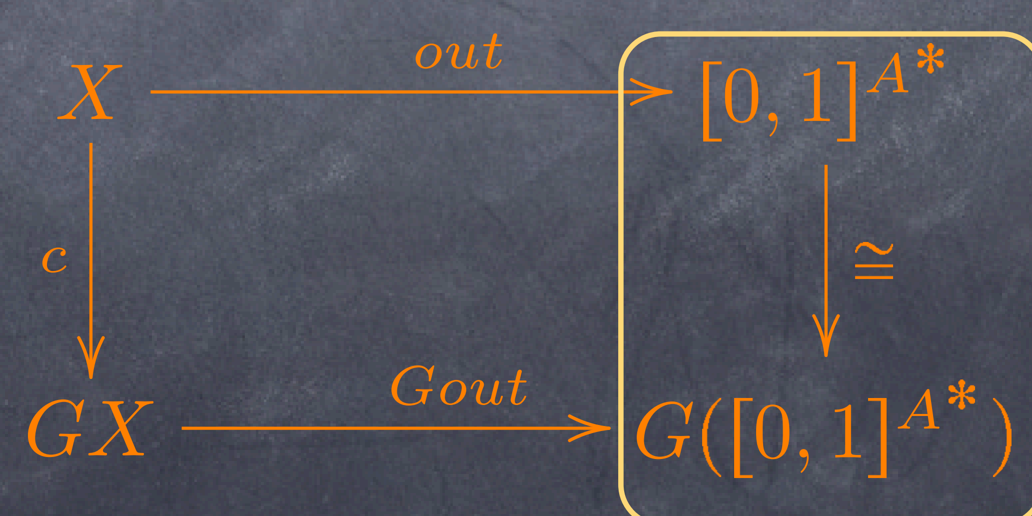
$\mathbf{Sets}^{\mathcal{D}_\omega}$

$$GX = [0,1] \times X^A$$

Its determinization

a G-coalgebra

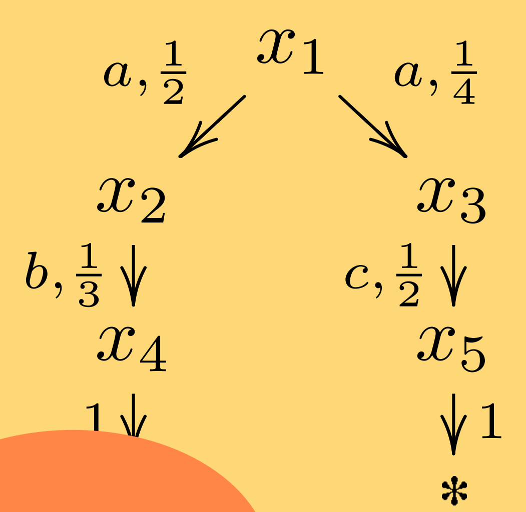




final G-coalgebra

Determinization of PTS

PTS example



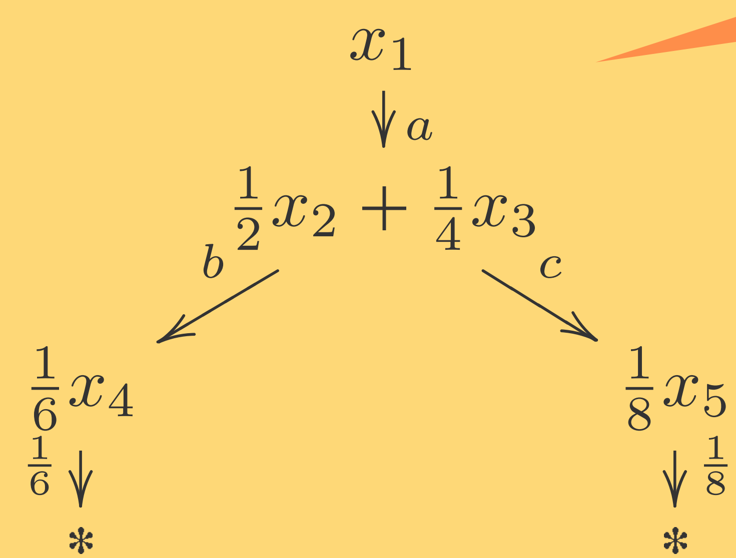
$\mathbf{Sets}^{\mathcal{D}_\omega}$

$$GX = [0,1] \times X^A$$

$$\begin{array}{ccc} X & \xrightarrow{\eta} & \mathcal{D}_\omega(X) \\ \downarrow c & & \downarrow (\delta \circ c)^\# \\ \mathcal{D}_\omega(1 + A \times X) & \xrightarrow{\delta} & G\mathcal{D}_\omega(X) \end{array}$$

