# The Theory of Traces for Nondeterminism and Probability

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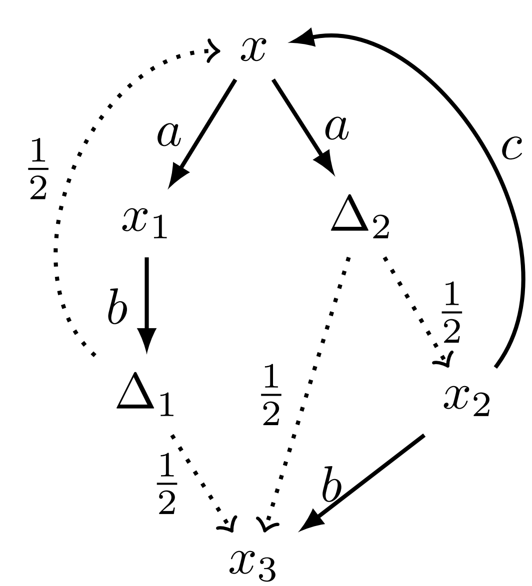


University of Pisa, 15.4.19

# It’s all about leaving a trace…

# Joint work with

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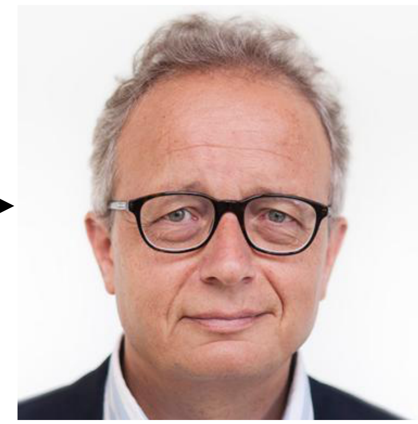
NII 国立情報学研究所 National Institute of Informatics



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I will tell you about:

Mathematical framework based on category theory for state-based systems semantics

**1.** The absolute basics of coalgebra

for nondeterministic/ probabilistic systems

**2.** Trace semantics via determinisation

systems with algebraic effects

**3.** …enabled by algebraic structure

# Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

states

object in the base category ***C***

type

functor on the base category ***C***

form a category too

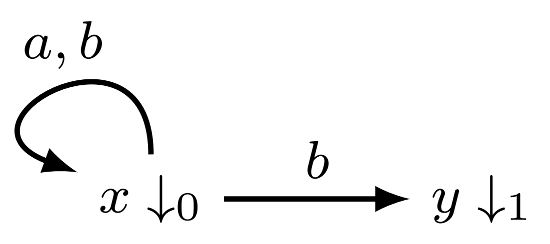
generic notion of behavioural equivalence (bisimilarity)

