# Semantics of Probabilistic Automata via Coalgebra

Ana Sokolova

UNIVERSITY of SALZBURG

Panhellenic Logic Symposium 12
29.6.19

# Semantics of Probabilistic Automata via Coalgebra

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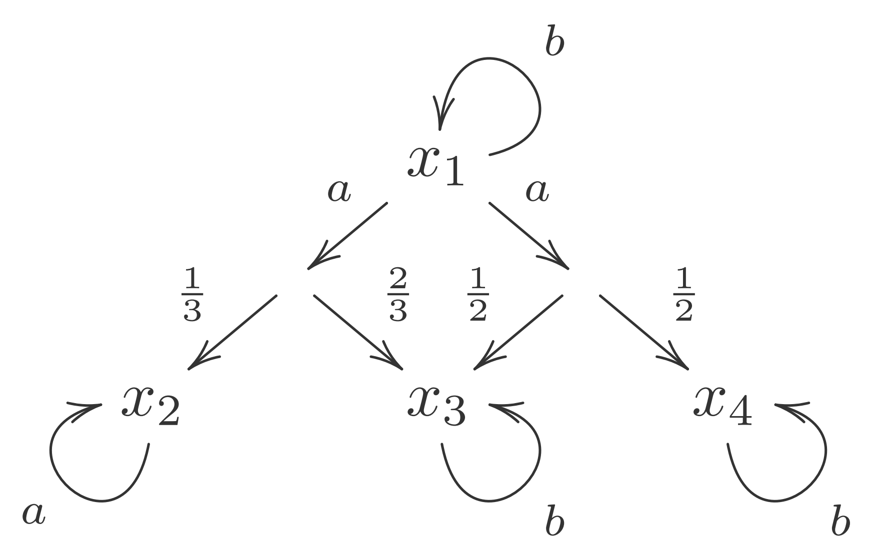
probabilistic automata

# The different natures of PA

probabilistic automata

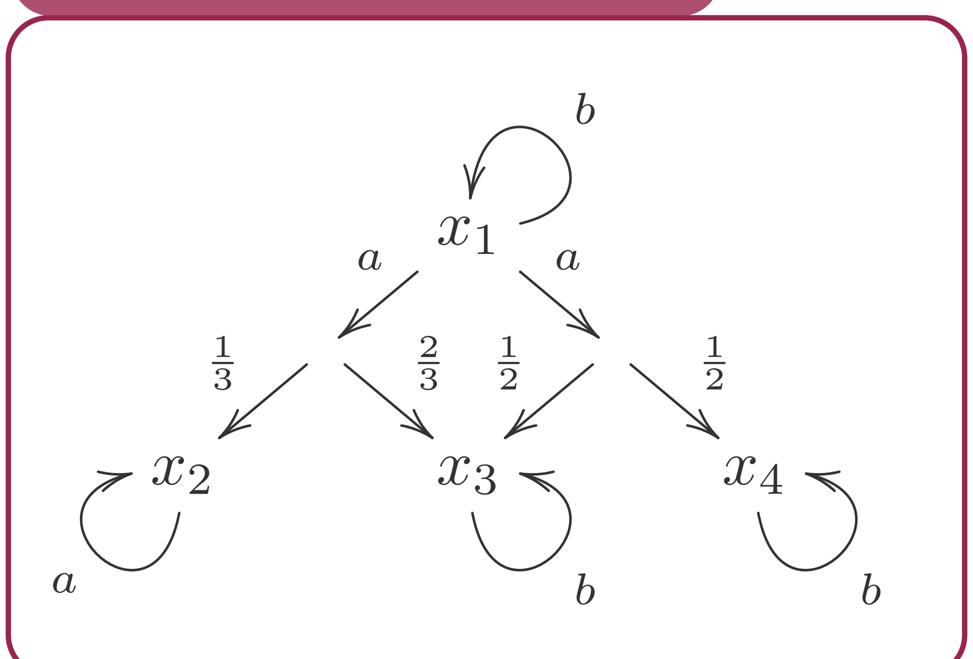
# The different natures of PA



probabilistic automata

# The different natures of PA

$$X \xrightarrow{c} (\mathcal{P}\mathcal{D}X)^A$$



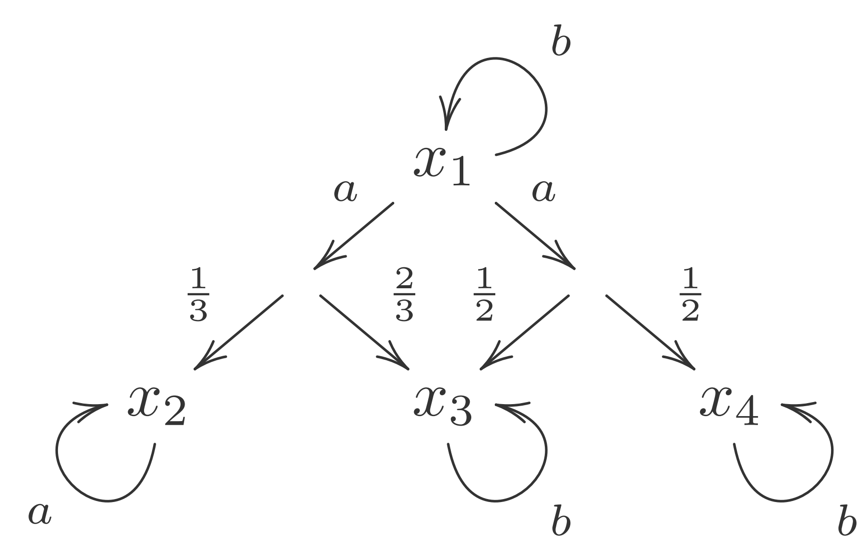
# The different natures of PA

probabilistic automata

$X \xrightarrow{c} (\mathcal{PD}X)^A$

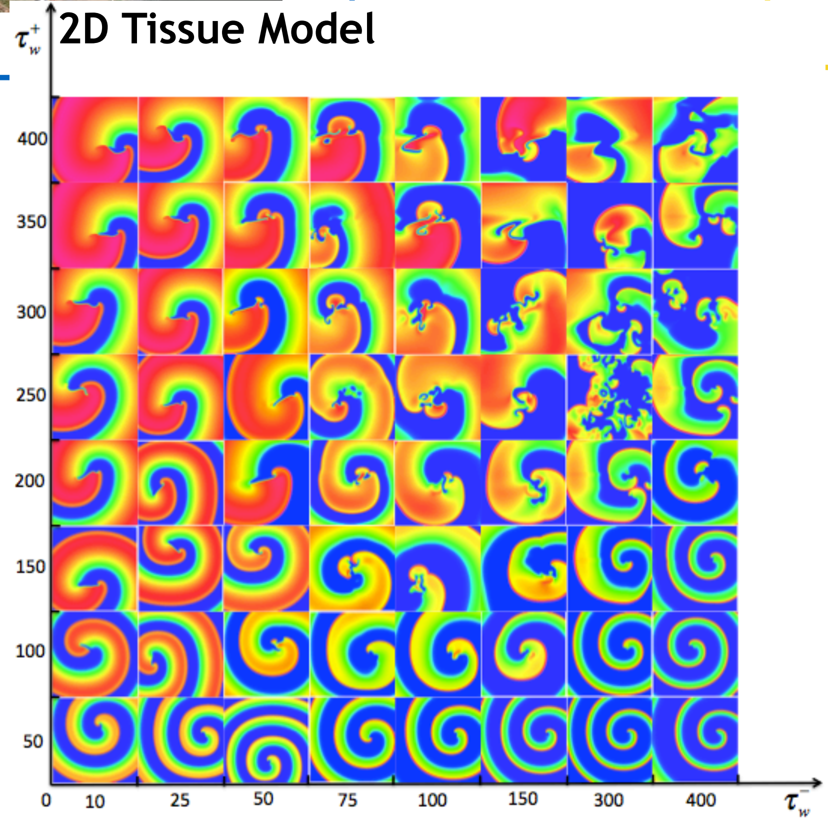
we write $s \xrightarrow{a} \mu$

for $\mu \in c(s)(a)$

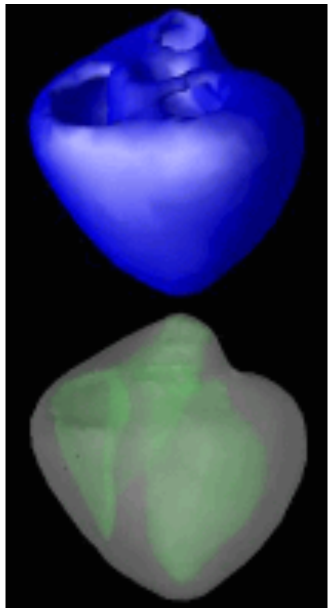


# Where do they appear ?


2D Tissue Model




3D Organ Model



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Model of a human heart Bartocci et al. TCS09, CAV11

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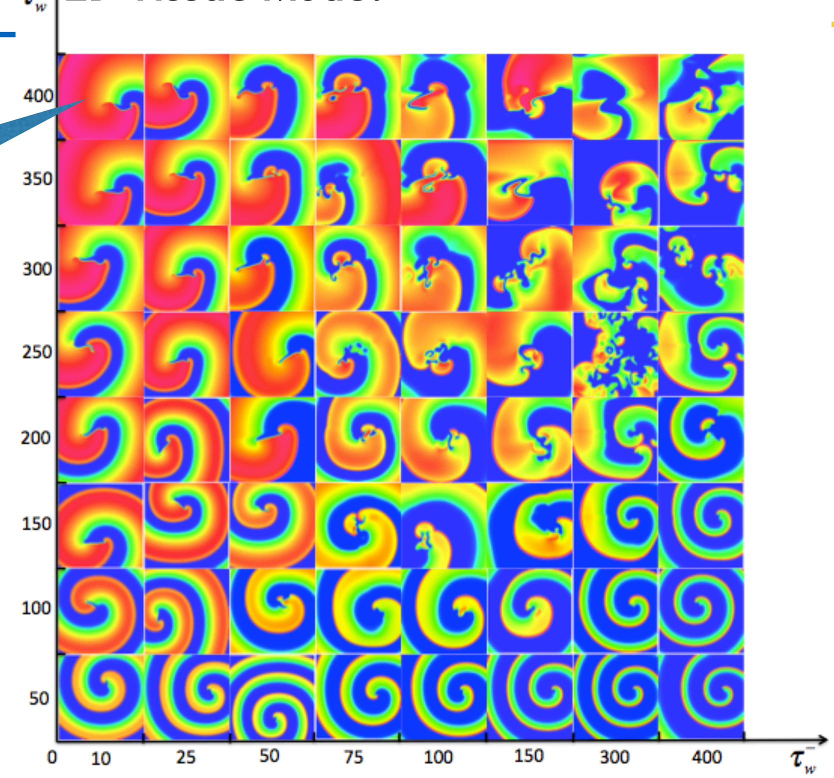


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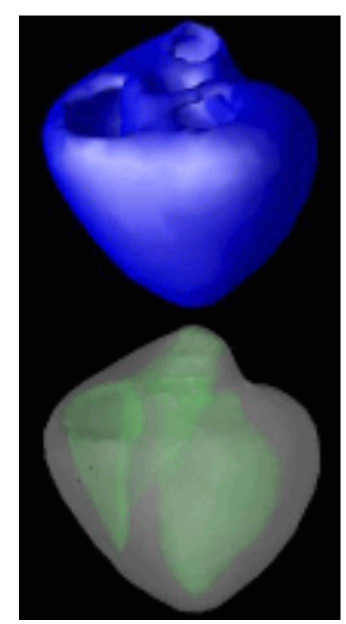

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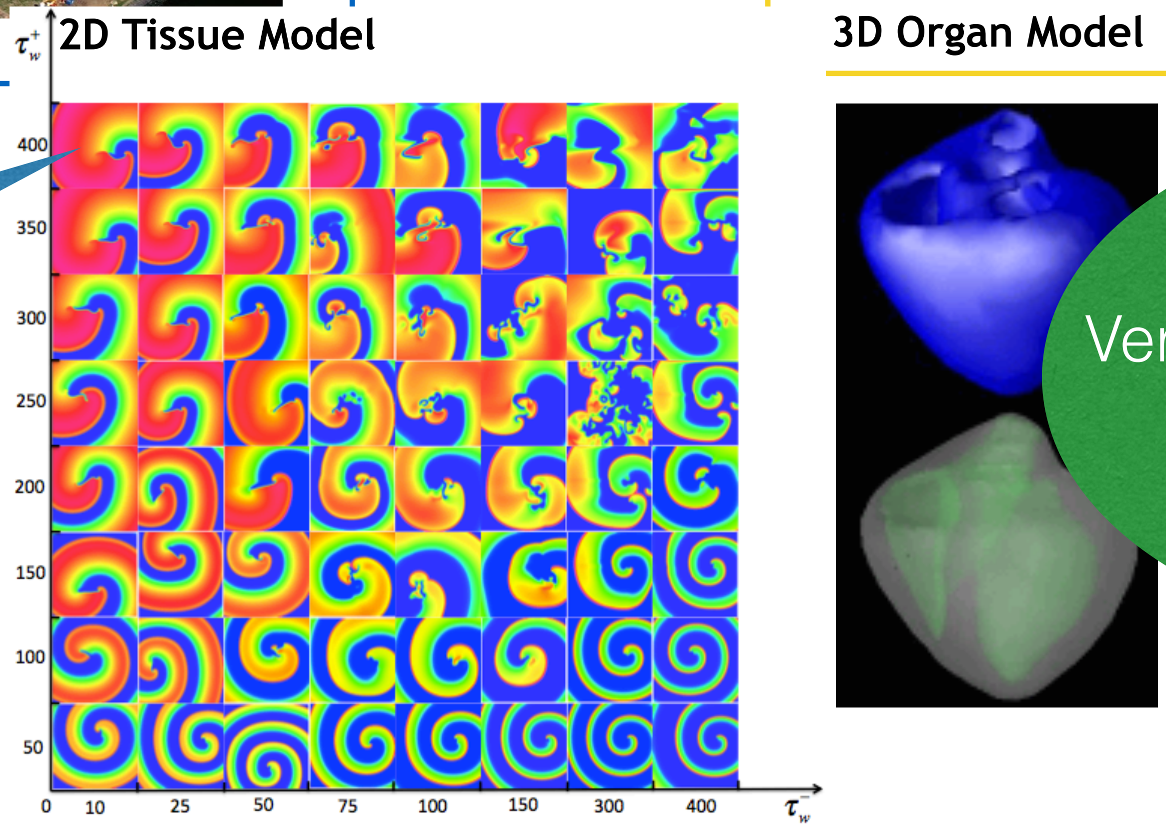
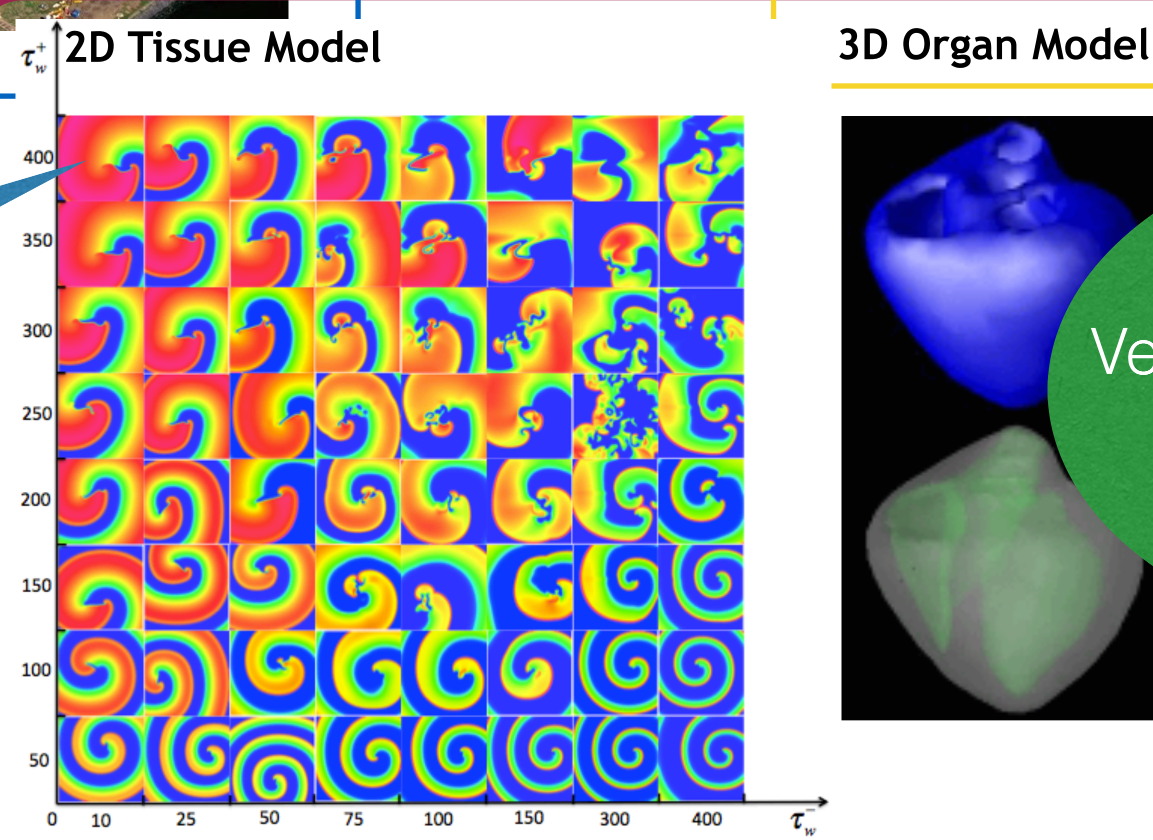



Verification requires clear semantics

# Where do they appear ?


Model of a human heart Bartocci et al. TCS09, CAV11

Verification requires clear semantics

and suffers from state-space explosion

# Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity
4. Trace semantics

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[Bonchi,Silva, S. CONCUR'17]

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# Behavioural Equivalences

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Two states are equivalent iff they admit the same traces (words).