Its determinization

a G-coalgebra



$$\begin{array}{ccc} X & \xrightarrow{out} & [0,1]^{A^*} \\ \downarrow c & & \downarrow \cong \\ GX & \xrightarrow{Gout} & G([0,1]^{A^*}) \end{array}$$

final G-coalgebra

A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

PCA
Eilenberg-Moore
algebras

A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

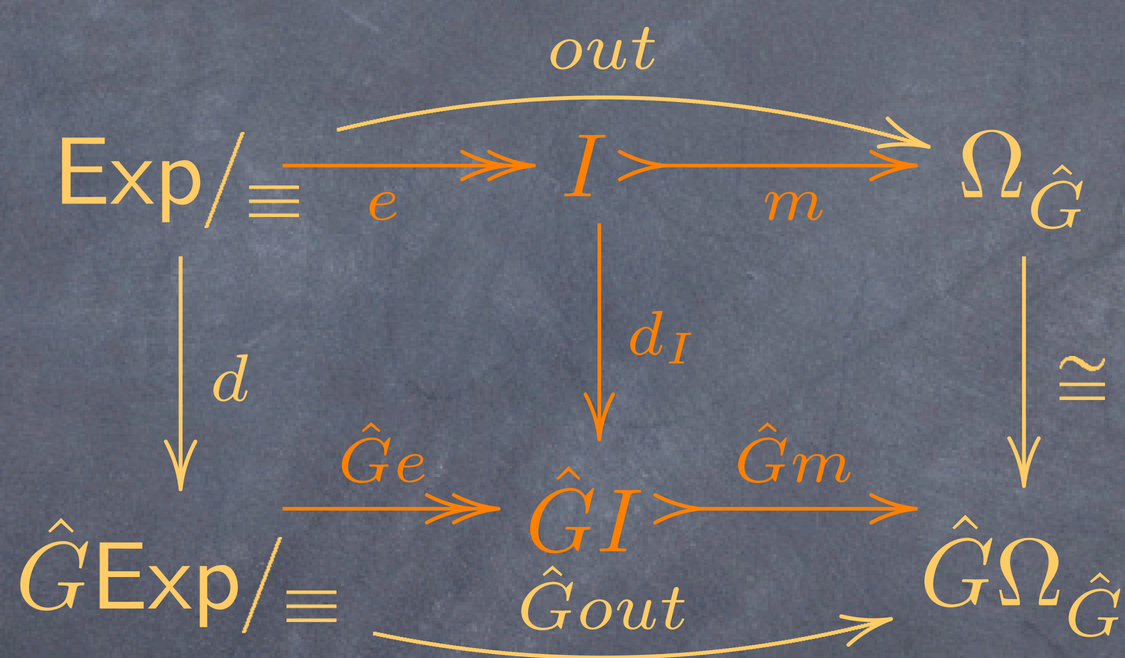
PCA
Eilenberg-Moore
algebras

$$
\begin{array}{ccc}
\mathsf{Exp}/_{\equiv} & \xrightarrow{\quad out \quad} & \Omega_{\hat{G}} \\
\Big\downarrow d & & \Big\downarrow \cong \\
\hat{G}\mathsf{Exp}/_{\equiv} & \xrightarrow{\quad \hat{G}out \quad} & \hat{G}\Omega_{\hat{G}}
\end{array}
$$

