Lumpability of Markov chains and reward processes

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TU/e

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- Conclusions

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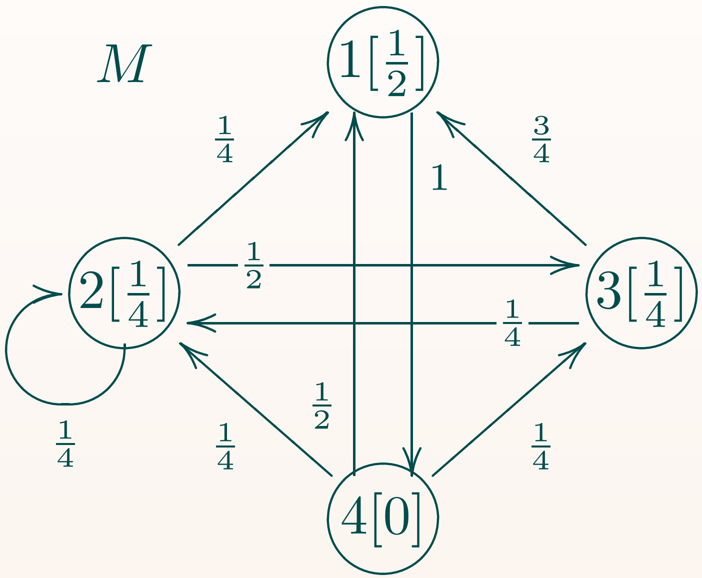

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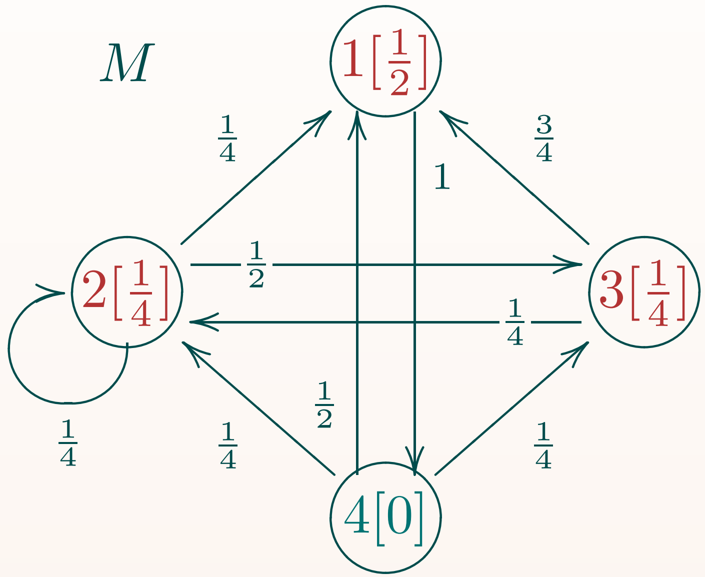

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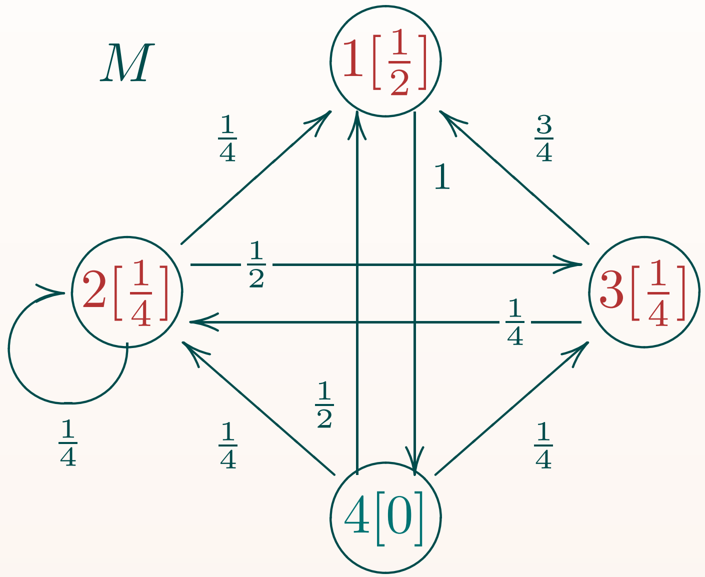

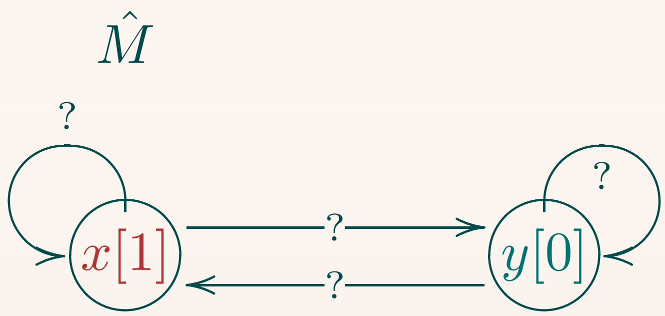

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- Markov chain, $X(n) = M = (S, P, \pi)$

$S -$ state space

$P -$ trans. prob.