$\approx$

# Examples

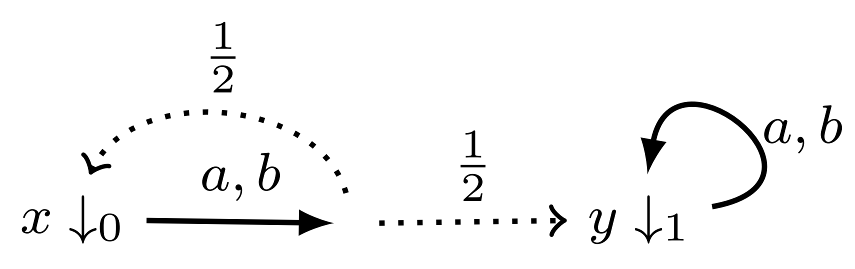
## NFA

$X \to 2 \times (\mathcal{P}X)^A$



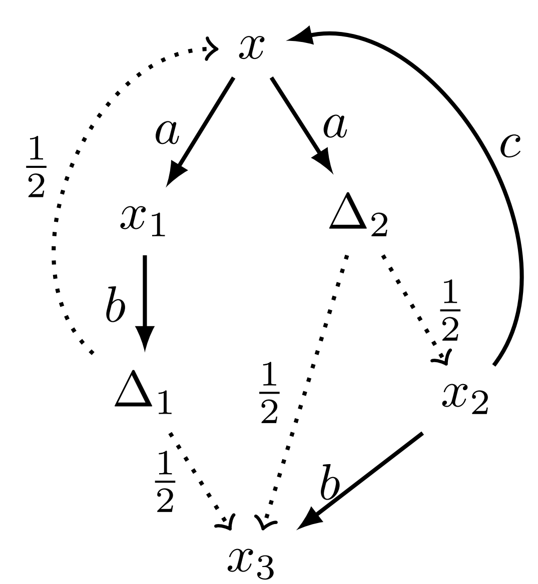

## Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$




## Simple NPA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$



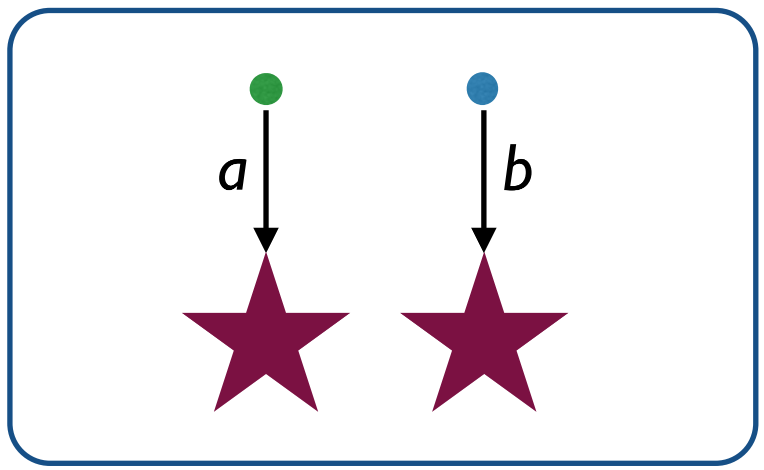

systems with nondeterminism and probability

# In general

Automata

with observations in O

$X \to O \times (MX)^A$




and M-effects

for a monad M

$\mu\colon MM \Rightarrow M$

$\eta\colon Id \Rightarrow M$

providing algebraic effects

we write $x{\downarrow}o,\ \ x \xrightarrow{a} t_x$

# In our examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$

$M = \mathcal{P}\mathcal{D}$ ???
for nondeterminism
and probability

# In our examples

NFA

$X \to 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

$M = \mathcal{D}_{\leq 1}$
for probability

Subdistributions

Simple PA

$X \to ? \times (\mathcal{C}X)^A$

$M = \mathcal{C}$
for nondeterminism
and probability !

Nonempty f.g. convex
subsets of
distributions

# Trace Semantics

language semantics

NFA = LTS + termination

X → 2 x ($\mathcal{P}$X)$^A$

$$\begin{tikzcd} x \downarrow_0 \arrow[loop left, "a, b"] \arrow[r, "b"] & y \downarrow_1 \end{tikzcd}$$

$$\mathrm{tr}\colon X \to 2^{A^*}$$

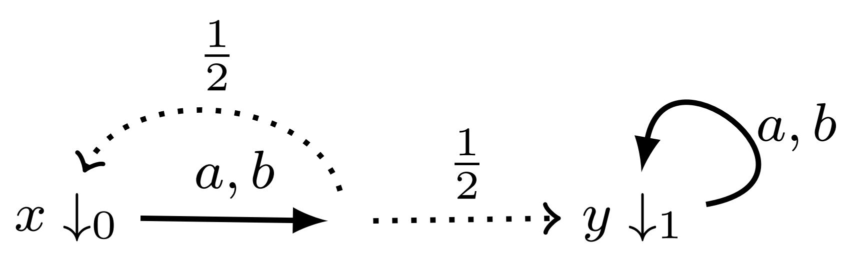
$$\mathrm{tr}(x) = (a \cup b)^* b = \{w \in \{a, b\}^* \mid w \text{ ends with a } b\}$$