A way out for traces

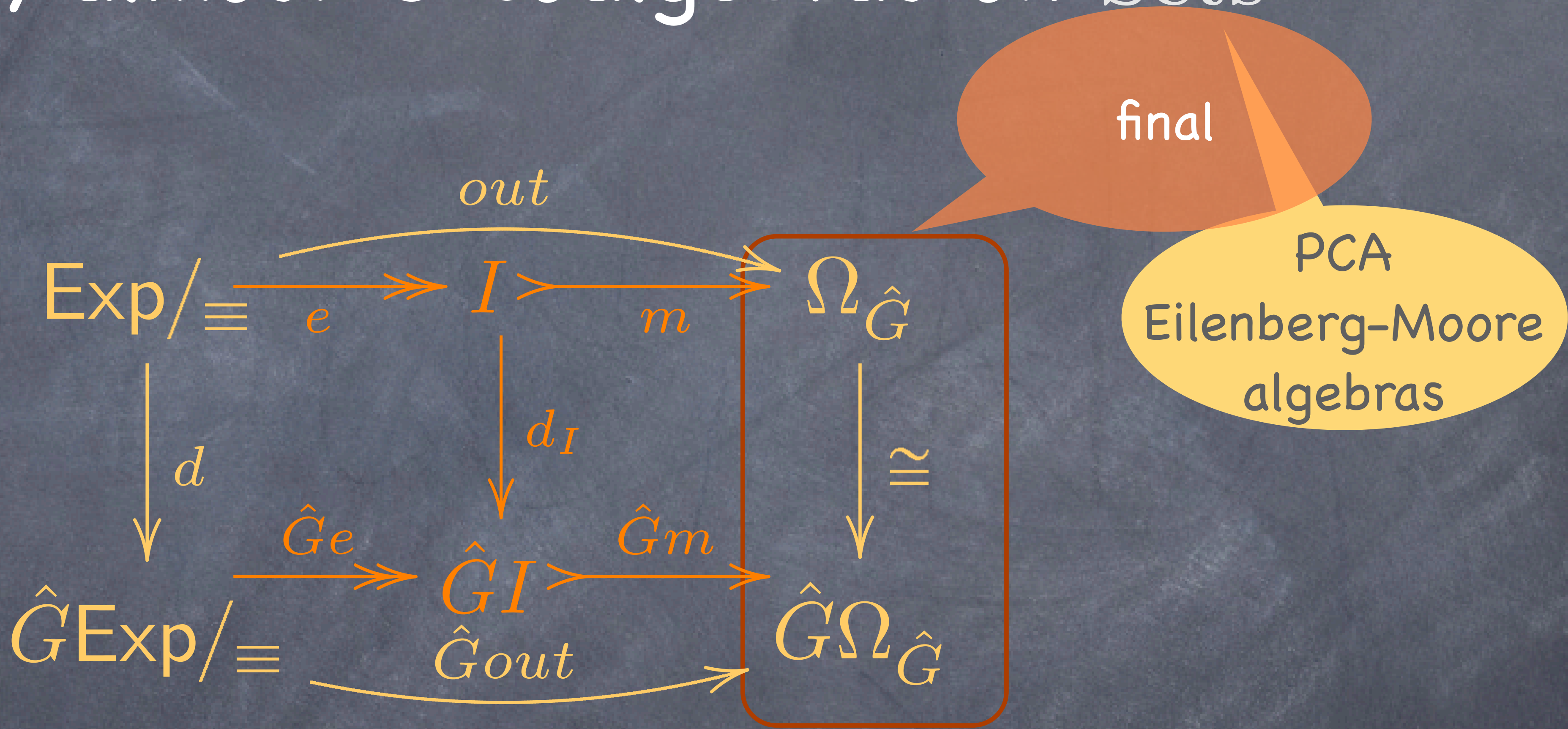
Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$