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- When is $\hat{X}(n)$ a markov chain ???

General lumpability

Th. The Markov chain $M = (S, P, \pi)$ is lumpable w.r.t a partition $L = \{C_1, \ldots, C_m\}$ on S iff there exists a matrix $\hat{P}$ of order m, such that for all $i, j \in \{1, \ldots, m\}$ and for all $k \geq 0$ it holds

$$\hat{P}^k(i,j) = \frac{\sum_{i' \in C_i} \pi(i') \sum_{j' \in C_j} P^k(i', j')}{\hat{\pi}(i)}$$

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Notation: $M \xrightarrow{L}_l \hat{M},\ \hat{M} = (L, \hat{P}, \hat{\pi})$ for

$\hat{\pi}(i) = \sum_{i' \in C_i} \pi(i')$

$M_1 \sim_l M_2$ if they have a common lumping.

General lumpability - rewards

Def. The MRP $M = (S, P, \pi, r)$ is lumpable w.r.t a partition $L = \{C_1, \ldots, C_m\}$ on S iff the chain (S, P, π) is.

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Recall: $pm(M) = \sum_{i \in S} r(i) \underline{\pi}(i)$

Th. If $M \xrightarrow{L}_{lp} \hat{M}$ then $pm(M) = pm(\hat{M})$.

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Prop. $M \xrightarrow{L}_{ol} \hat{M} \Rightarrow M \xrightarrow{L}_{l} \hat{M}$

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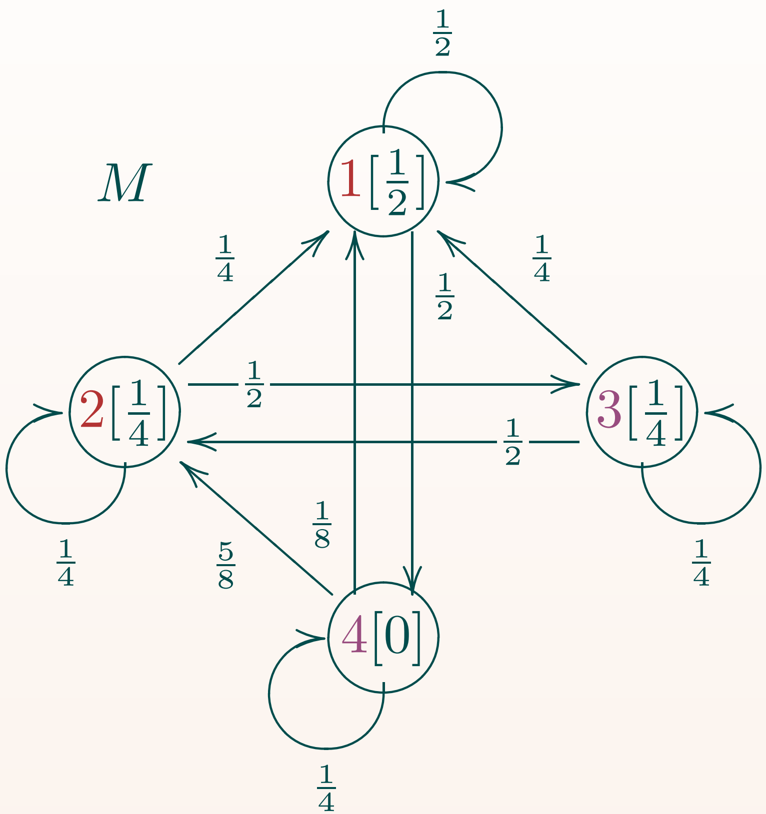

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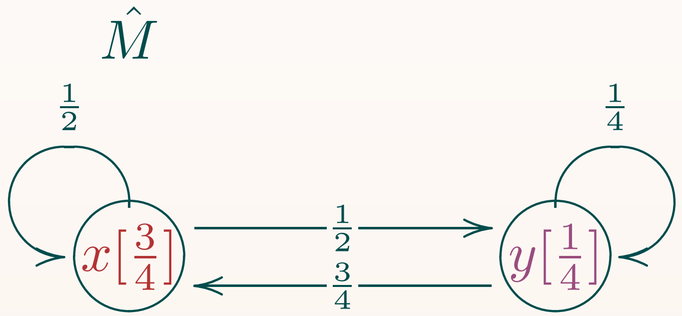

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Note: In the exact case for rewards