# Trace Semantics

Rabin PA = RPTS + termination

probabilistic language semantics

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$



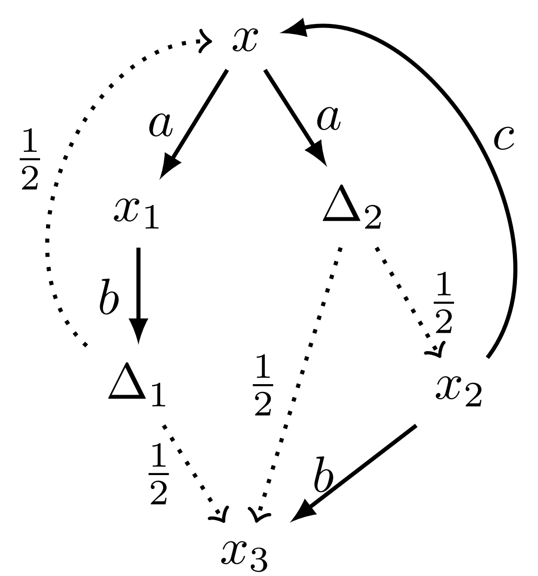

$$\mathrm{tr}(x) = \left(a \mapsto \frac{1}{2}, aa \mapsto \frac{3}{4}, \ldots\right)$$

$$\mathrm{tr}\colon X \to [0,1]^{A^*}$$

# Trace Semantics

## Simple NPA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$




nondet. probabilistic language semantics ?

Existing definitions are “local” given in terms of schedulers

$$\mathrm{tr}(x) = ???$$

$$\mathrm{tr}\colon X \to ?^{A^*}$$

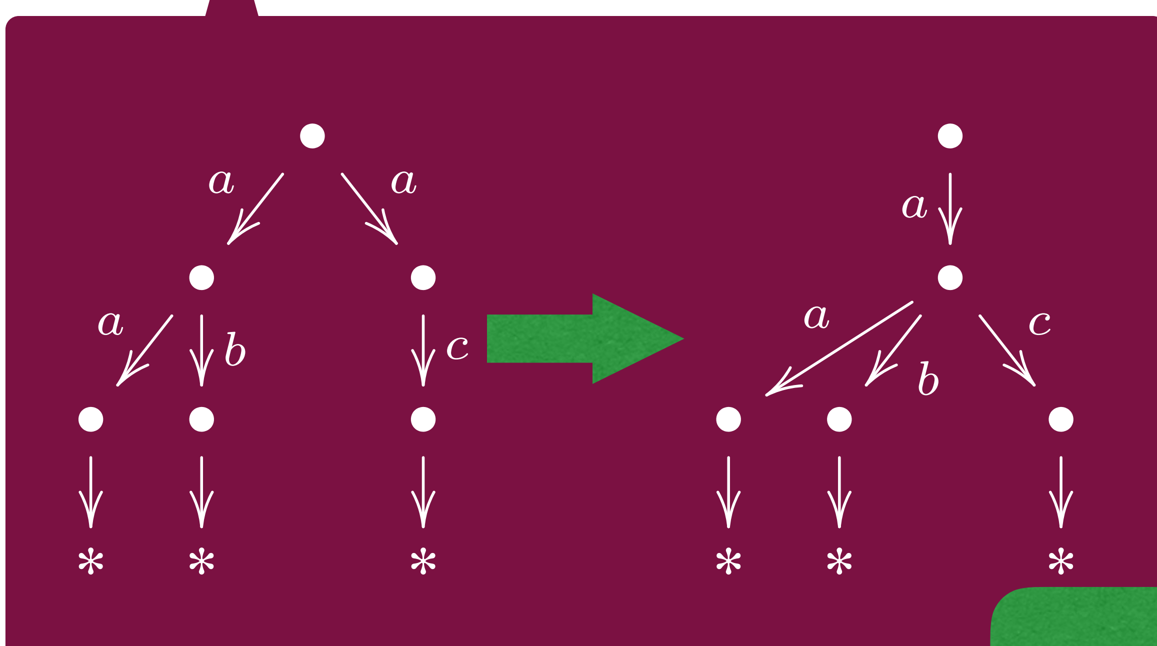
# Trace semantics coalgebraically?