PCA Eilenberg-Moore algebras



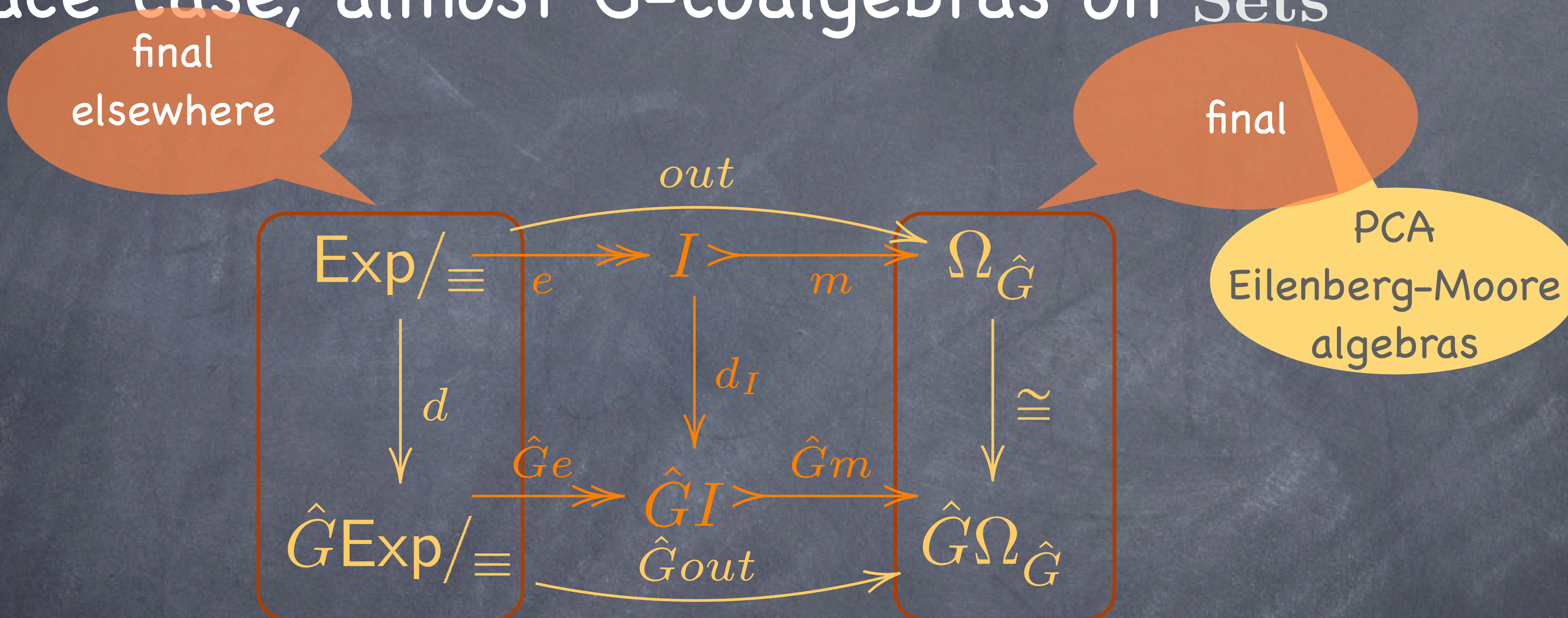
A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$



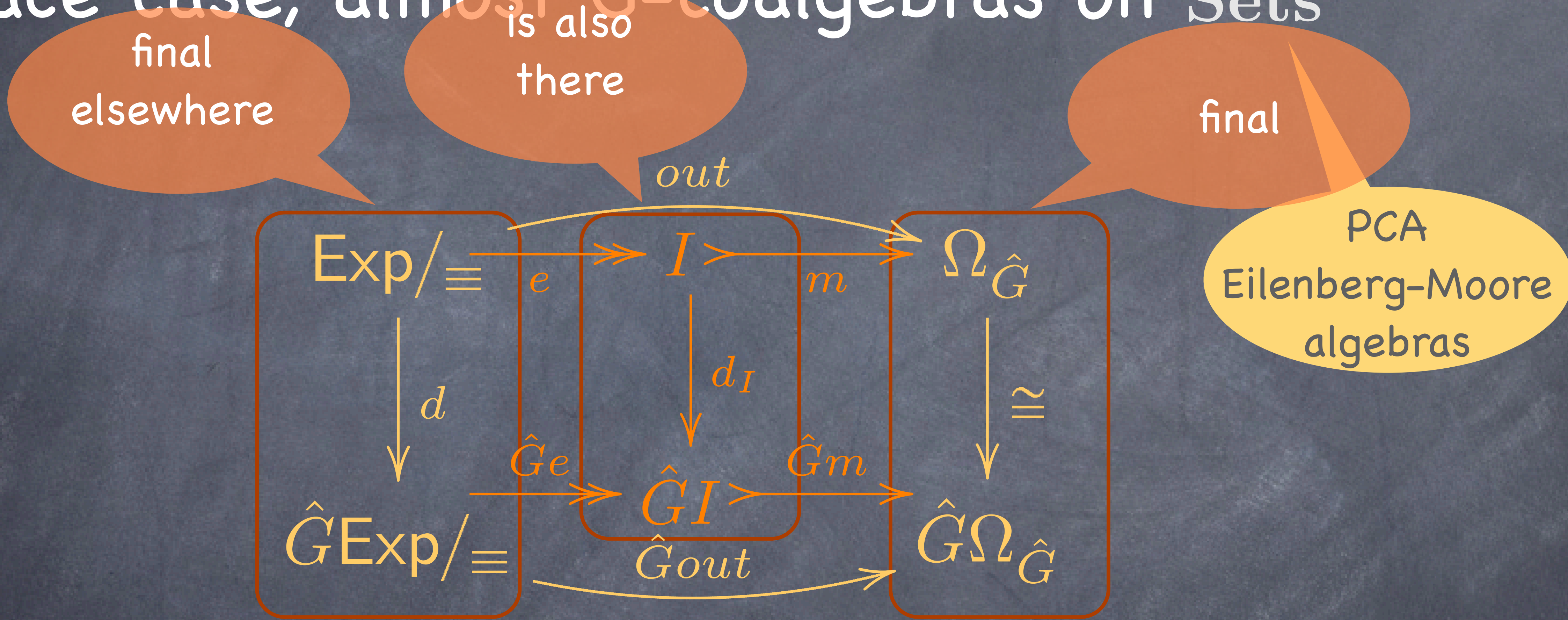
A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$