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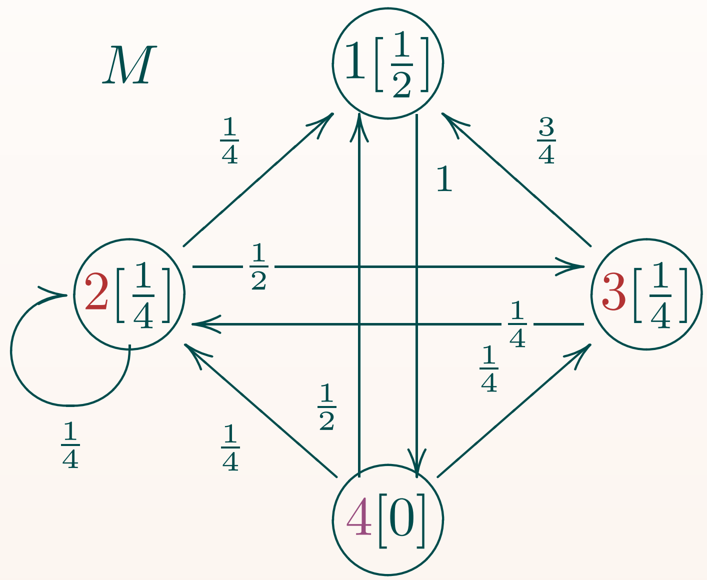

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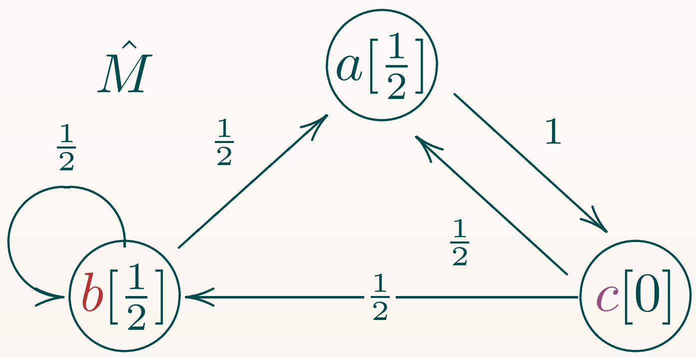

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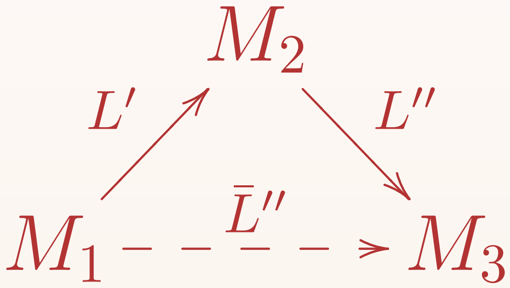

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$$\begin{array}{ccc} & M_2 & \\ {\scriptstyle L'}\nearrow & & \searrow{\scriptstyle L''} \\ M_1 & \overset{\bar{L}''}{\dashrightarrow} & M_3 \end{array}$$

Prop. p-transitivity implies transitivity

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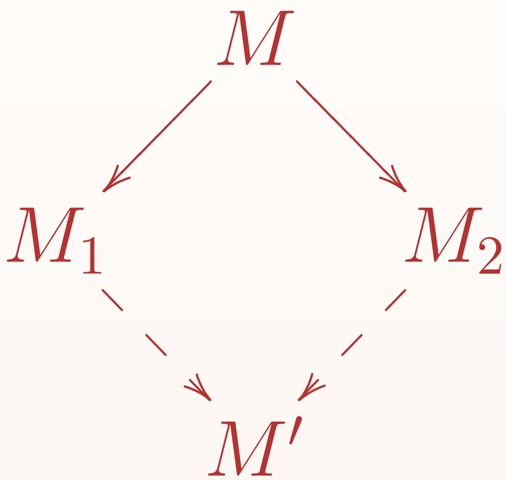
◇ - property:



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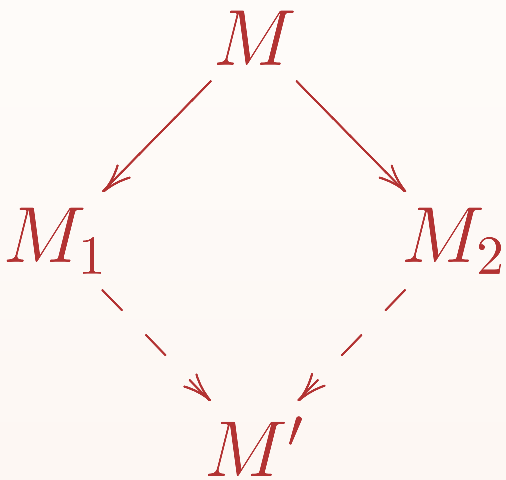