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trace equivalence

An equivalence relation $R \subseteq X \times X$ is a bisimulation of the LTS $X \to (\mathcal{P}X)^A$ iff whenever $(x, y) \in R$ for all $a \in A$

$$x \xrightarrow{a} x' \quad \Rightarrow \quad \exists y'. y \xrightarrow{a} y' \wedge (x', y') \in R.$$

Bisimilarity, denoted by $\sim$, is the largest bisimulation.

bisimilarity

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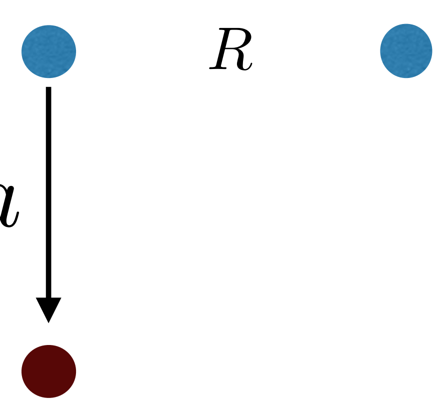


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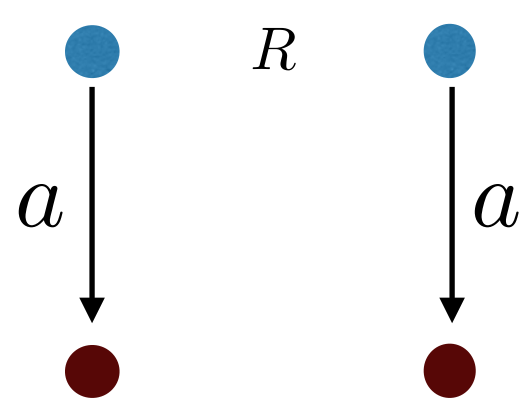


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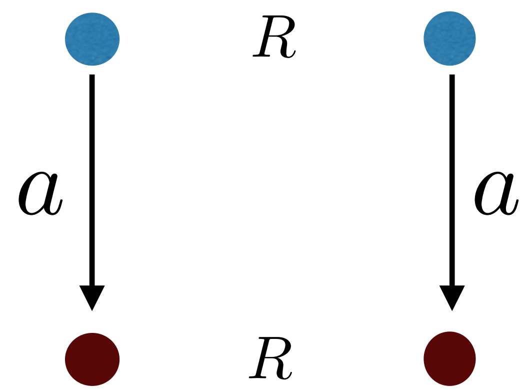


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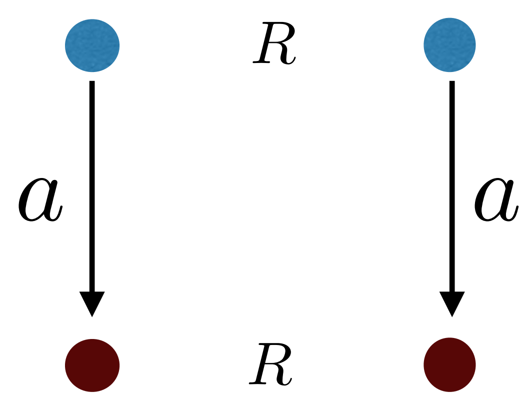


largest bisimulation

bisimilarity

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[Bonchi,Silva, S. CONCUR'17]

trace and testing theory

[Bonchi, S., Vignudelli LICS'19]

# Bisimilarity

An equivalence relation $R$ on the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ is a bisimulation iff whenever $(s,t) \in R$ for all $a \in A$ and $\mu \in \mathcal{D}X$

$$s \xrightarrow{a} \mu \Longrightarrow \exists \nu \in \mathcal{D}X.\ t \xrightarrow{a} \nu \wedge \mu \equiv_R \nu$$

where $\mu \equiv_R \nu$ iff $\mu[C] = \nu[C]$ for all $R$-equivalence classes $C$, with $\mu[C] = \sum_{x \in C} \mu(x)$.

Bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim$, is the largest bisimulation.

# Bisimilarity

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bisimulation

$R$

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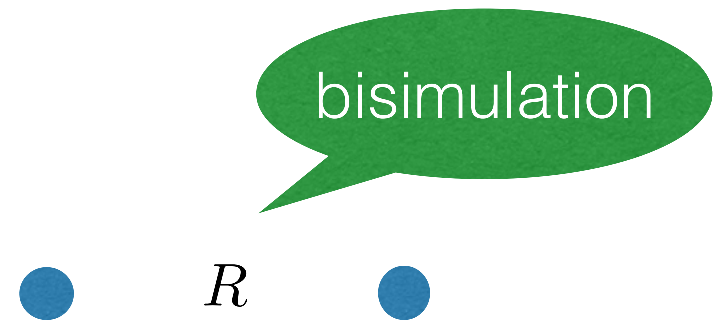

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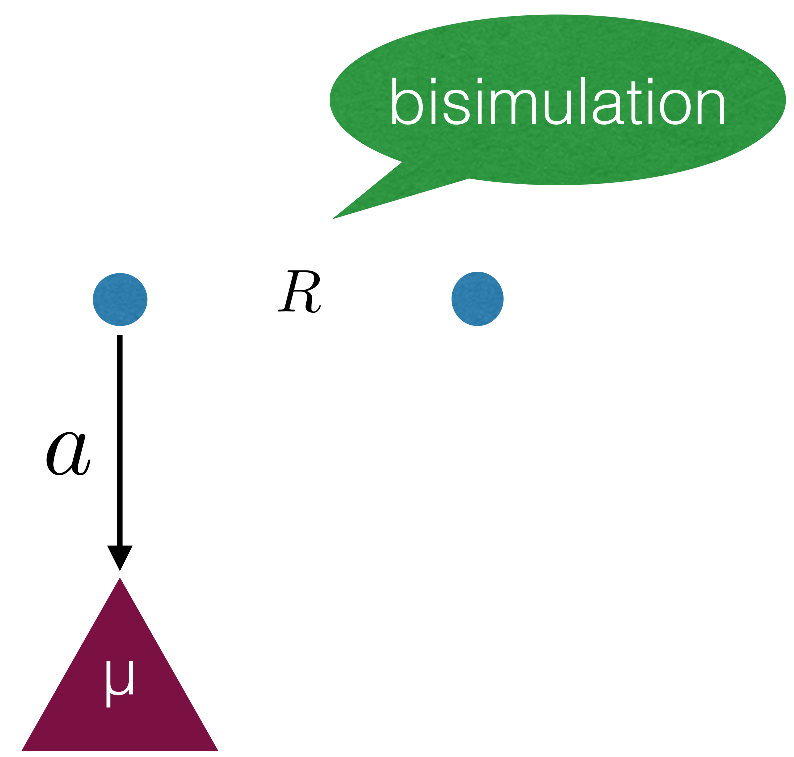

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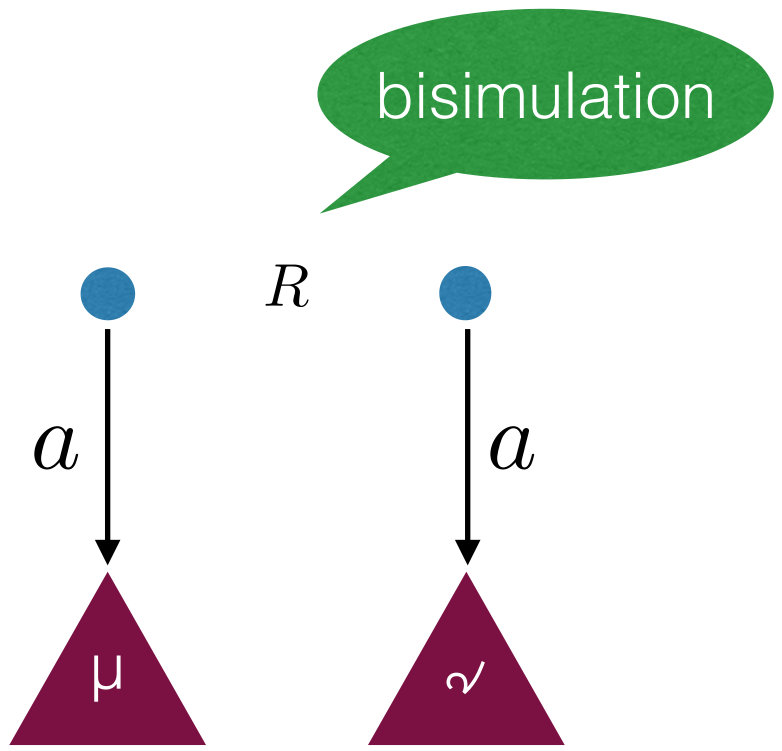

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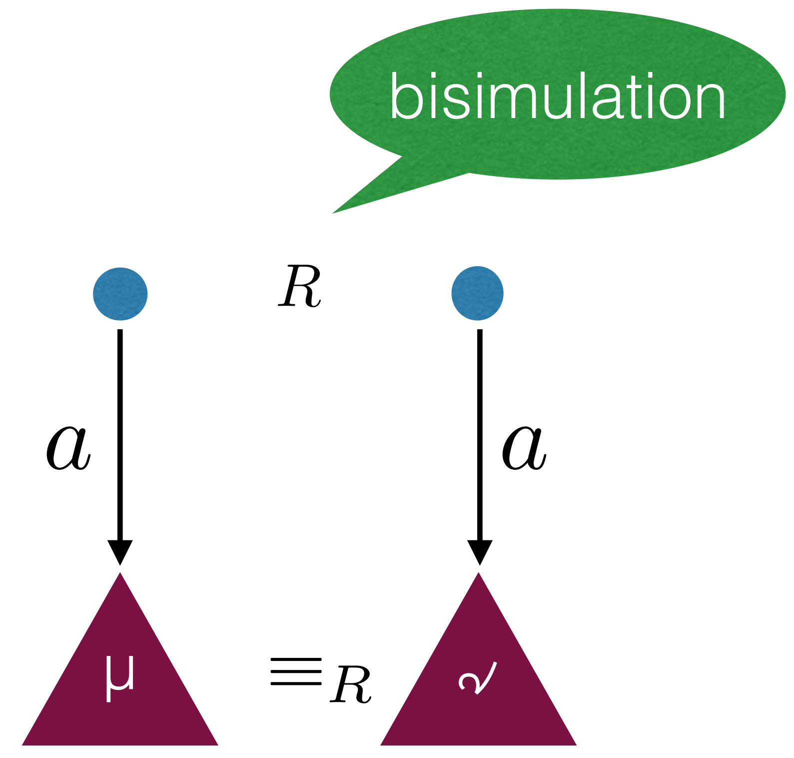

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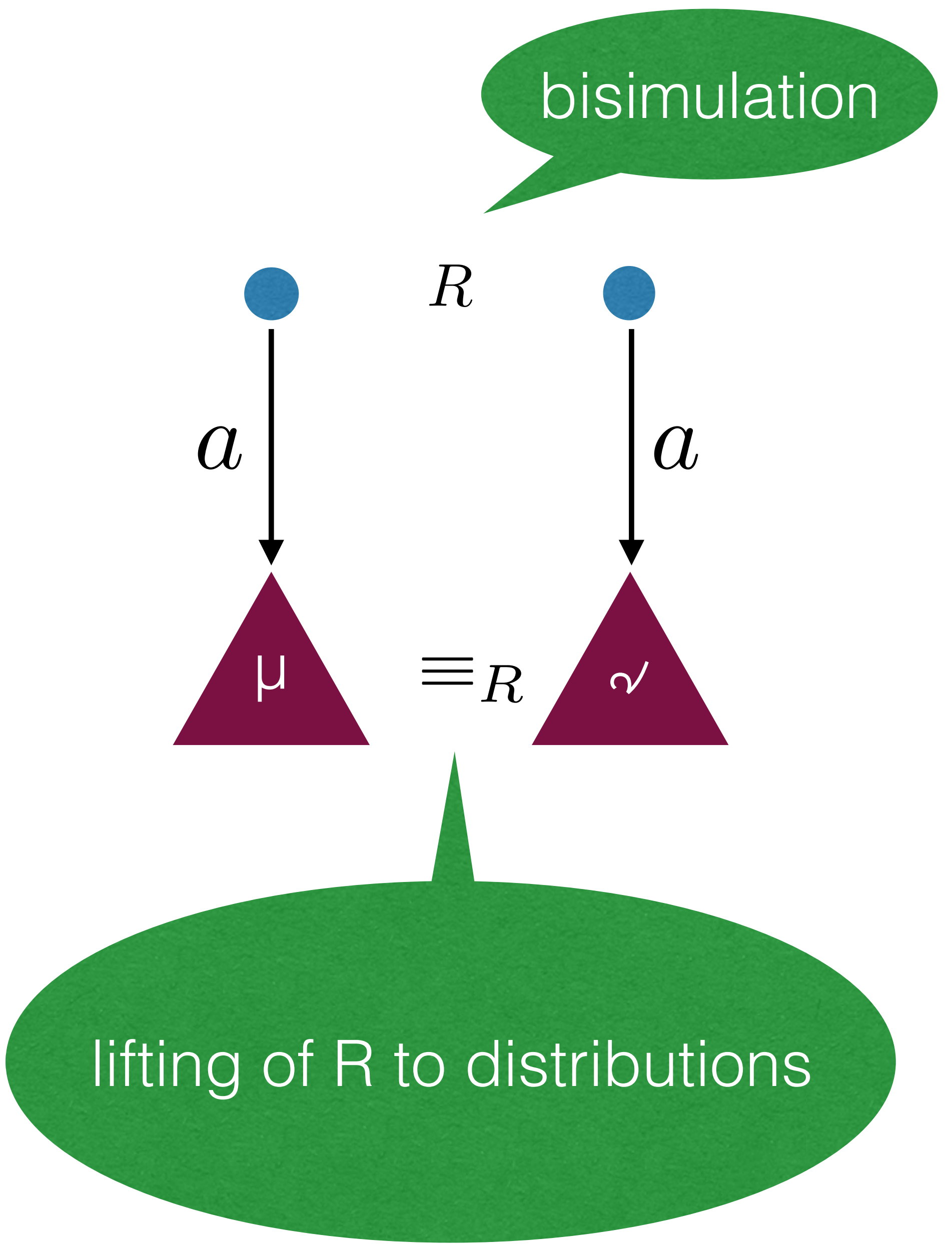

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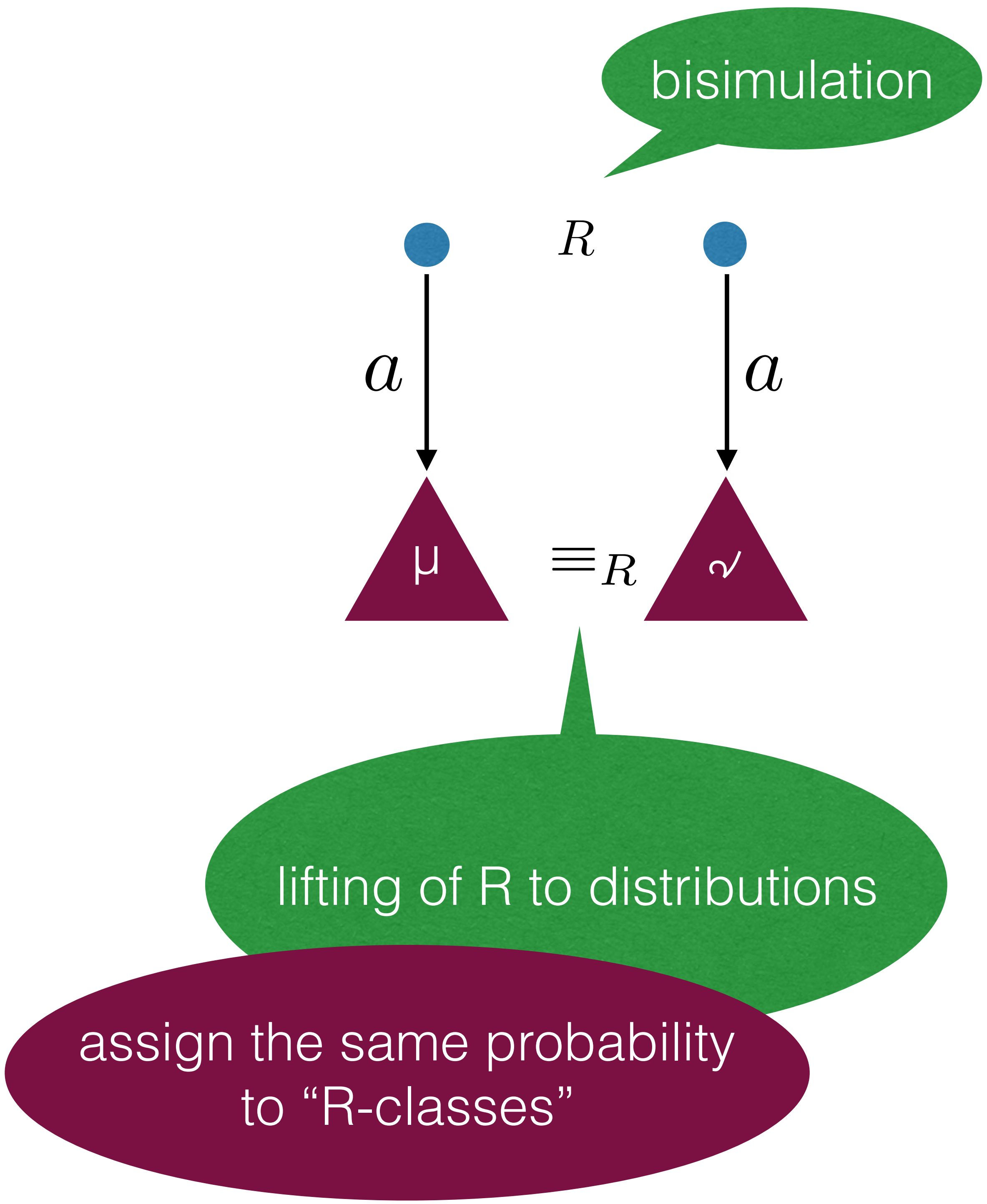
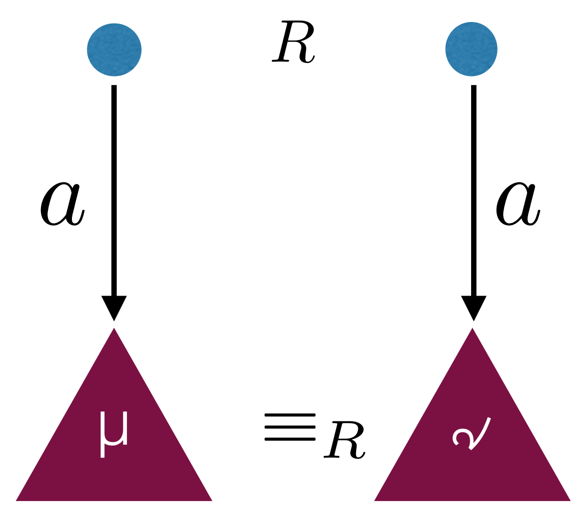


# Bisimilarity

$\sim$ largest bisimulation

bisimulation




lifting of R to distributions

assign the same probability to “R-classes”

# Convex bisimilarity

An equivalence relation $R \subseteq X \times X$ is a convex bisimulation of the PA $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$ iff whenever $(x, y) \in R$, for all $a \in A$ and $\mu \in \mathcal{D}X$

$$x \xrightarrow{a} \mu \quad \Rightarrow \quad \exists \nu.\, \mu \equiv_R \nu \wedge \nu = \sum_{i=1}^{n} p_i \nu_i \wedge y \xrightarrow{a} \nu_i.$$

Convex bisimilarity on $c\colon X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_c$, is the largest bisimulation.