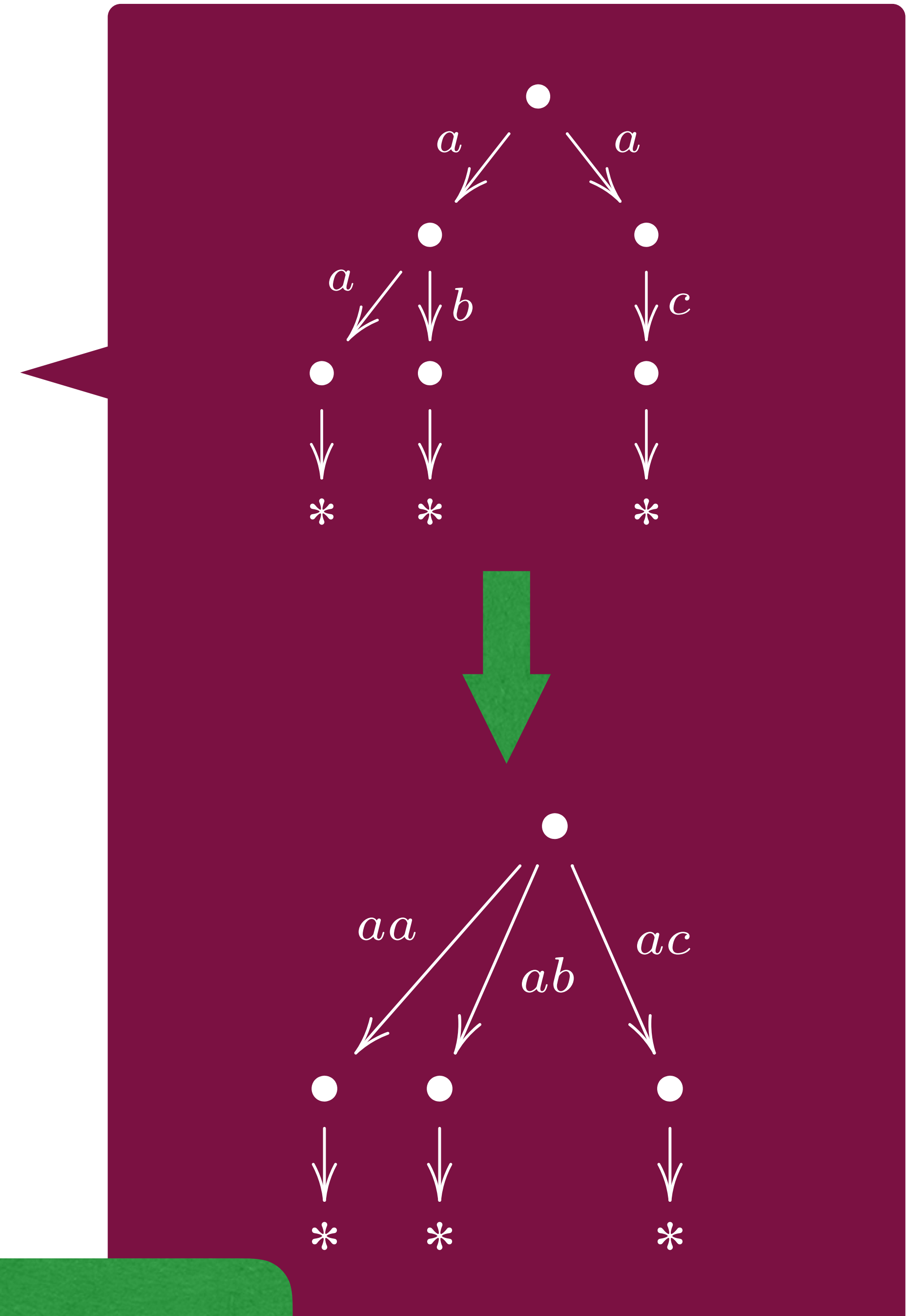
NFA / LTS

Two ideas:

(1) unfold branching + transitions on words

(2) trace = bisimilarity after determinisation




monads !

# Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

Hasuo, Jacobs, S. LMCS '07

(2) modelling in an Eilenberg-Moore category

Silva, Bonchi, Bonsangue, Rutten FSTTCS'10

algebras of a monad M

Jacobs, Silva, S. JCSS'15

(1) and (2) are related

# Traces via determinisation

**Automaton with M-effects**

$X \to O \times (MX)^A$

O has to be an M-algebra !

**Determinisation**

$MX \to O \times (MX)^A$

MX carries the free M-algebra

trace = bisimilarity after determinisation

Algebras for M

ideally we have a presentation

$\mathrm{tr}\colon X \to O^{A^*}$

Eilenberg-Moore algebras

$$\mathrm{tr}(x)(a_1 a_2 \dots a_n) = o \;\Leftrightarrow\; x \xrightarrow{a_1} t_1 \xrightarrow{a_2} t_2 \cdots t_{n-1} \xrightarrow{a_n} t_n \;\wedge\; t_n{\downarrow}o$$

# Eilenberg-Moore Algebras

abstractly

$\mathcal{EM}(M)$

- objects

$$\begin{array}{c} MA \\ \downarrow a \\ A \end{array}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal] & MA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} MMA \arrow[r, "\mu"] \arrow[d, "Ma"'] & MA \arrow[d, "a"] \\ MA \arrow[r, "a"] & A \end{tikzcd}$$

- morphisms

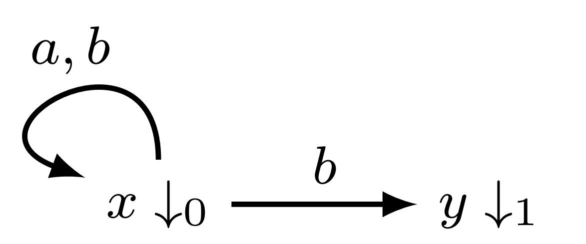
$$\begin{array}{c} MA \\ \downarrow a \\ A \end{array} \xrightarrow{h} \begin{array}{c} MB \\ \downarrow b \\ B \end{array}$$

$$\begin{tikzcd} MA \arrow[r, "Mh"] \arrow[d, "a"'] & MB \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

# Traces via determinisation

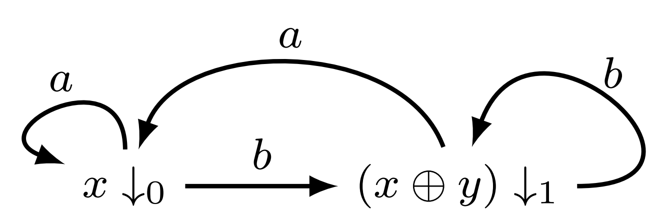
## NFA

$X \to 2 \times (\mathcal{P}X)^A$




## DFA

$\mathcal{P}X \to 2 \times (\mathcal{P}X)^A$




$$\frac{x \xrightarrow{a} t_x,\; y \xrightarrow{a} t_y}{x \oplus y \xrightarrow{a} t_x \oplus t_y}$$

$$\frac{x{\downarrow}o_x,\; y{\downarrow}o_y}{x \oplus y \downarrow o_x \oplus o_y}$$

Algebras for $\mathcal{P}$

finite powerset !

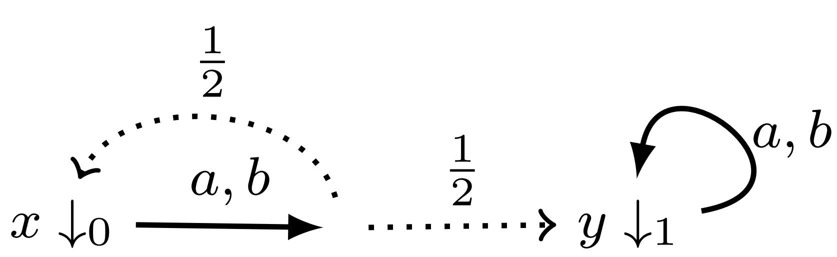
join semilattices with bottom

$2 = \mathcal{P}1$

# Traces via determinisation

## Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1} X)^A$




## DPA

$\mathcal{D}_{\leq 1} X \to [0,1] \times (\mathcal{D}_{\leq 1} X)^A$

$$x \downarrow_0$$
$$\Big\downarrow a, b$$
$$(x +_{\frac{1}{2}} y) \downarrow_{\frac{1}{2}}$$
$$\Big\downarrow a, b$$
$$(x +_{\frac{1}{4}} y) \downarrow_{\frac{3}{4}}$$
$$\Big\downarrow a, b$$
$$\vdots$$