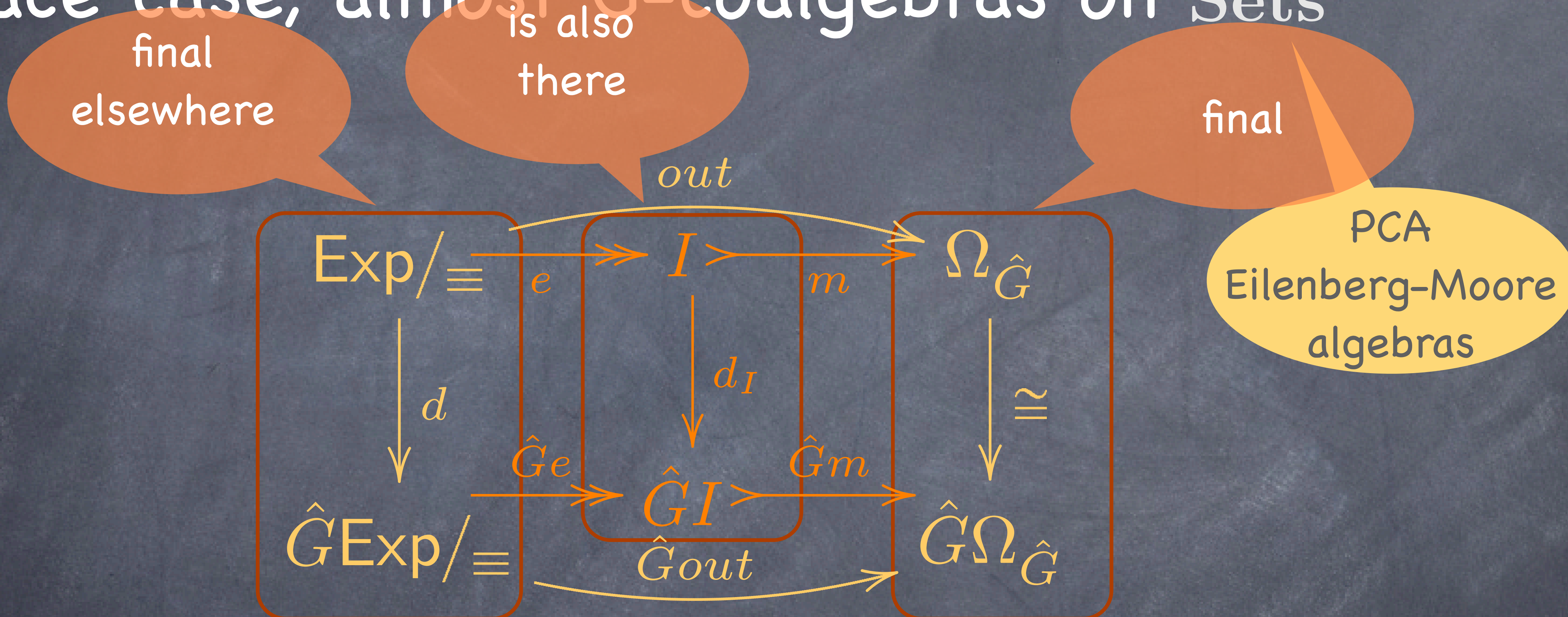
A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$