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convex combination

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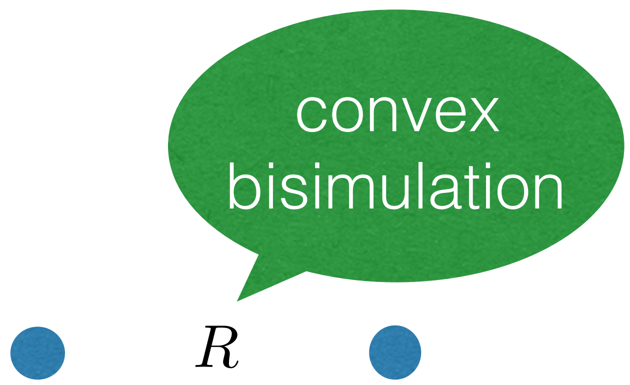
# Convex bisimilarity



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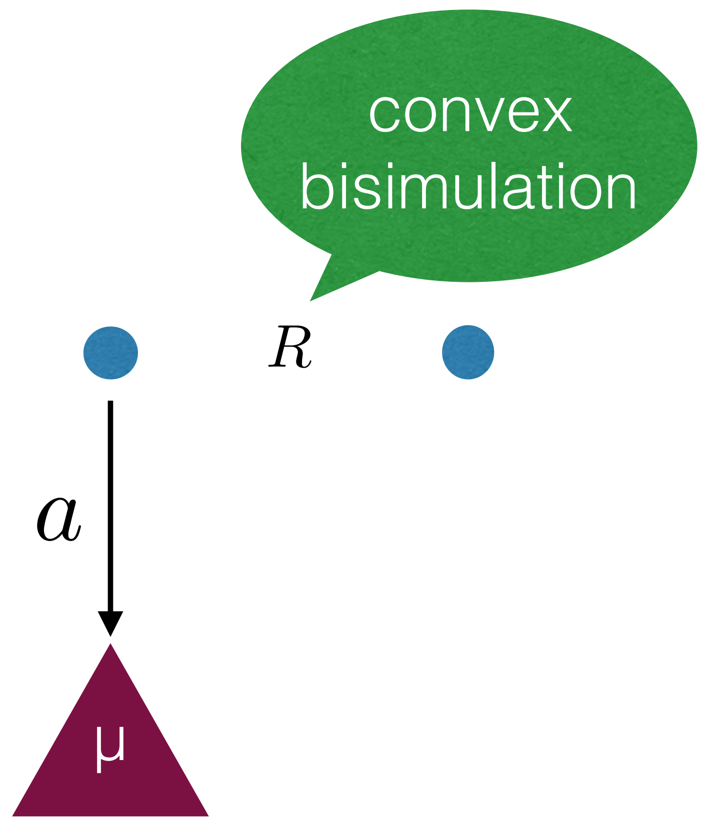

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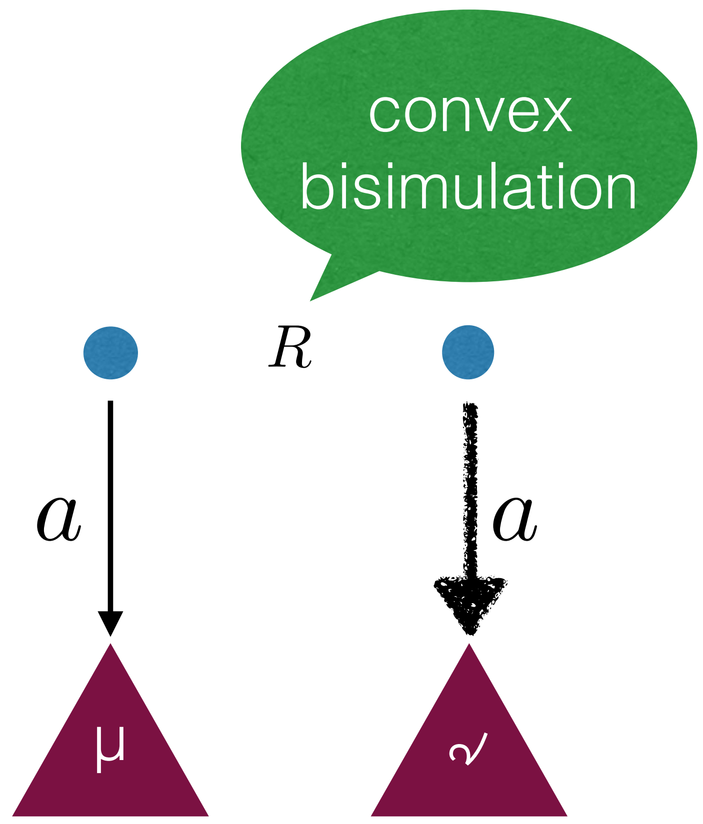

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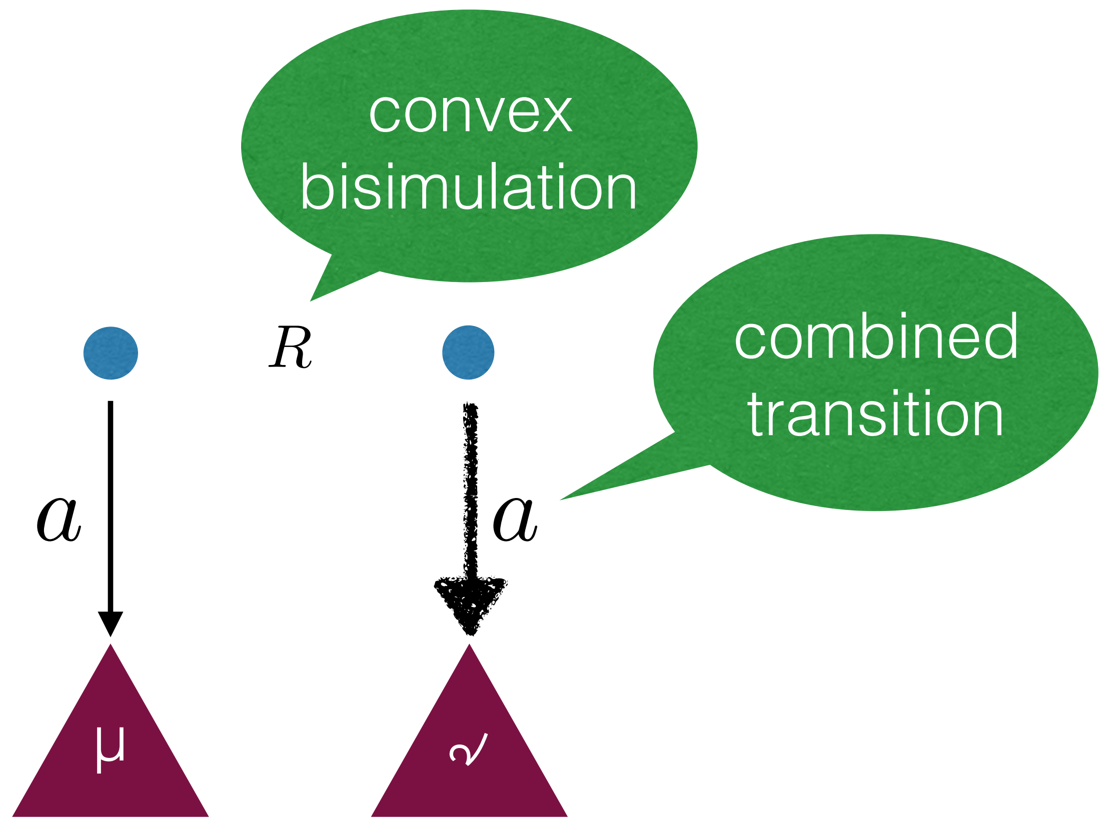

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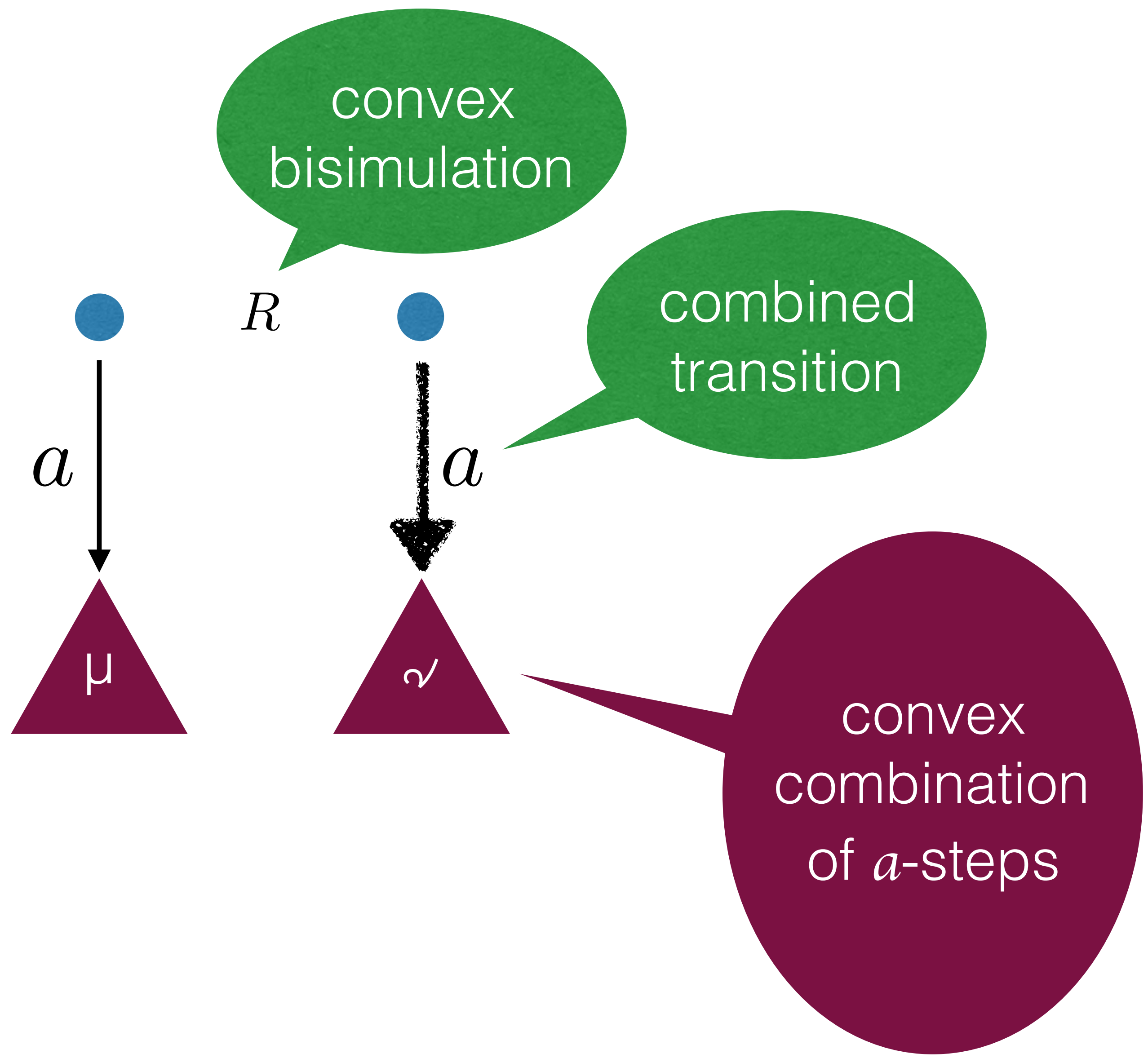

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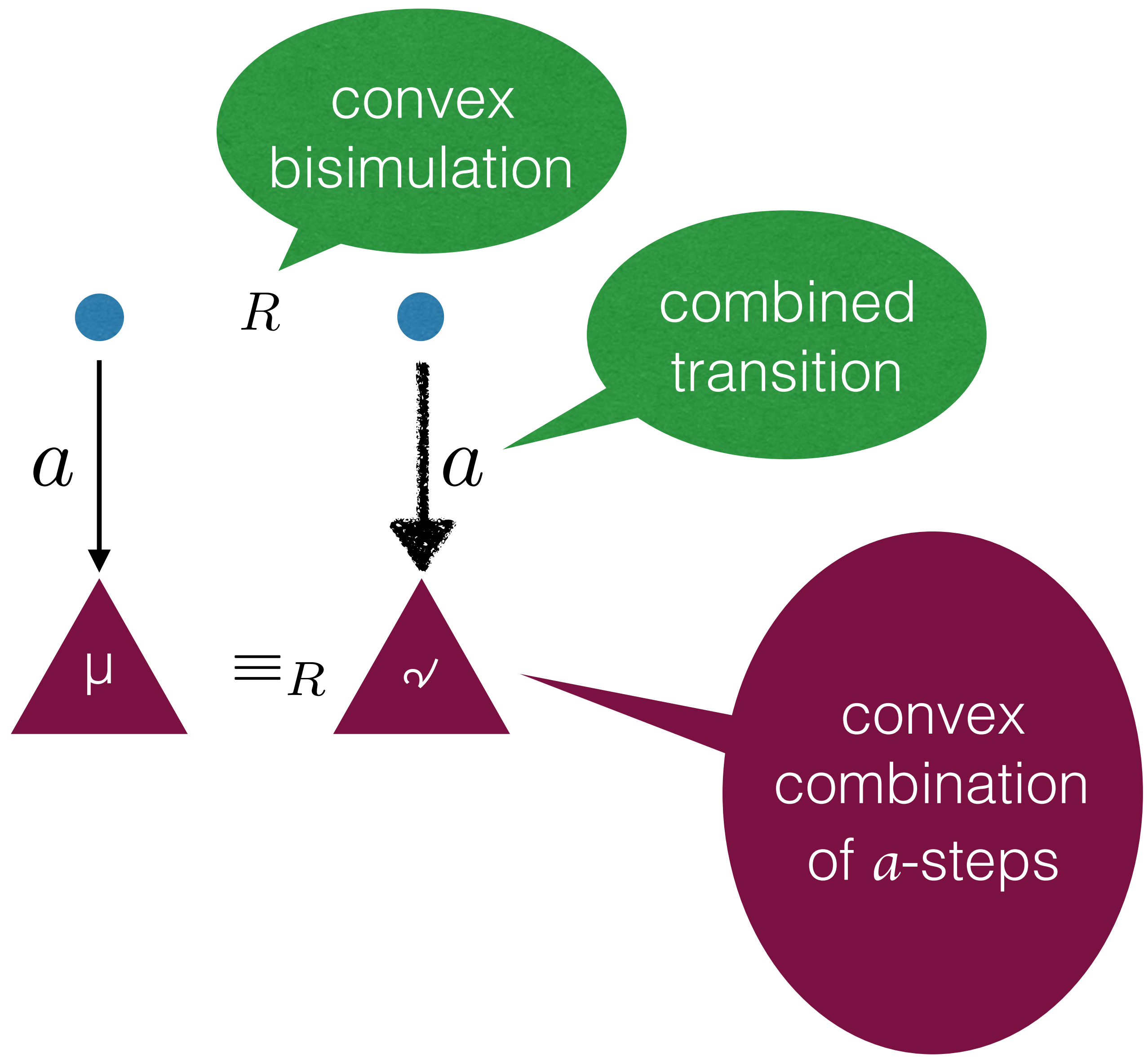

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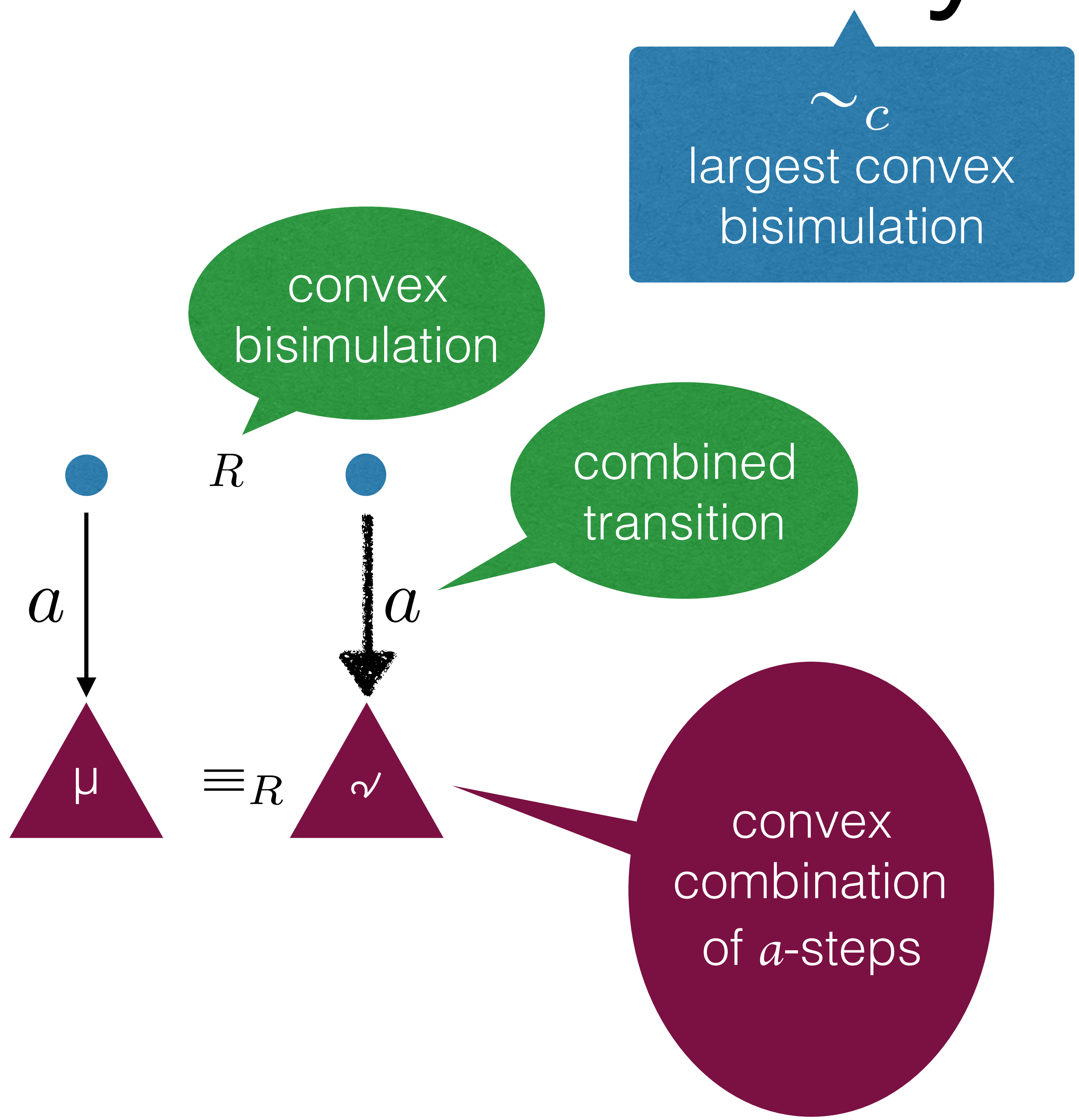

# Distribution bisimilarity

An equivalence relation $R$ on the carrier of the belief-state transformer $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$ is a distribution bisimulation iff whenever $(\mu, \nu) \in R$ for all $a \in A$

$$\mu \xrightarrow{a} \mu' \implies \exists \nu' \in \mathcal{D}X.\ \nu \xrightarrow{a} \nu' \wedge (\mu', \nu') \in R.$$

Distribution bisimilarity on $c\colon \mathcal{D}X \to (\mathcal{P}\mathcal{D}X)^A$, denoted by $\sim_d$, is the largest distribution bisimulation.