Algebras for $\mathcal{D}_{\leq 1}$

positive convex algebras

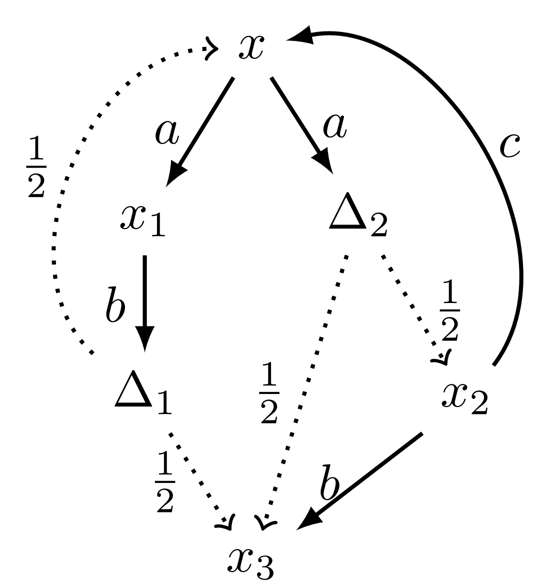
finitely supported subdistributions!

$[0,1] = \mathcal{D}_{\leq 1} 1$

# Traces via determinisation

## Simple NPA

$X \to ? \times (\mathcal{C}X)^A$




## DNPA

$\mathcal{C}X \to ? \times (\mathcal{C}X)^A$

$$x_1 \xrightarrow{a} x_1 \oplus (x_3 +_{\frac{1}{2}} x_2)$$

$? = \mathcal{C}1$

Algebras for $C$

convex semilattices

finitely generated convex sets of distr…

# Presentation for $\mathscr{C}$

Algebras for $\mathscr{C}$

convex semilattices

finitely generated convex sets of distr…

Bonchi, S., Vignudelli ‘19

$$\mathbb{A} = (A, \oplus, +_p)$$

$p \in (0,1)$

$$(x \oplus y) \oplus z \stackrel{(A)}{=} x \oplus (y \oplus z)$$
$$x \oplus y \stackrel{(C)}{=} y \oplus x$$
$$x \oplus x \stackrel{(I)}{=} x$$

semilattice

S., Woracek ’15, ’17, ‘18

$$(x +_q y) +_p z \stackrel{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z)$$
$$x +_p y \stackrel{(C_p)}{=} y +_{1-p} x$$
$$x +_p x \stackrel{(I_p)}{=} x$$

convex algebra

$$(x \oplus y) +_p z \stackrel{(D)}{=} (x +_p z) \oplus (y +_p z)$$

distributivity

# Three variants for “$\mathcal{C}$”

Algebras for “$\mathcal{C}$”

nonempty f.g. convex subsets of **sub**distr…

Bonchi, S., Vignudelli ‘19

We explore the whole space and
prove coincidence with “local” trace semantics

I.pointed convex semilattices

Intervals in [0,1] with min-max, Minkowski, and [0,0] = “$\mathcal{C}$”1

II. with bottom

[0,1] with max, $+_p$ and 0 = “$\mathcal{C}$”1

III. with top

[0,1] with min, $+_p$ and 0 = “$\mathcal{C}$”1

Many general properties
follow
also a sound
up-to context
proof technique

Three things to take home:

**1.** Semantics via determinisation
is easy for automata with M-effects

**2.** Having a presentation for M gives us syntax

**3.** Having the syntax makes determinisation natural !

combining
nondeterminism
and probability
becomes easy

Thank You !