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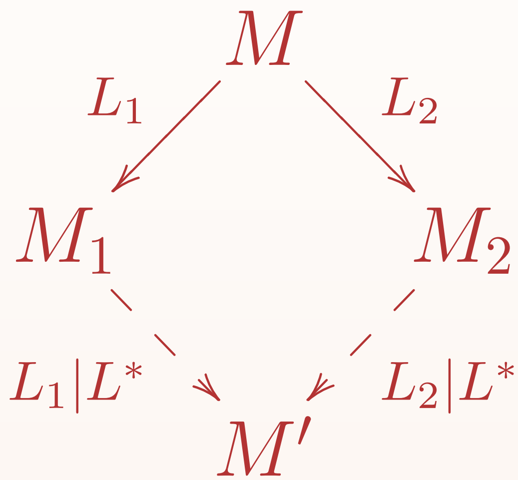

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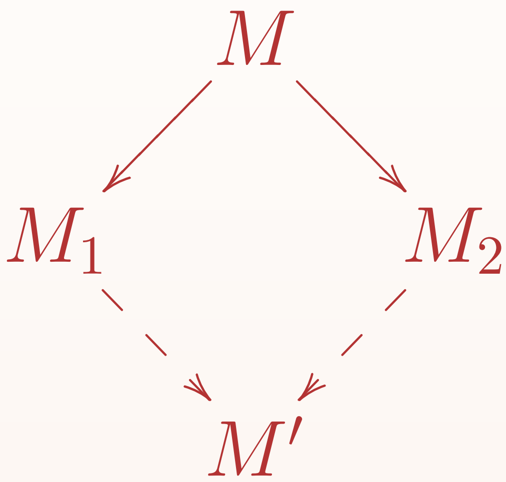

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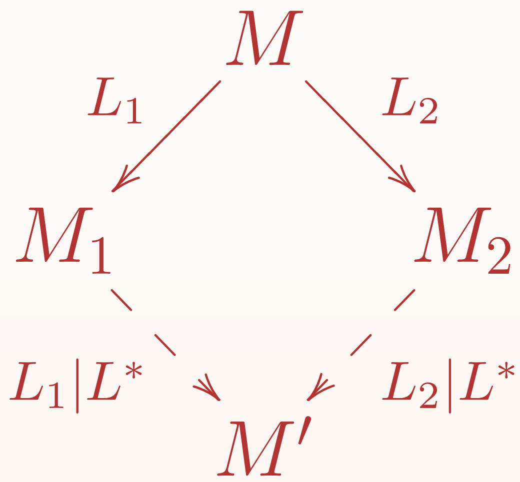

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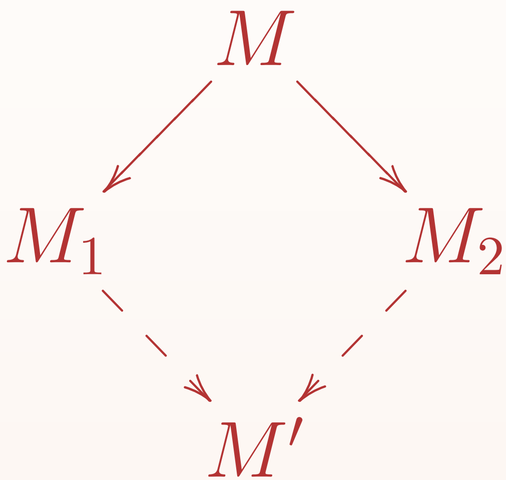

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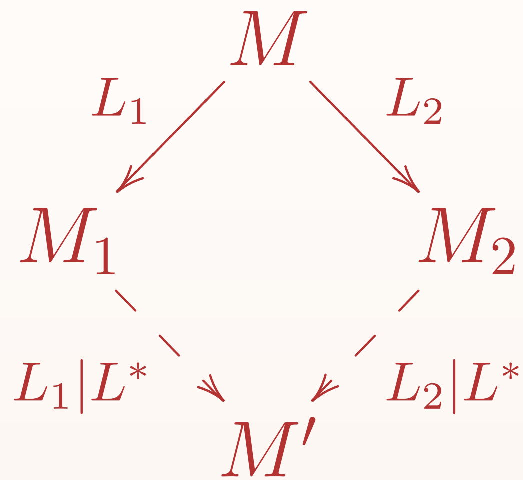

Prop. $*\diamond$ implies $\diamond$.

Prop. $\diamond$ and transitivity imply $\sim$ equivalence.

Results - lumpability relations

	ordinary	exact	general
p-transitivity	yes	no	yes
transitivity	yes	no	yes
$\diamond$	yes	no	yes
$*\diamond$	yes	no	no
$\sim$ is equiv.	yes	no	yes

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The same holds with or without rewards.

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