# Distribution bisimilarity

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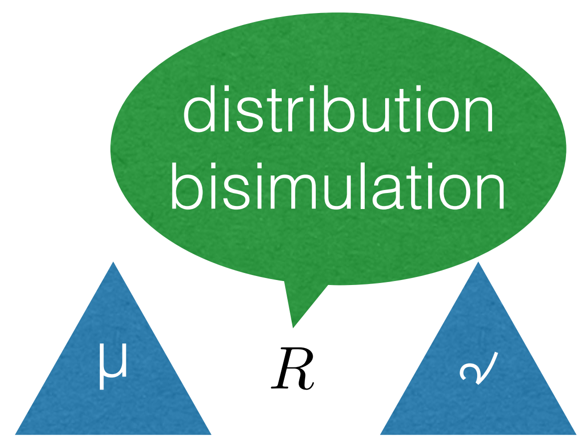
distribution bisimulation

$R$

# Distribution bisimilarity



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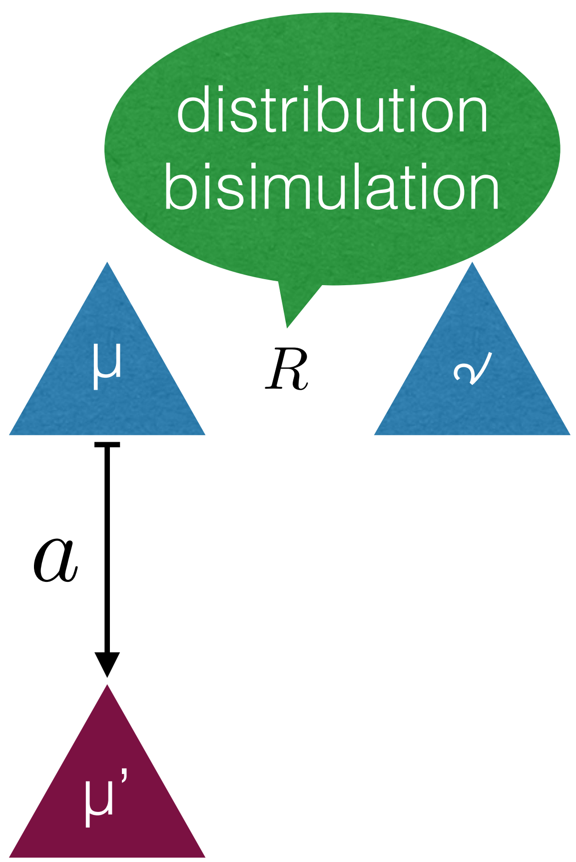

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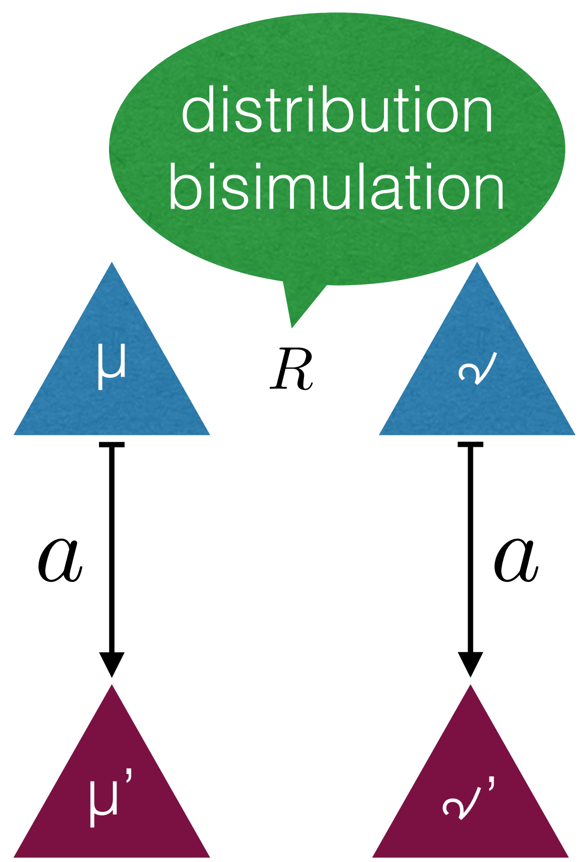

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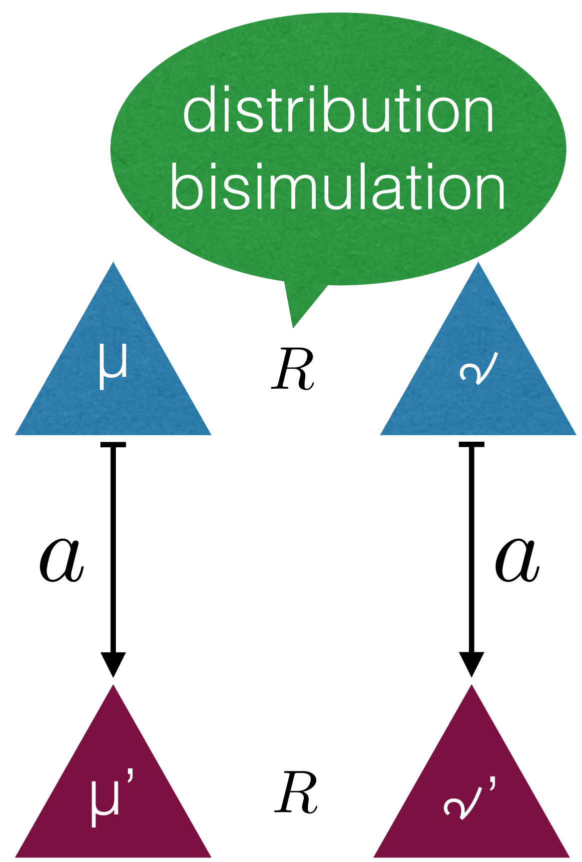

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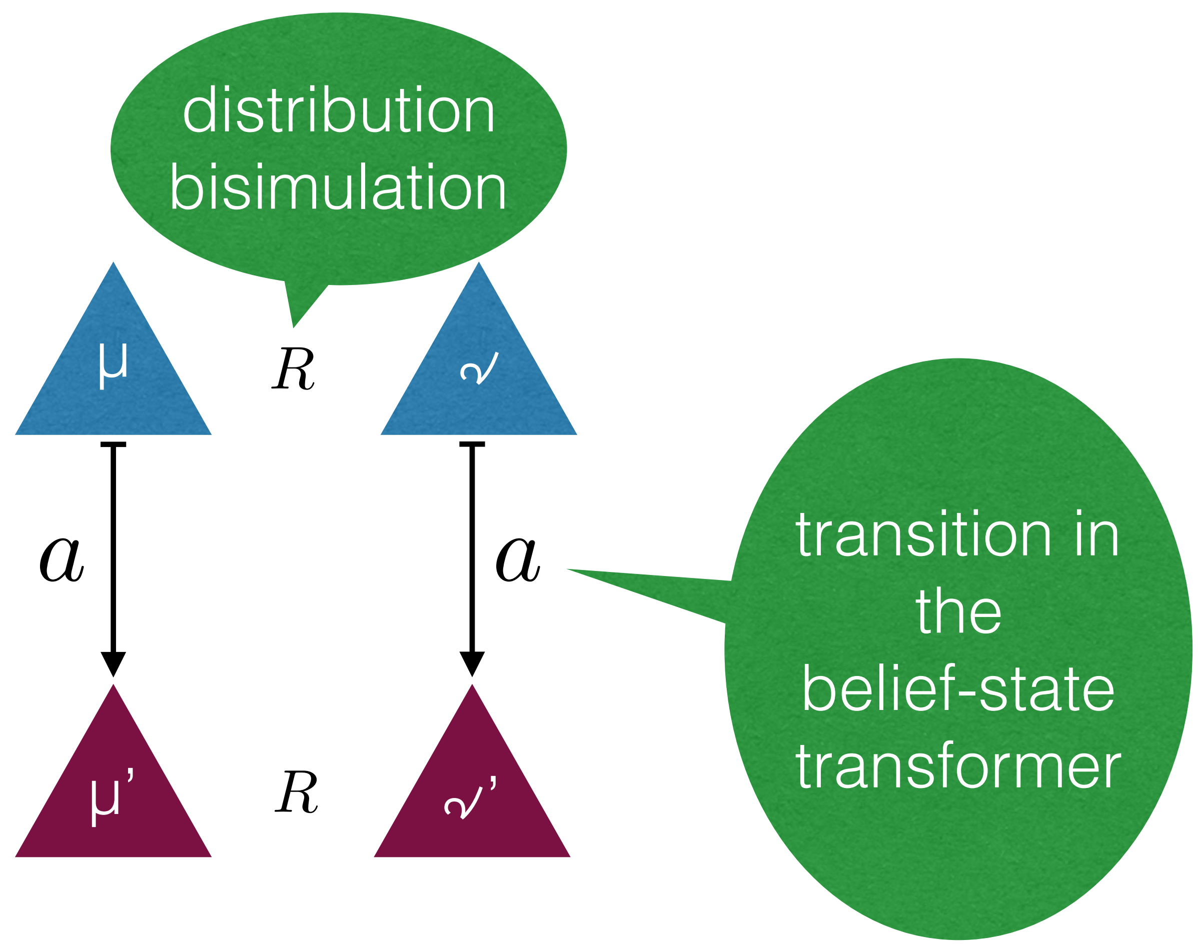

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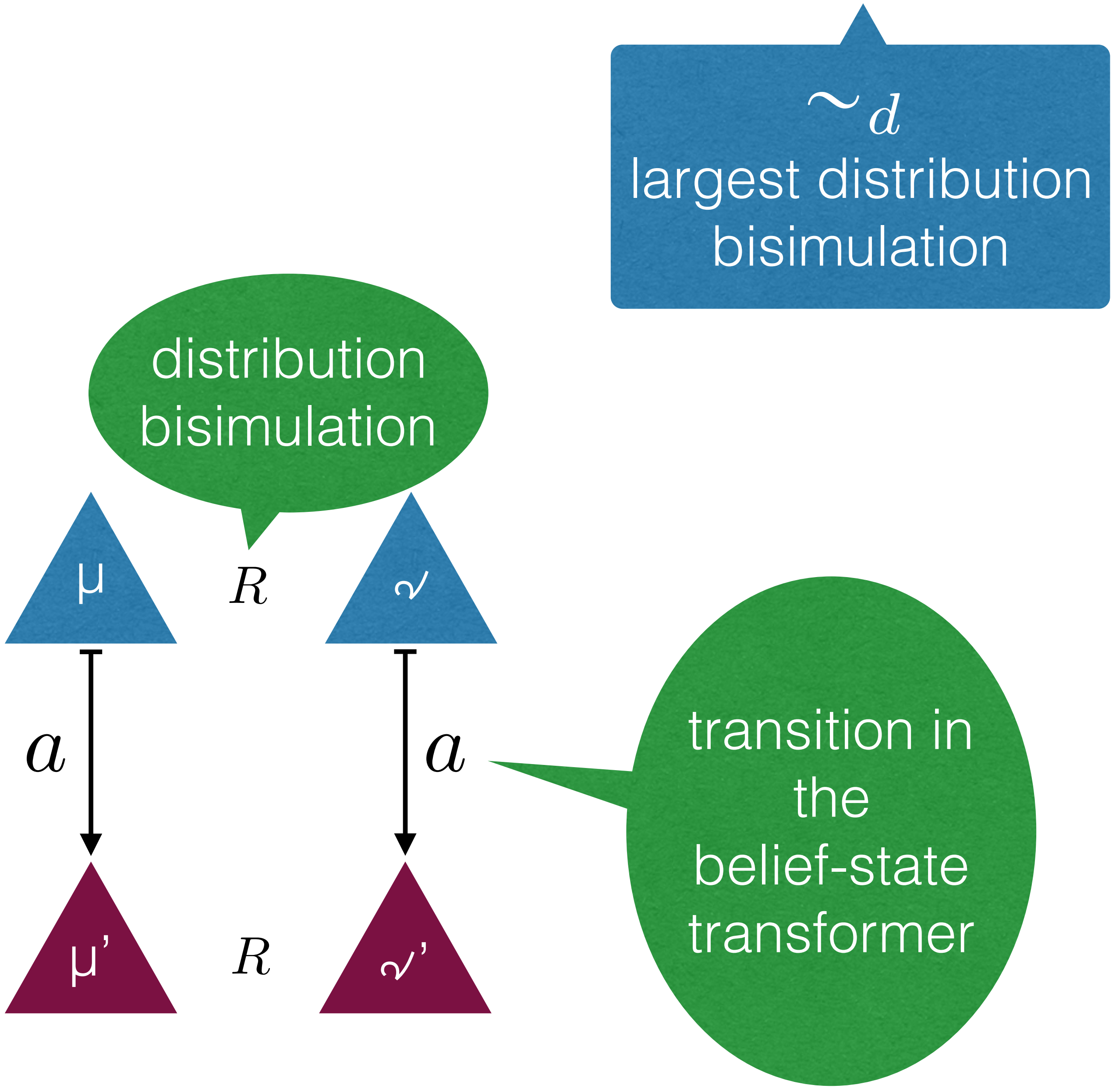
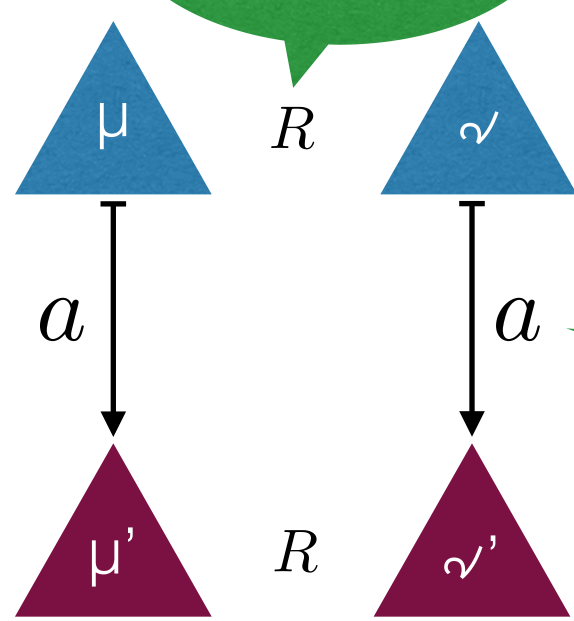


# Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation




transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

# Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

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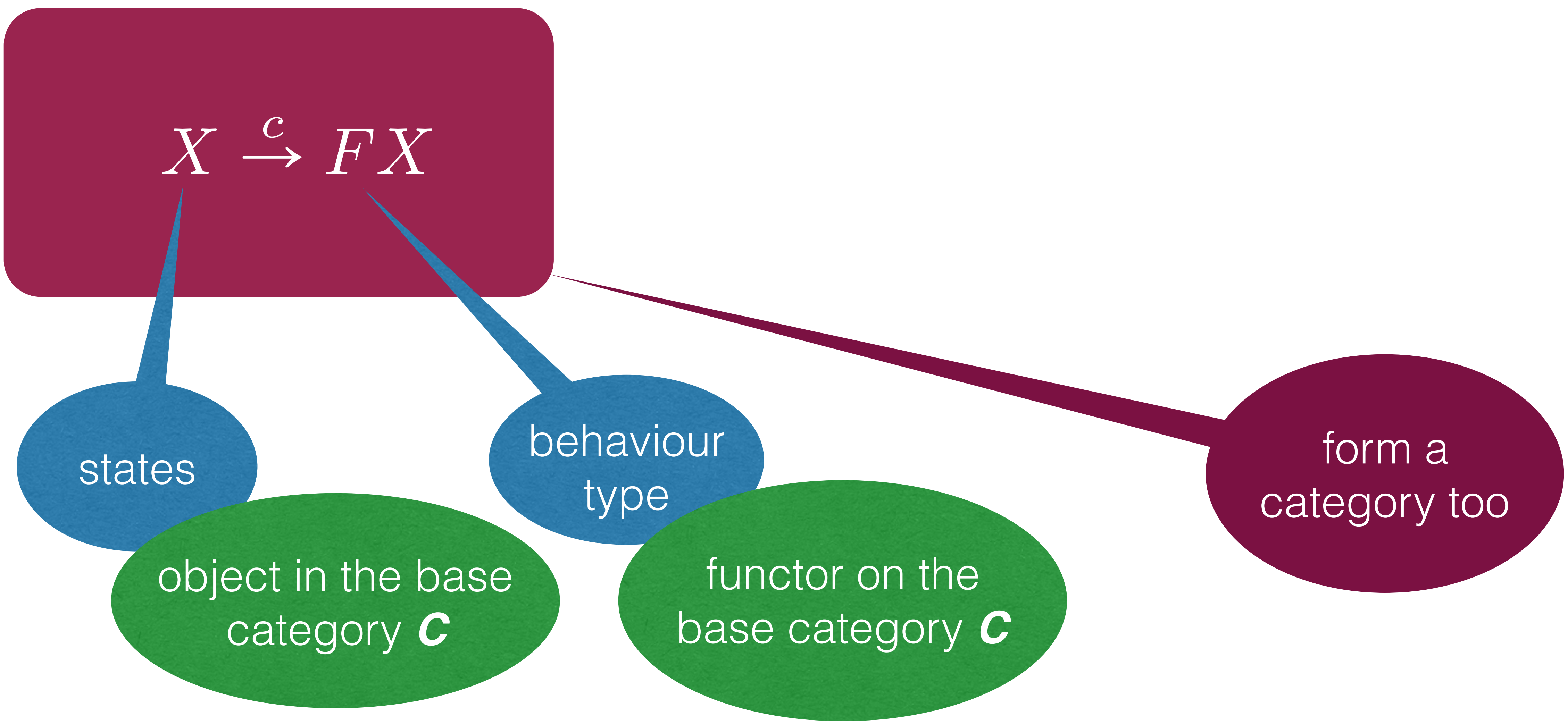

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form a category too

$$\mathbf{CoAlg}_{\mathbf{C}}(F)$$

# Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

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object in the base category ***C***

behaviour type

functor on the base category ***C***

generic notion of behavioural equivalence

$\approx$

form a category too

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

# The category of F-coalgebras

$$\boxed{\mathbf{CoAlg}_{\mathbf{C}}(F)}$$

Objects = coalgebras

Arrows = coalgebra homomorphisms

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$\mathbf{CoAlg_C}(F)$

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behaviour-preserving maps

Objects = coalgebras

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behaviour-
preserving maps

Objects = coalgebras

Arrows = coalgebra homomorphisms

$$X \xrightarrow{c} FX$$

$$h \colon X \to Y$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ FX \arrow[r, "Fh"'] & FY \end{tikzcd}$$

# The category of F-coalgebras

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

behaviour-preserving maps

Objects = coalgebras

Arrows = coalgebra homomorphisms

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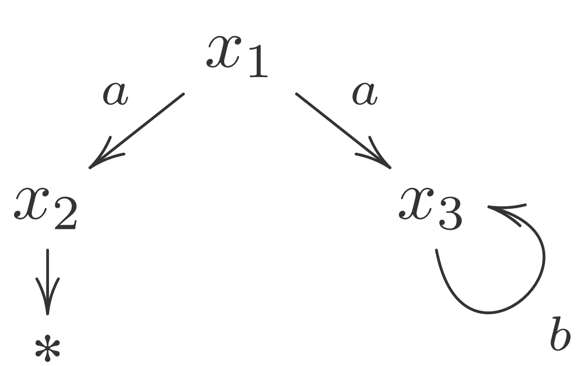
$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ FX \arrow[r, "Fh"'] & FY \end{tikzcd}$$

Two states $x, y \in X$ are behaviourally equivalent, notation $x \approx y$ iff there exists a coalgebra homomorphism $h\colon X \to Y$ from $c\colon X \to FX$ to some coalgebra $d\colon Y \to FY$ such that $h(x) = h(y)$.