A way out for traces

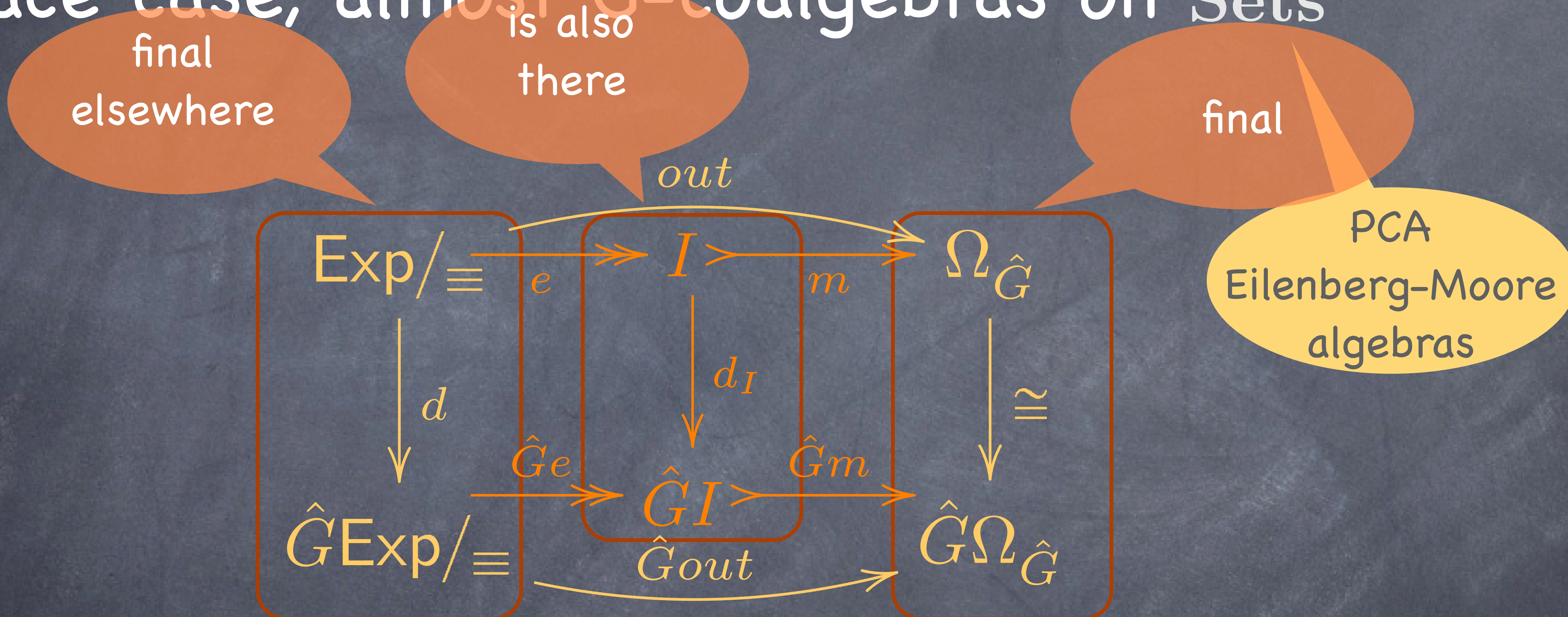
Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$



Hence, e is iso, and out is injective

A way out for traces

Trace case, almost G-coalgebras on $\mathbf{Sets}^{\mathcal{D}_\omega}$



Hence, e is iso, and out is injective

Moreover: $tr = out \circ [-]$

Conclusions

- we present a solution to a concrete problem

  sound and complete axiomatization of traces for PTS

  bisimilarity expressions and axioms plus one new axiom

- in a coalgebraic setting
- it opens many generalization questions...
- all about algebra and coalgebra

Conclusions

- we present a solution to a concrete problem

sound and complete axiomatization of traces for PTS

bisimilarity expressions and axioms plus one new axiom

- in a coalgebraic setting
- it opens many generalization questions...
- all about algebra and coalgebra

Thank you !