# Examples

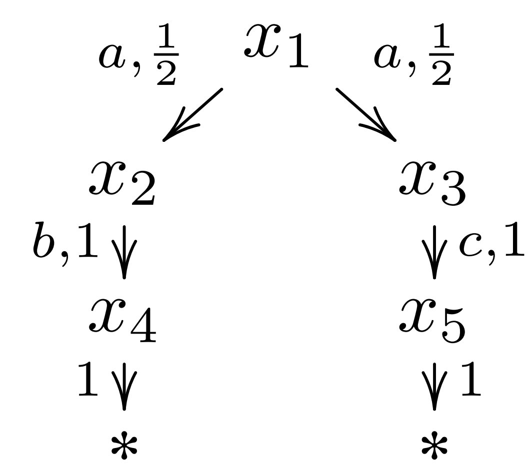
## NFA

$$X \to 2 \times (\mathcal{P}X)^A$$



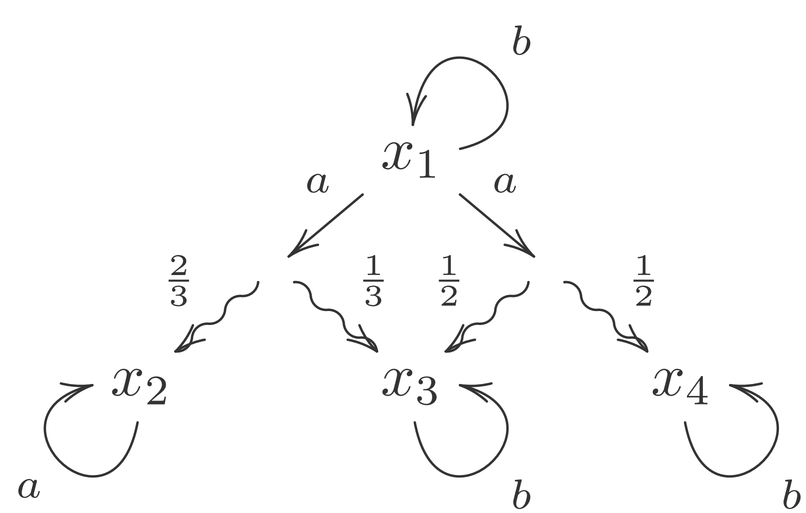

## Generative PTS

$$X \to \mathcal{D}(1 + A \times X)$$




## PA

$$X \to (\mathcal{P}\mathcal{D}X)^A$$



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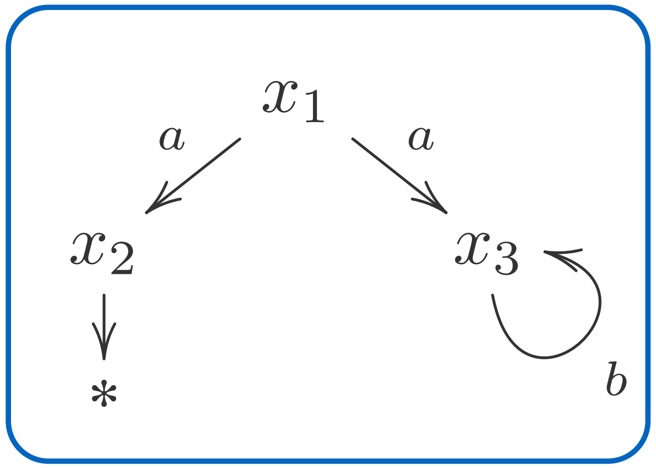


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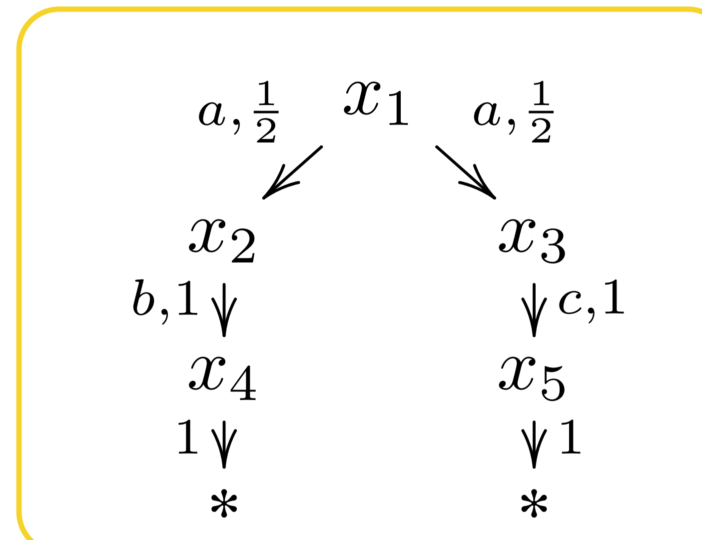
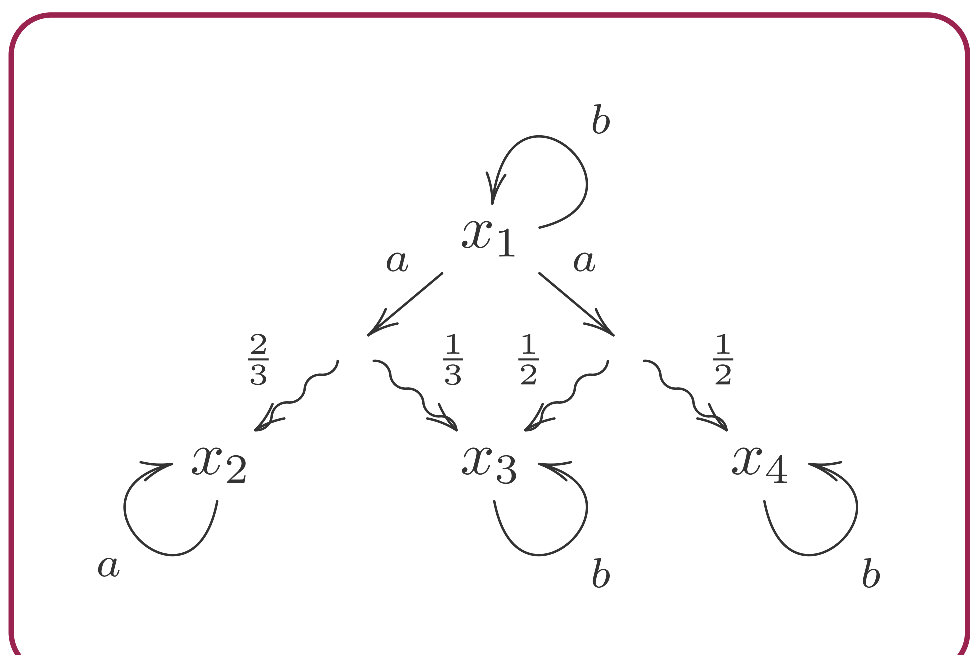



## PA

$X \to (\mathcal{P}\mathcal{D}X)^A$

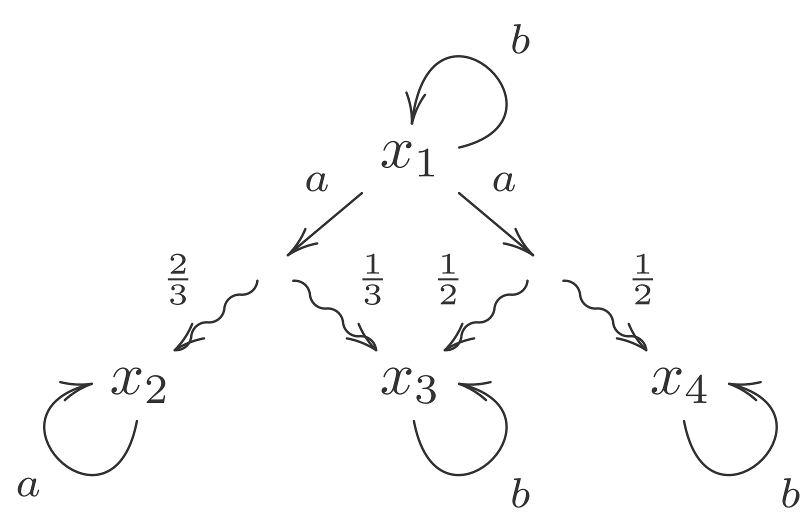



all on
**Sets**

# PA coalgebraically

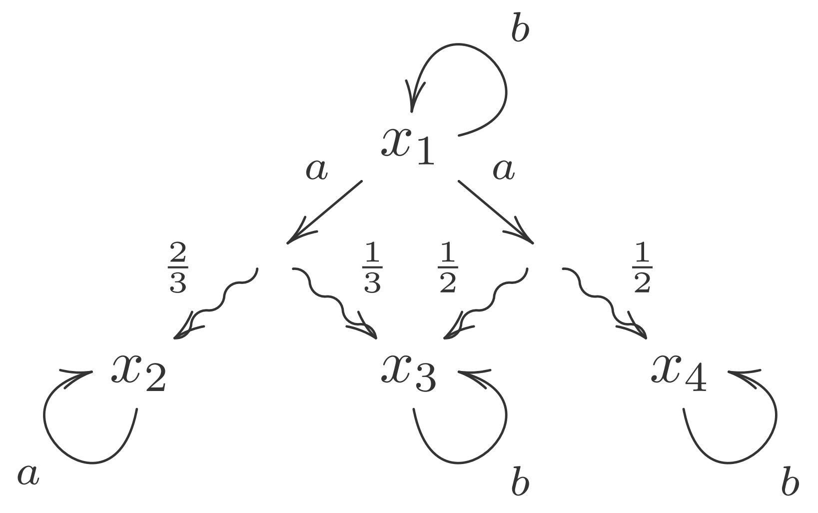
# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



# PA coalgebraically

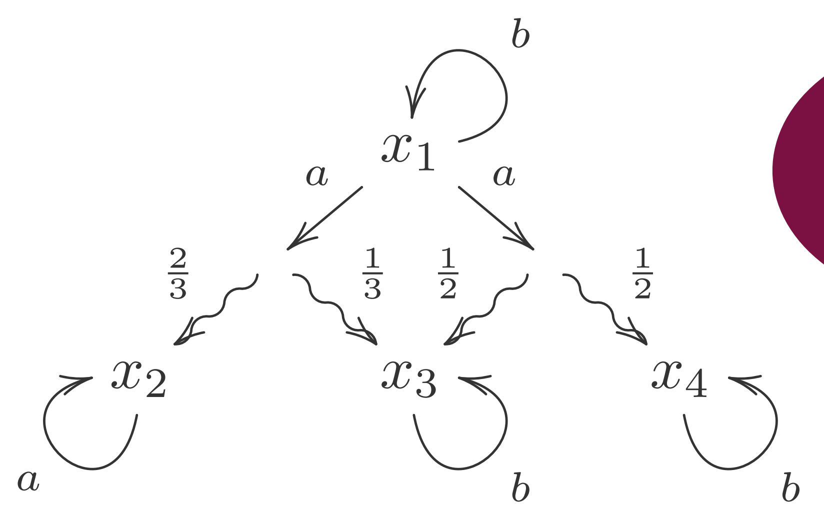
$X \to (\mathcal{P}\,\mathcal{D}X)^A$




on **Sets**

# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$

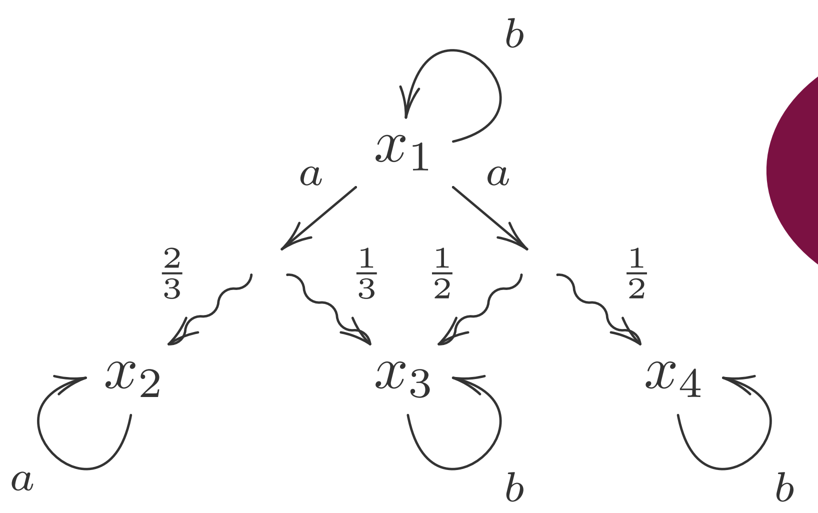



on **Sets**

$\sim \; = \; \approx$

# PA coalgebraically

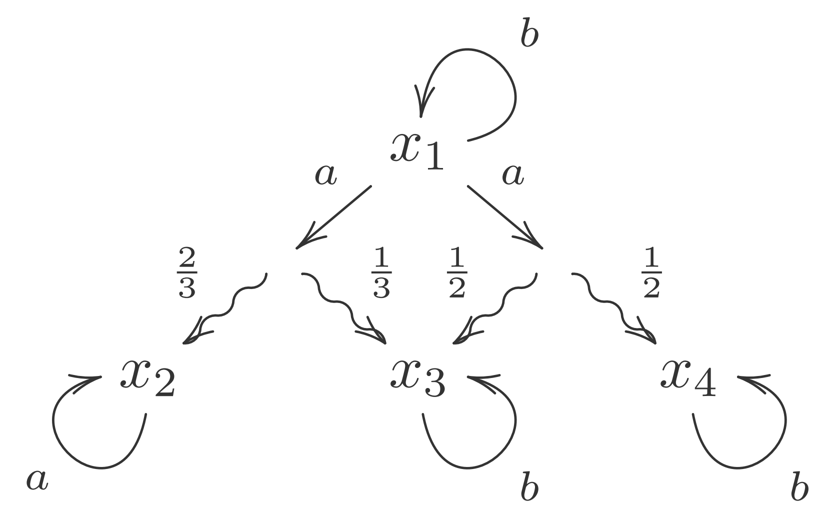
$X \to (\mathcal{P}\mathcal{D}X)^A$




on **Sets**

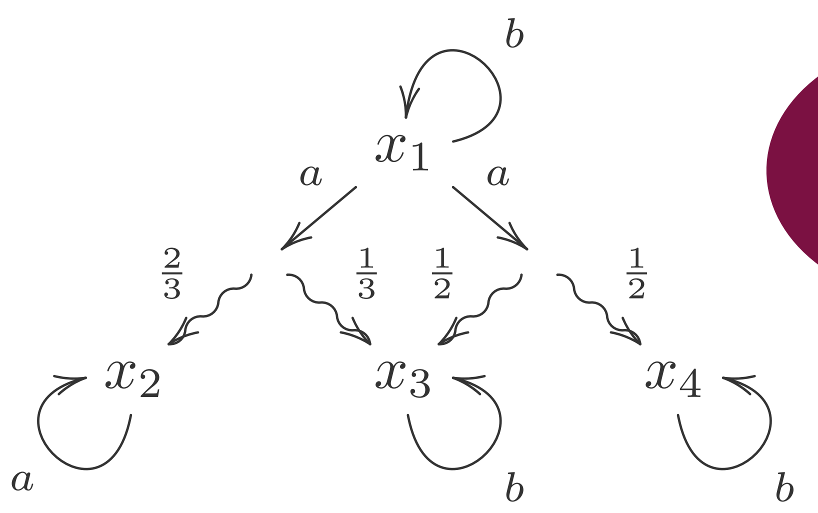
$\sim \, = \, \approx$

$X \to (\mathcal{C}X)^A$



# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$

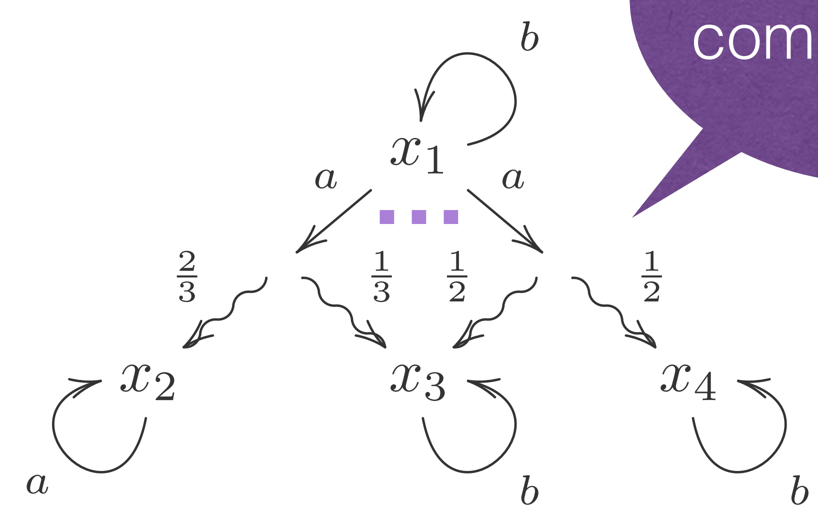



on **Sets**

$\sim \; = \; \approx$

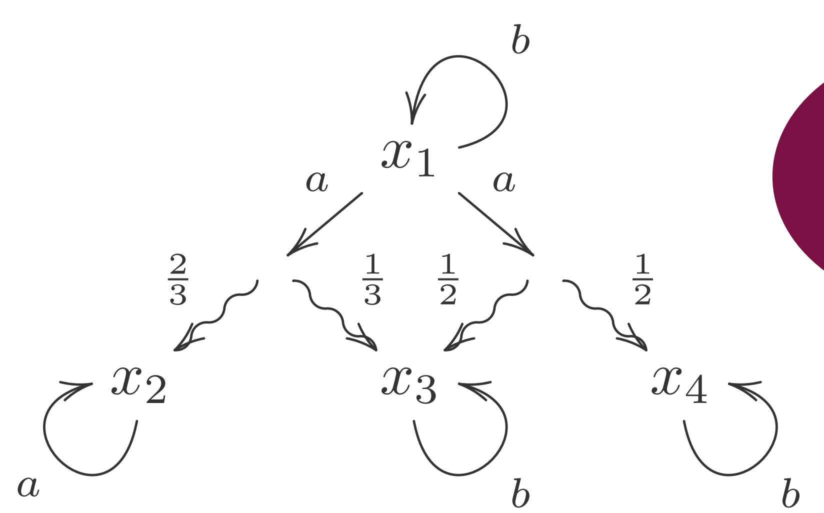
$X \to (\mathcal{C}X)^A$

and all convex combinations



# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



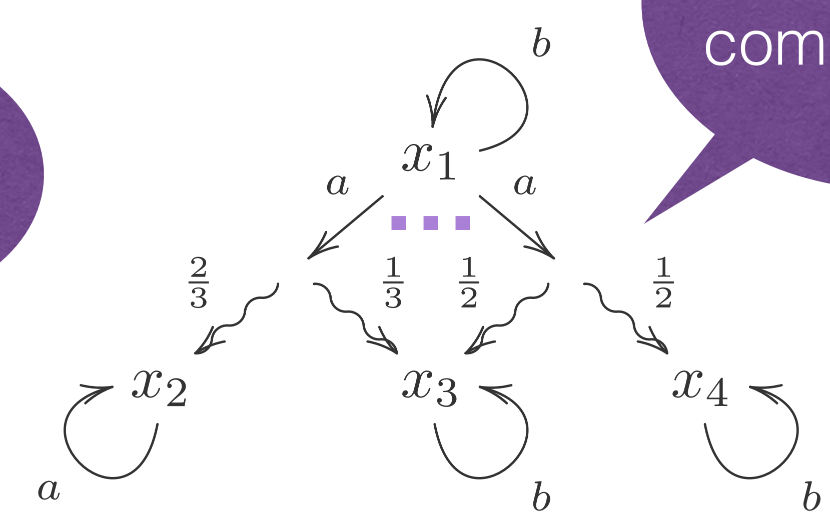

on **Sets**

$\sim \; = \; \approx$

$\sim_c \; = \; \approx$

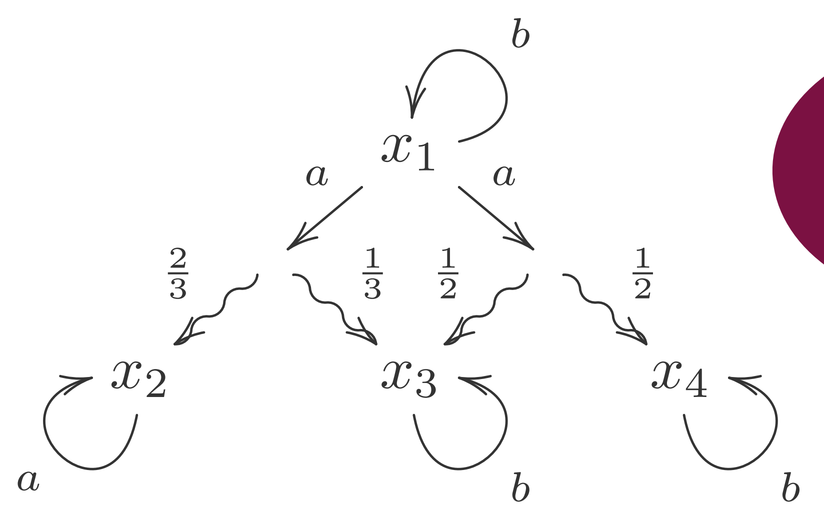
$X \to (\mathcal{C}X)^A$

and all convex combinations



# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



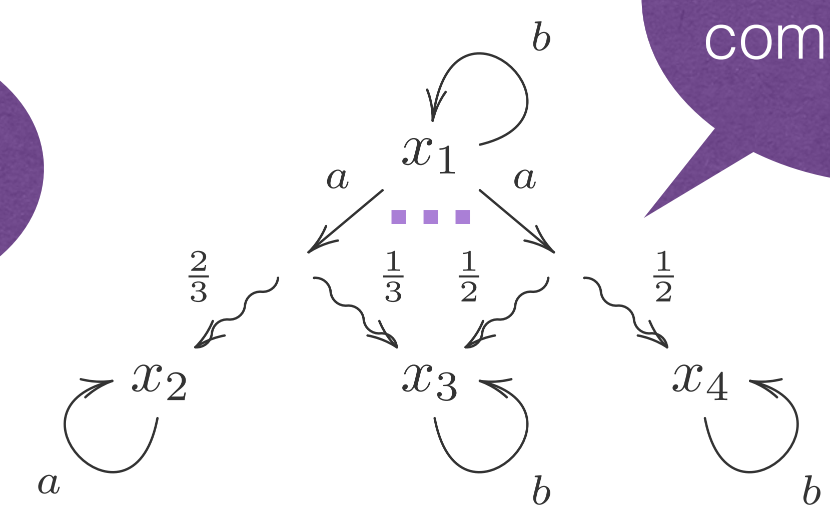

on **Sets**

$\sim \; = \; \approx$

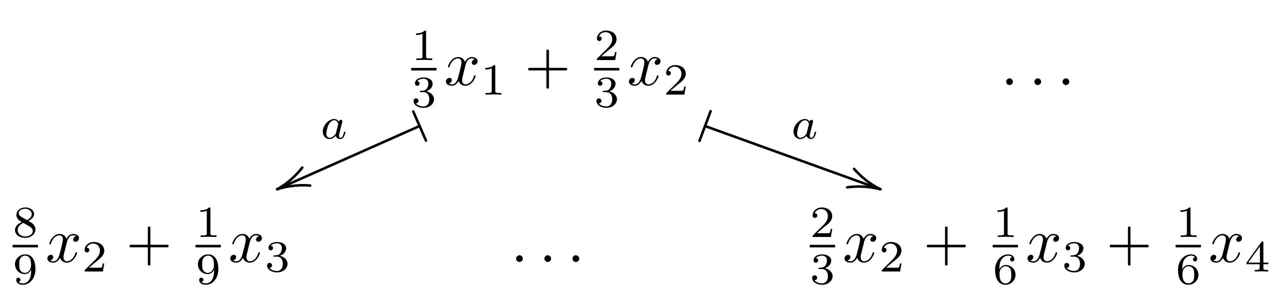
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$

and all convex combinations

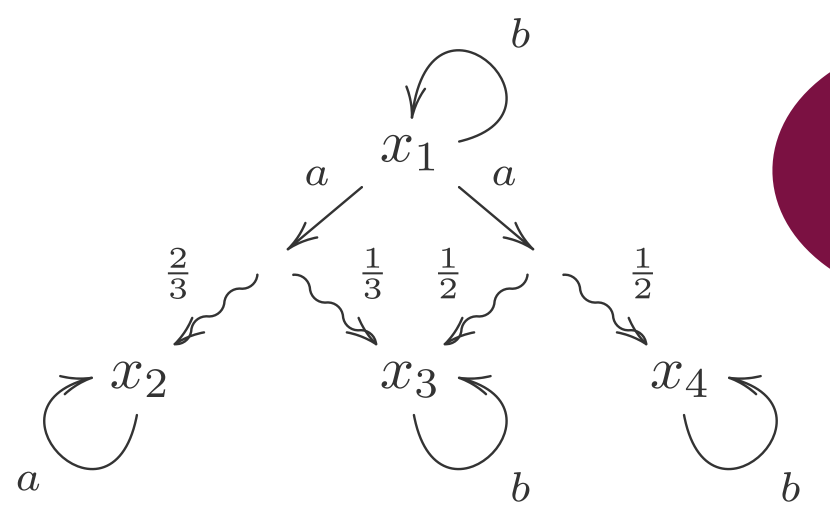



$X \to (\mathcal{P}_c X+1)^A$



# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$

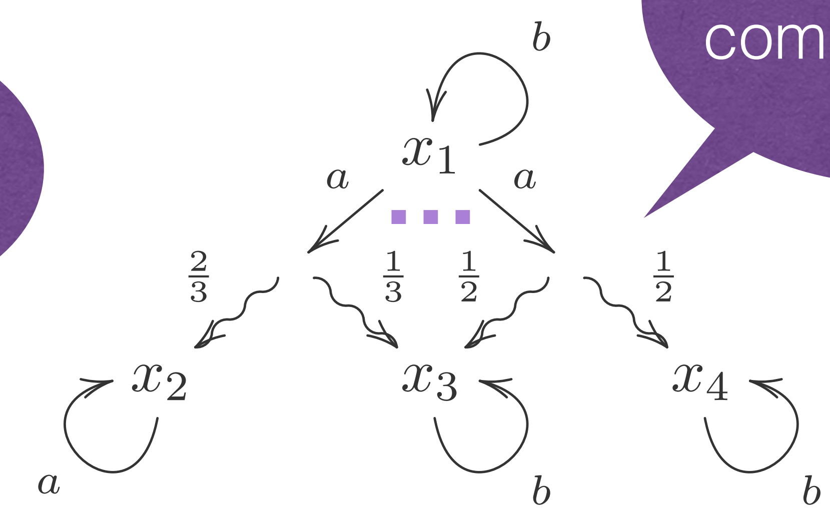



on **Sets**

$\sim \; = \; \approx$

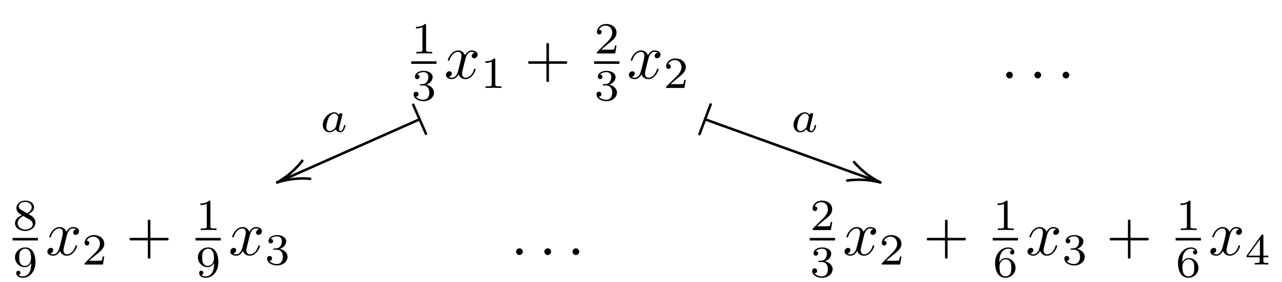
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$




and all convex combinations

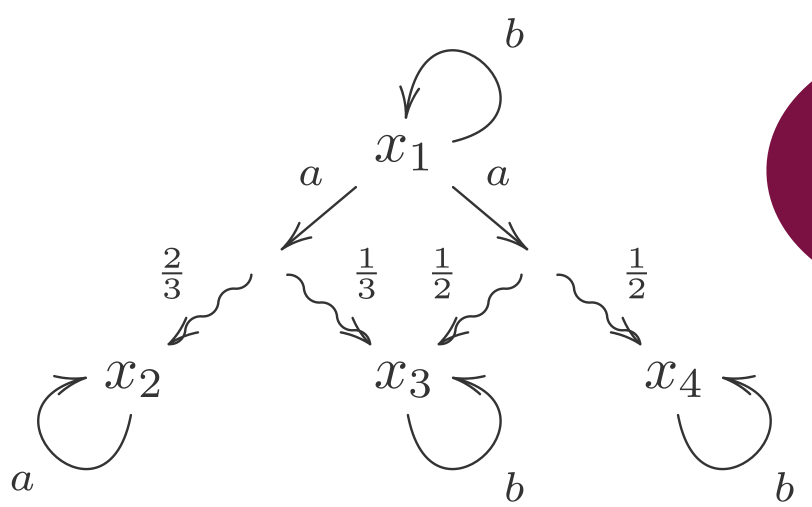
$X \to (\mathcal{P}_c X+1)^A$




on convex algebras

# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



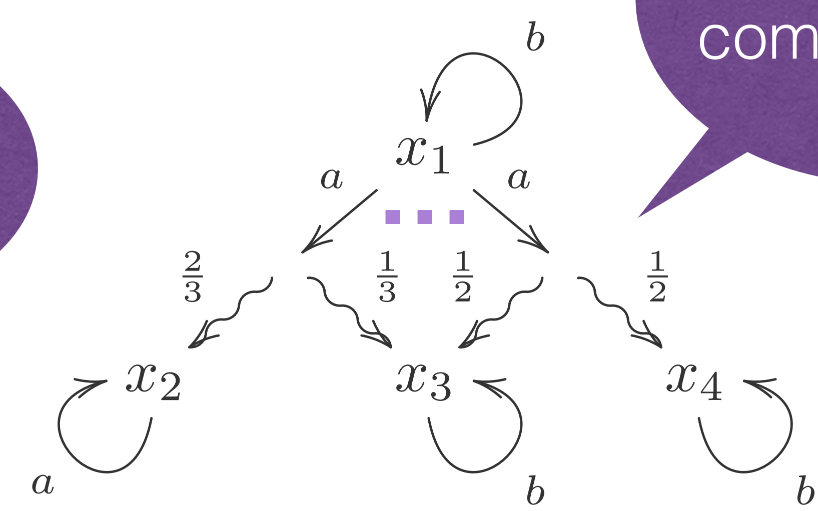

on **Sets**

$\sim \; = \; \approx$

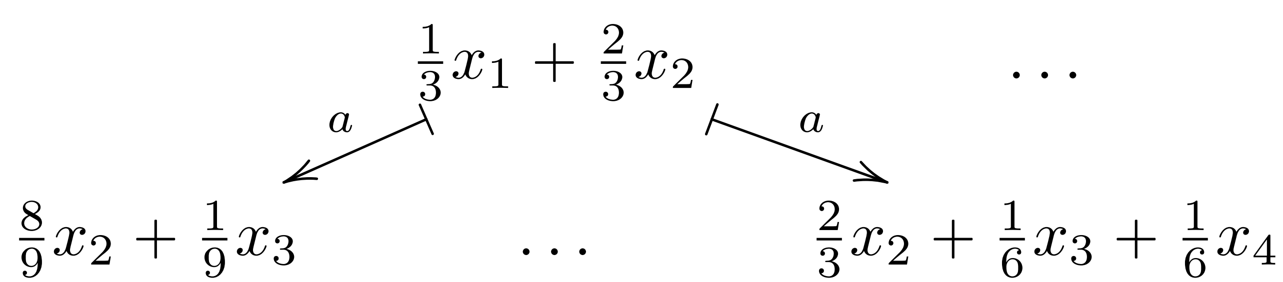
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$

and all convex combinations




$X \to (\mathcal{P}_c X+1)^A$

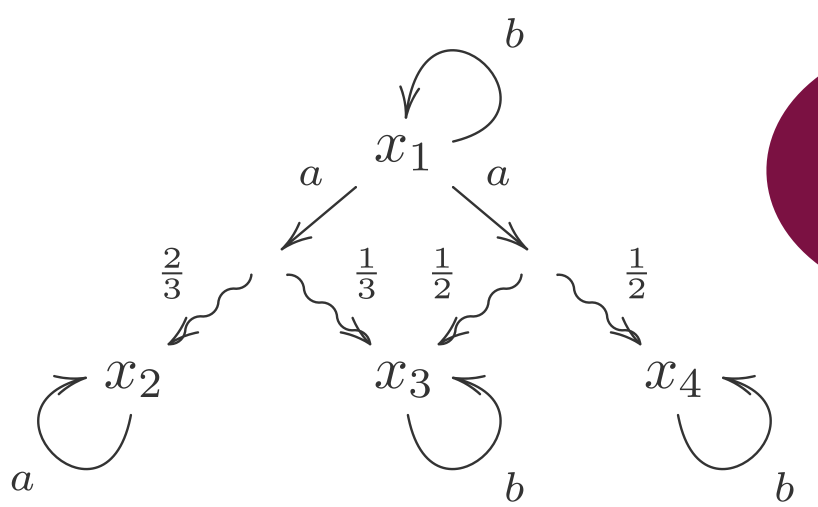



on convex algebras

$\mathcal{EM}(\mathcal{D})$

# PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}X)^A$



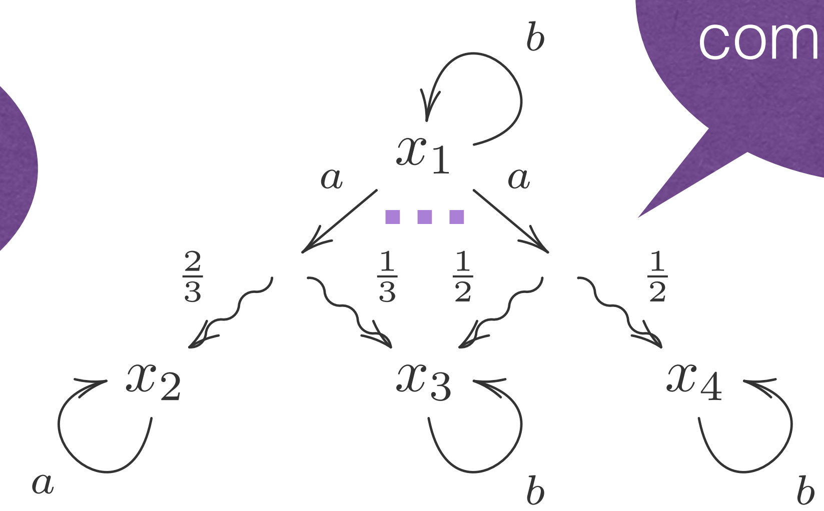

on **Sets**

$\sim \; = \; \approx$

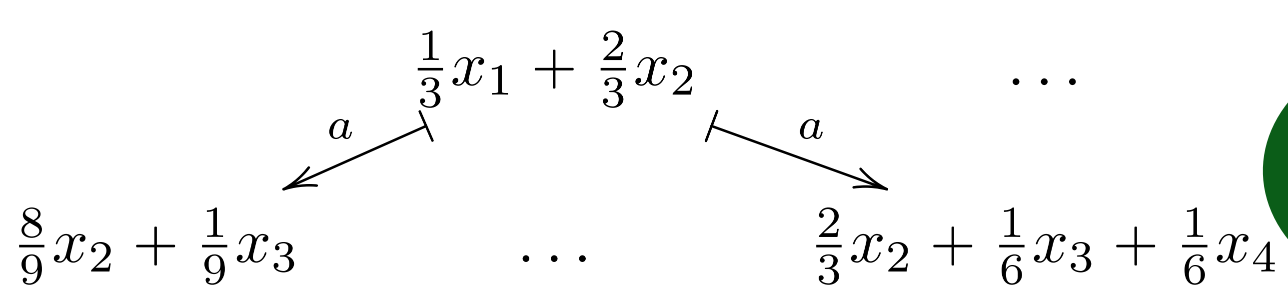
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}X)^A$

and all convex combinations




$X \to (\mathcal{P}_c X+1)^A$




on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \; = \; \approx$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

- algebras

$$\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$$

- convex (affine) maps

$$\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $\quad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps $\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$

$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many
finitary operations

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$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many finitary operations

convex combinations

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$\mathcal{EM}(\mathcal{D})$

# Convex algebras

infinitely many finitary operations

convex combinations

- algebras $\boxed{(A, \sum_{i=1}^{n} p_i(-)_i)}$ $\quad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$

- convex (affine) maps $\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$

satisfying

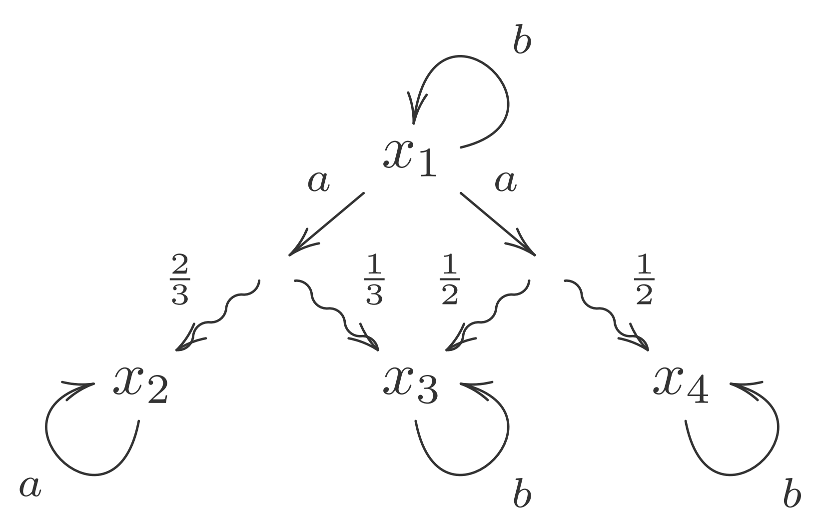
- Projection $\quad \sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\quad \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

# Belief-state transformer

# Belief-state transformer

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



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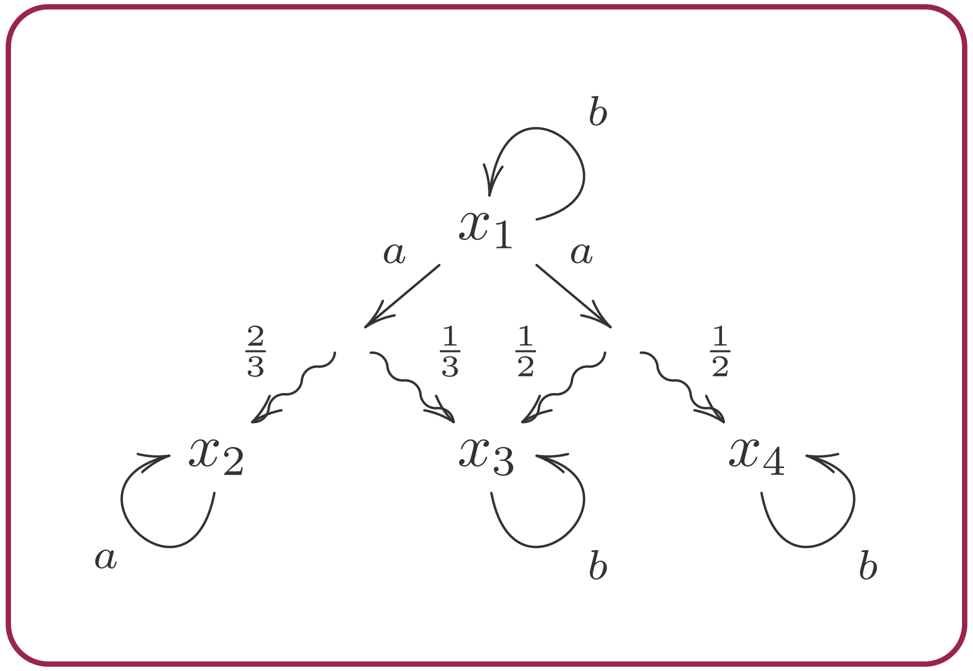


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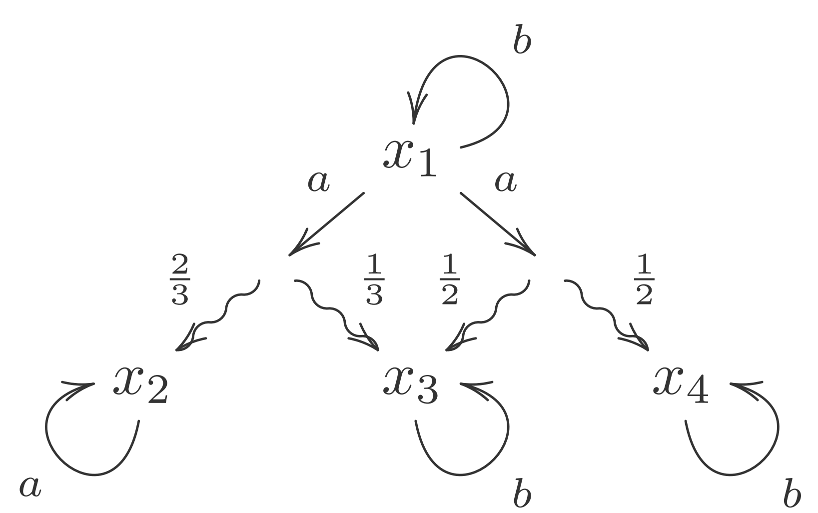
$X \to (\mathcal{P}\mathcal{D}X)^A$

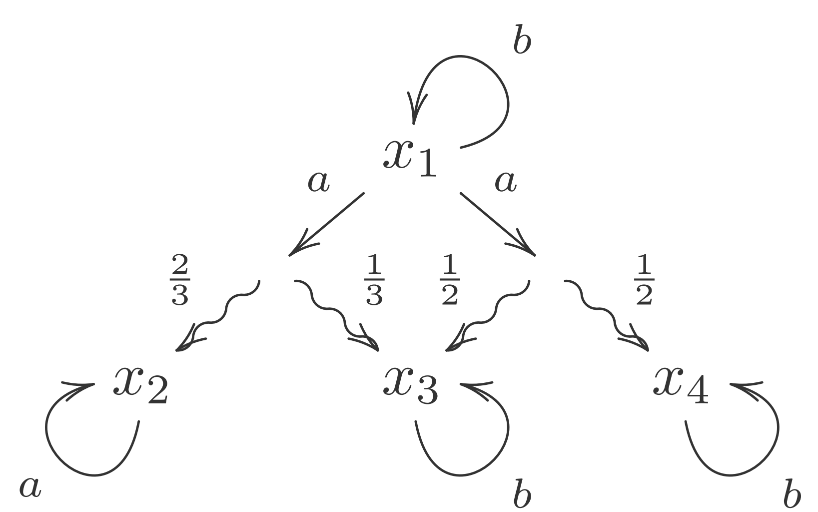



$$
\begin{tikzcd}
& \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \cdots \\
\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \cdots & \frac{8}{9}x_2 + \frac{1}{9}x_3
\end{tikzcd}
$$

# Belief-state transformer

PA

$X \to (\mathcal{PD}X)^A$




belief-state transformer

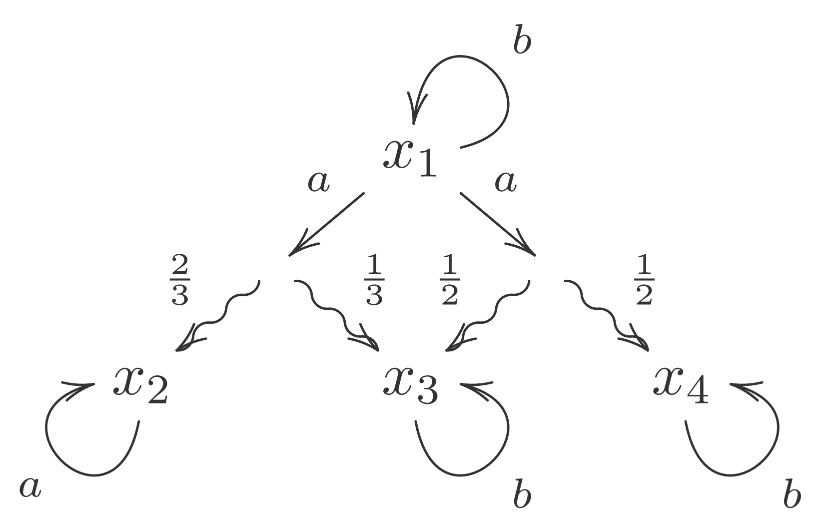
$$\frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \;\overset{a}{\longleftarrow\!\!\shortmid}\; \frac{1}{3}x_1 + \frac{2}{3}x_2 \;\overset{a}{\shortmid\!\!\longrightarrow}\; \frac{8}{9}x_2 + \frac{1}{9}x_3$$

$$\ldots \qquad \ldots$$

# Belief-state transformer

## PA

$X \to (\mathcal{P}\mathcal{D}X)^A$




belief-state transformer

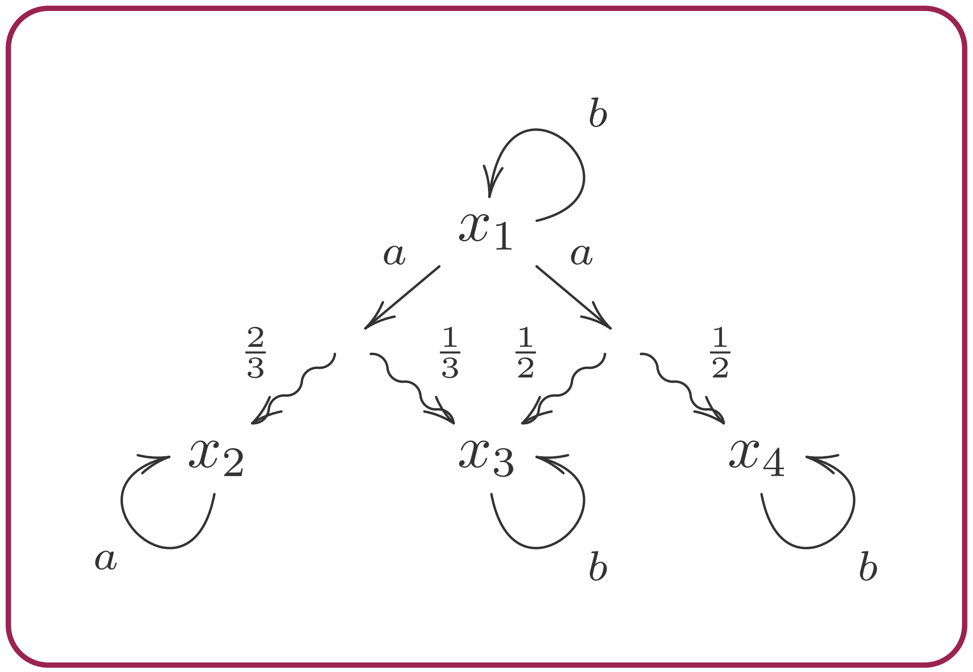
belief state

$$\begin{tikzcd}[column sep=small] & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \cdots \\ \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \cdots & \frac{8}{9}x_2 + \frac{1}{9}x_3 \end{tikzcd}$$

# Belief-state transformer

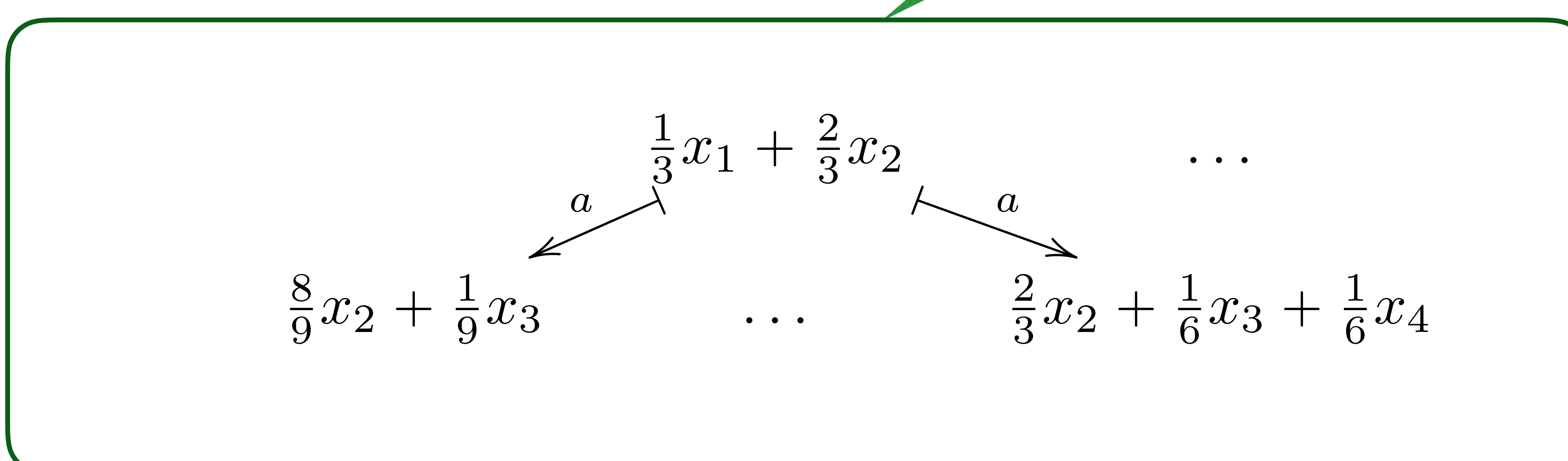
## PA

$X \to (\mathcal{PD}X)^A$

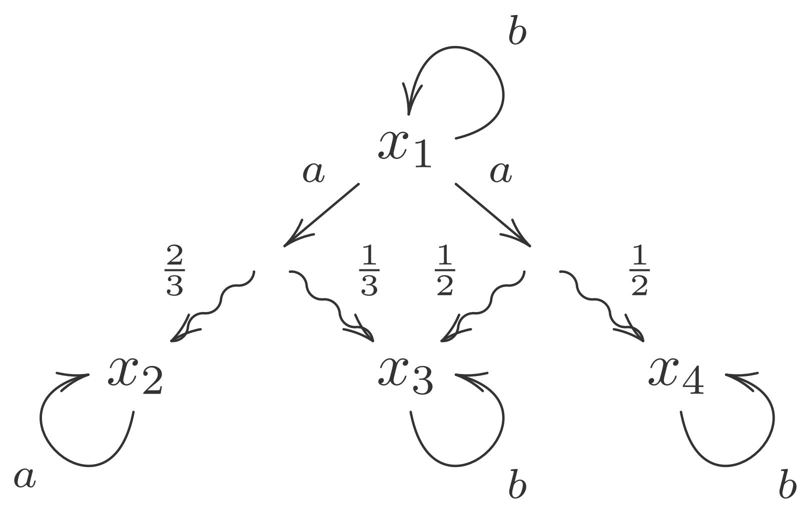



how does it emerge?

what is it?

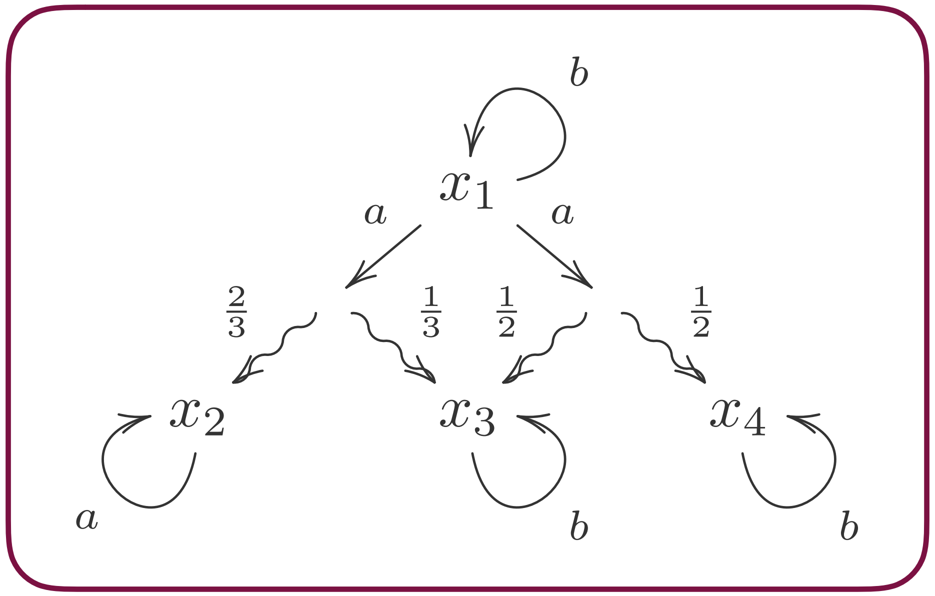


# Belief-state transformer




$$
\begin{tikzcd}[column sep=small]
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \ldots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \ldots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4
\end{tikzcd}
$$

# Belief-state transformer

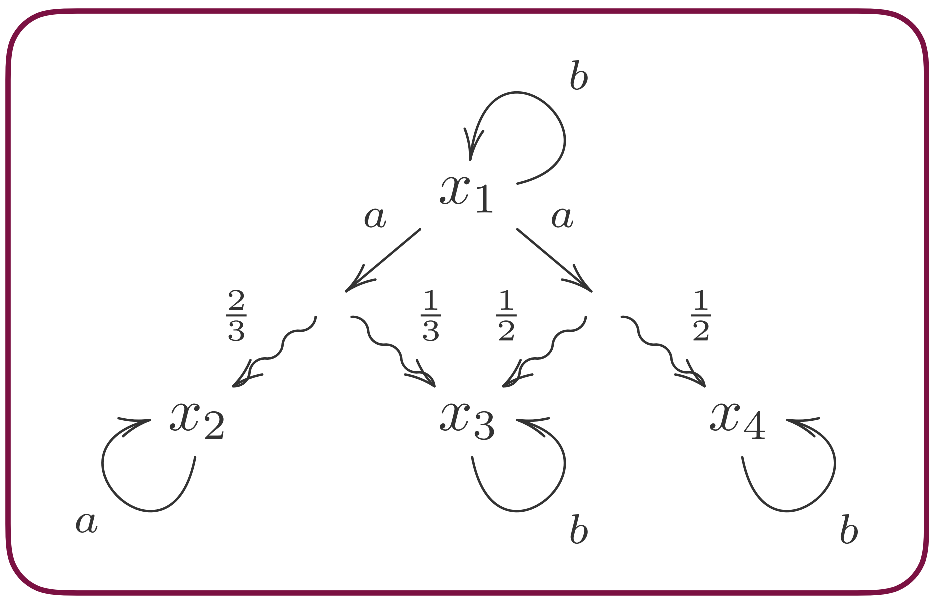


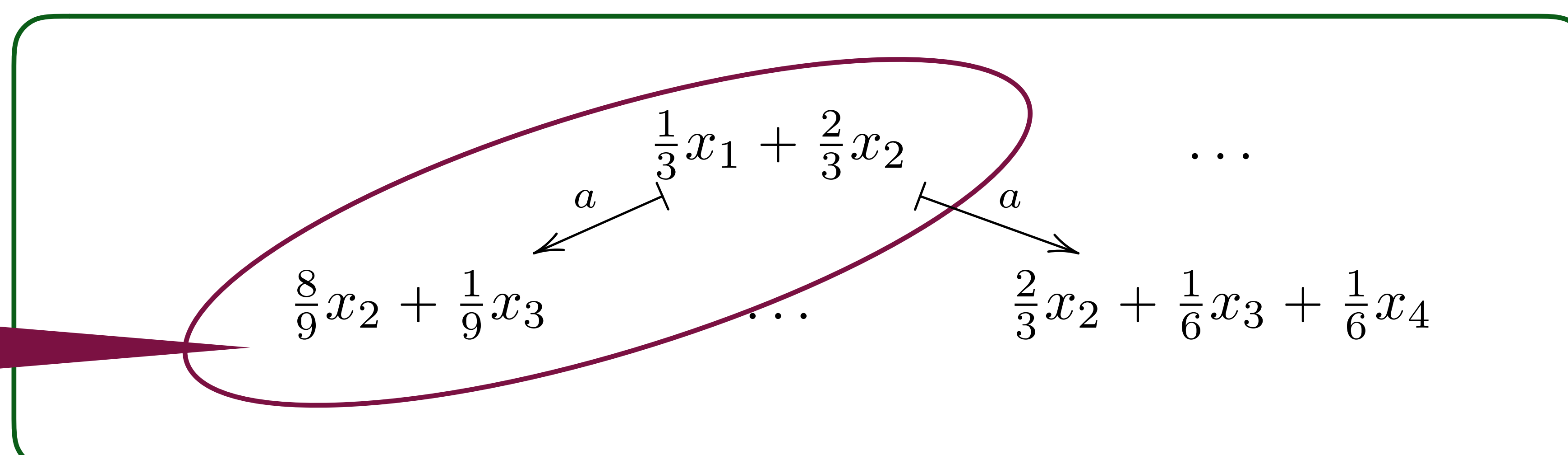

$$\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \qquad \cdots$$

$$\tfrac{8}{9}x_2 + \tfrac{1}{9}x_3 \xleftarrow{\;a\;} \tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \xrightarrow{\;a\;} \tfrac{2}{3}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{6}x_4$$

$$\cdots$$

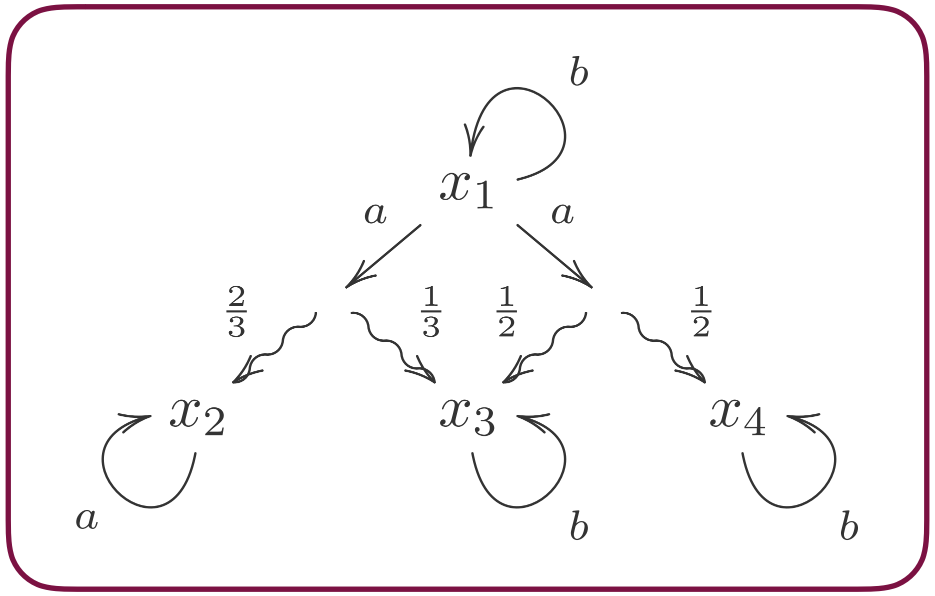
# Belief-state transformer



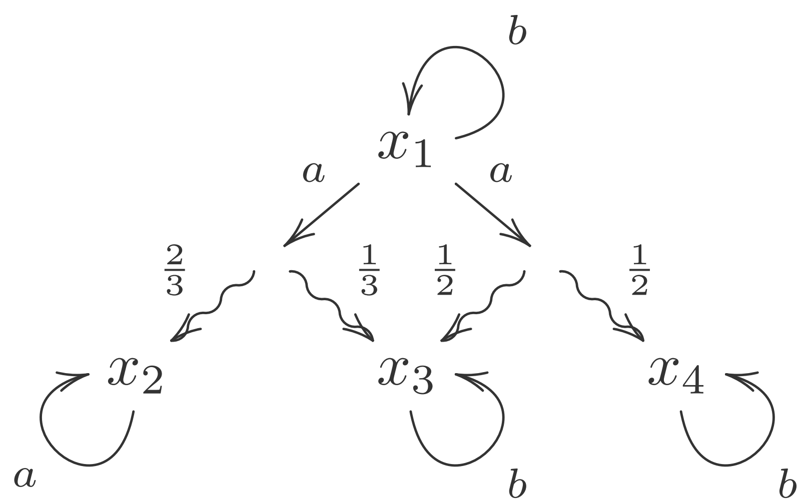

$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer



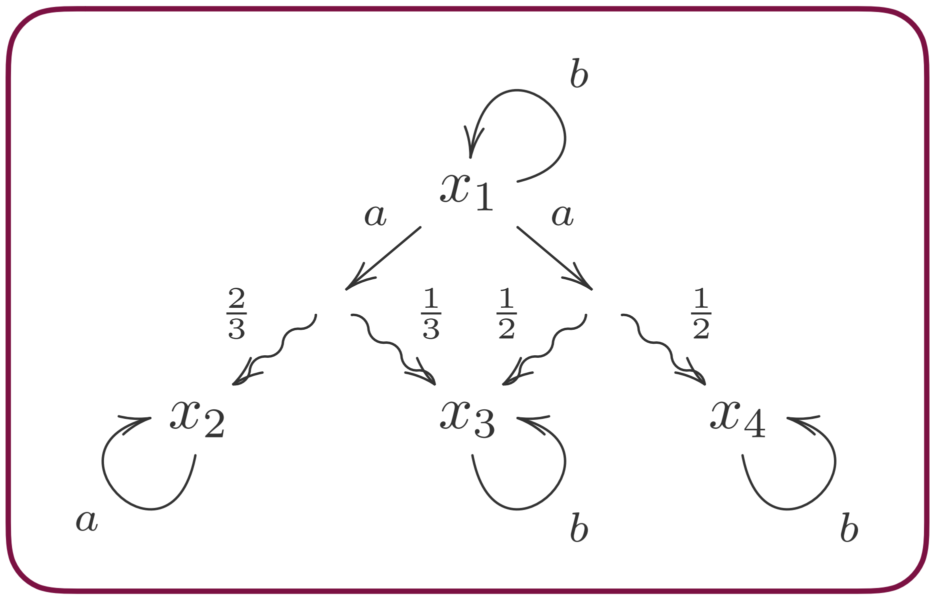

$$\begin{tikzcd}[column sep=small]
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \dots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \dots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}$$

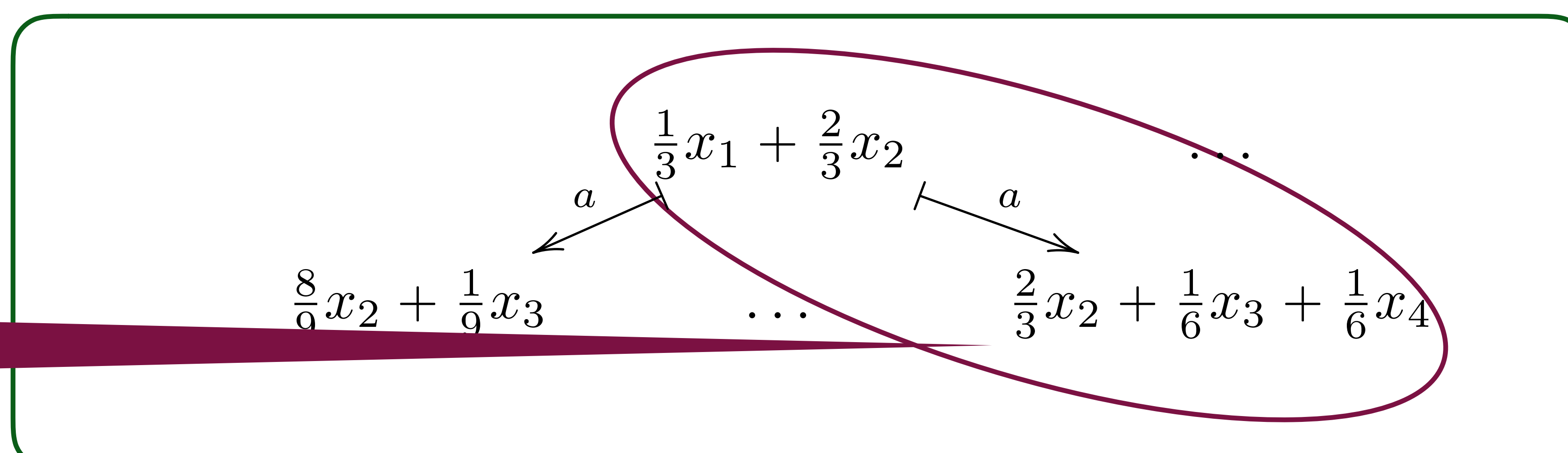
# Belief-state transformer




$$\begin{tikzcd} & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \cdots \\ \frac{8}{9}x_2 + \frac{1}{9}x_3 & \cdots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & \end{tikzcd}$$

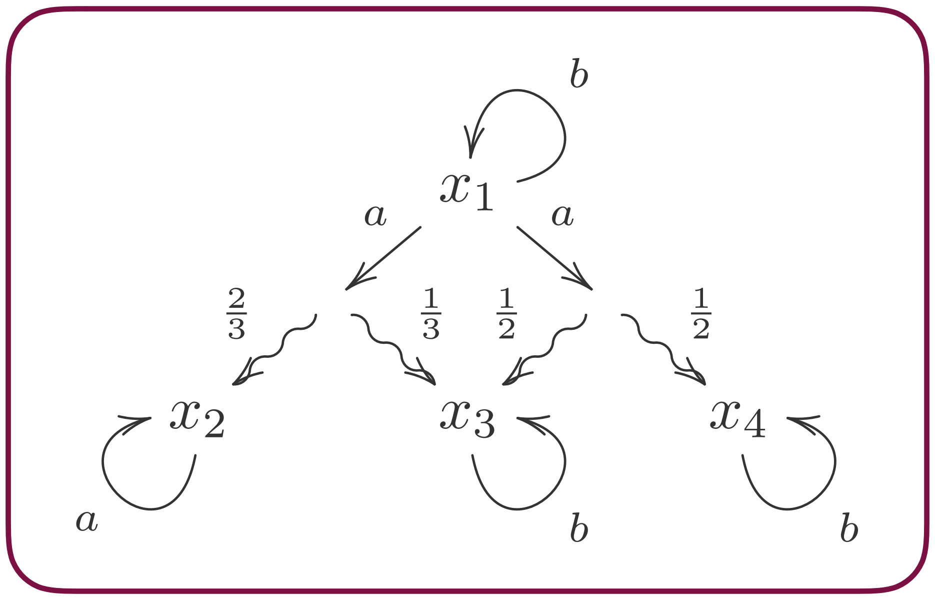
# Belief-state transformer



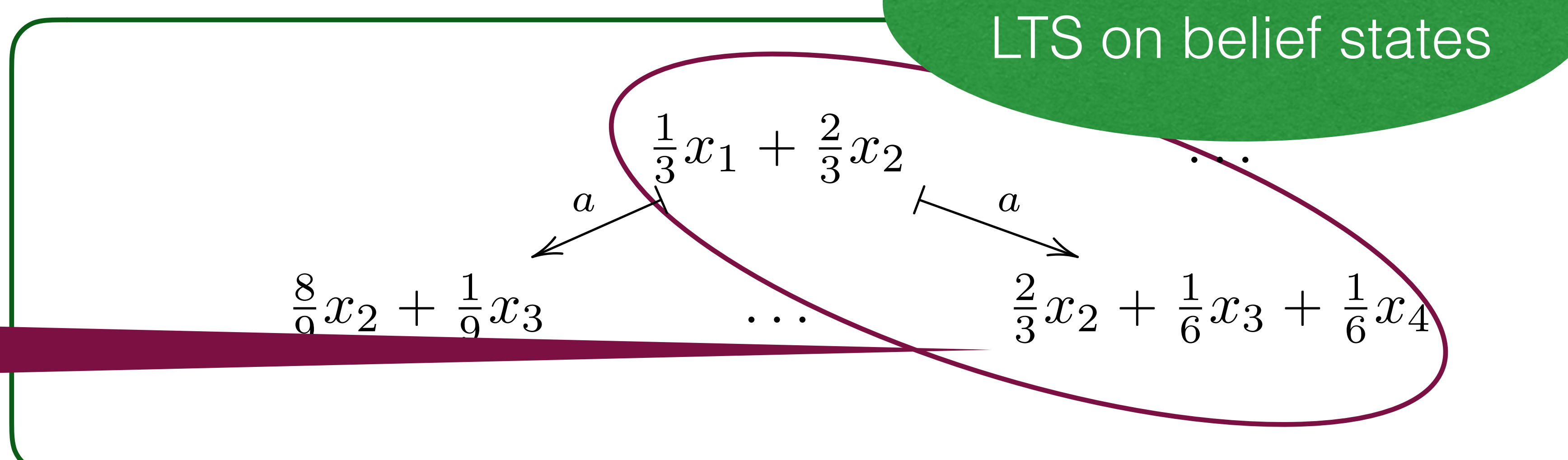




$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer




very infinite
LTS on belief states

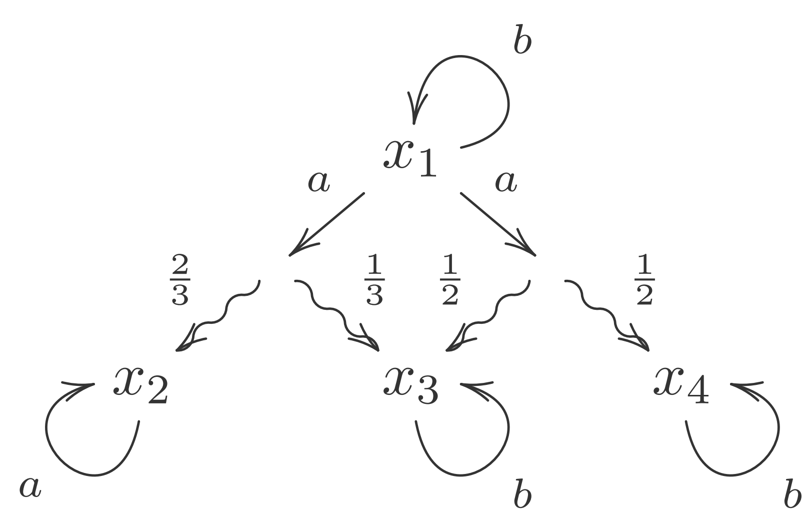



$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

# Belief-state transformer

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$




how does it emerge?

what is it?

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \ldots$$

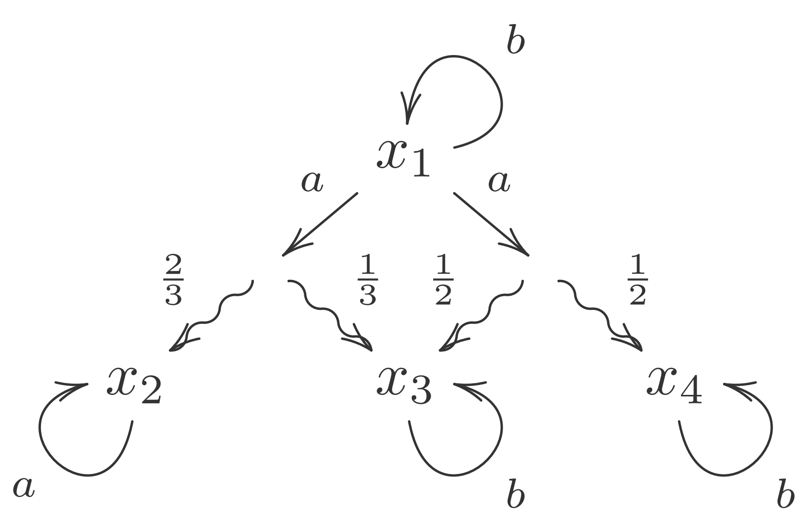
$$\frac{8}{9}x_2 + \frac{1}{9}x_3 \xleftarrow{a} \frac{1}{3}x_1 + \frac{2}{3}x_2 \xrightarrow{a} \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4$$

$$\frac{8}{9}x_2 + \frac{1}{9}x_3 \qquad \ldots \qquad \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4$$

# Belief-state transformer

PA

$X \to (\mathcal{PD}X)^A$




how does it emerge?

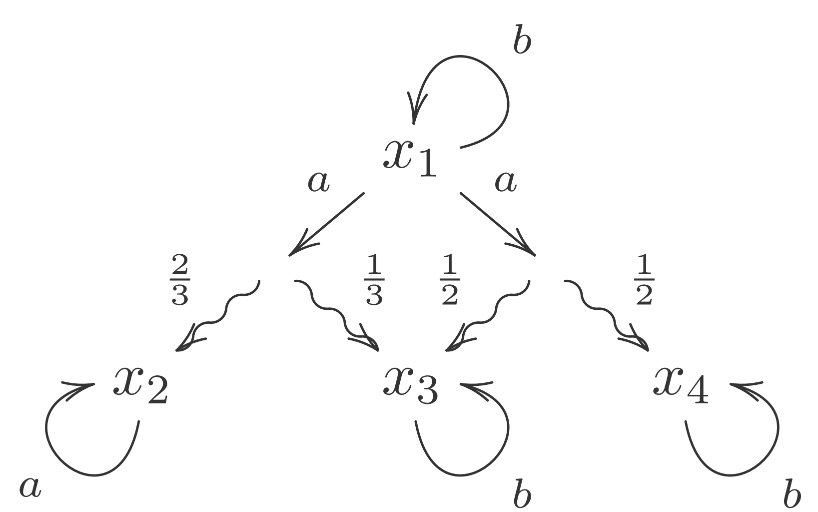
coalgebra over free convex algebra

$$\begin{array}{ccccc} & \frac{1}{3}x_1 + \frac{2}{3}x_2 & & & \dots \\ \swarrow^{a} & & \searrow^{a} & & \\ \frac{8}{9}x_2 + \frac{1}{9}x_3 & & \dots & & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \end{array}$$

# Belief-state transformer

PA

$X \to (\mathcal{PD}X)^A$




via a generalised determinisation

coalgebra over free convex algebra

$$
\begin{tikzcd}
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \ldots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \ldots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}
$$

# Determinisations

# Determinisations

NFA

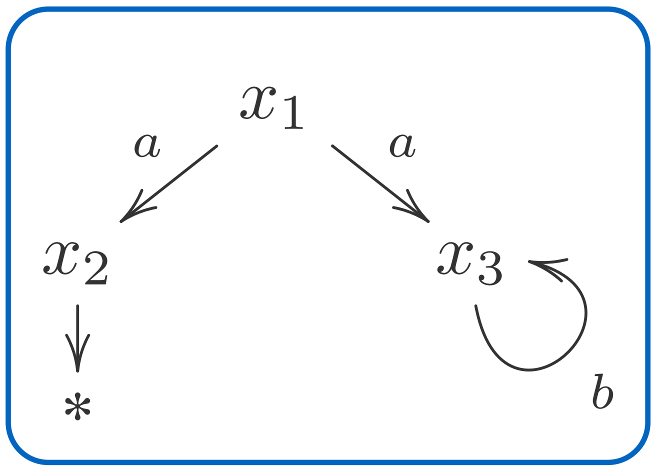
$$X \to 2 \times (\mathcal{P}X)^A$$

$$\begin{tikzcd} & x_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\ x_2 \arrow[d] & & x_3 \arrow[loop right, "b"] \\ * & & \end{tikzcd}$$

# Determinisations

NFA

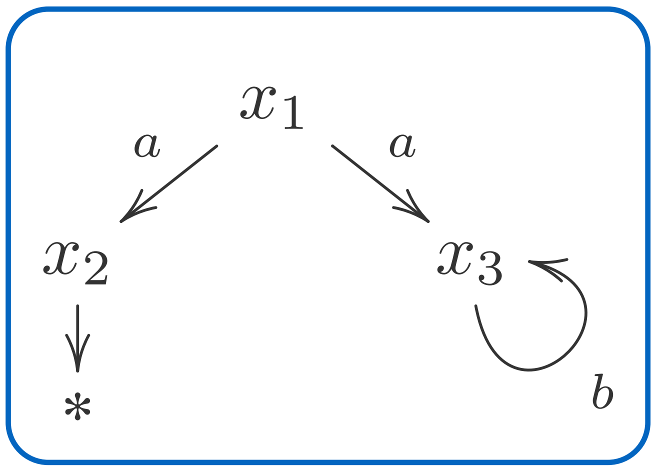
$X \to 2 \times (\mathcal{P}X)^A$

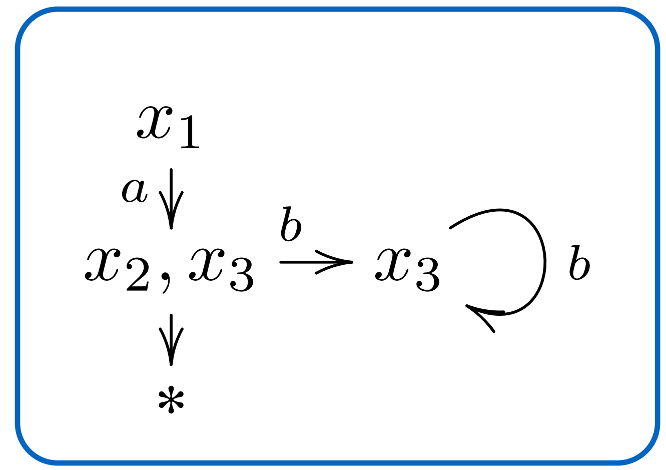


# Determinisations

NFA

$$X \to 2 \times (\mathcal{P}X)^A$$

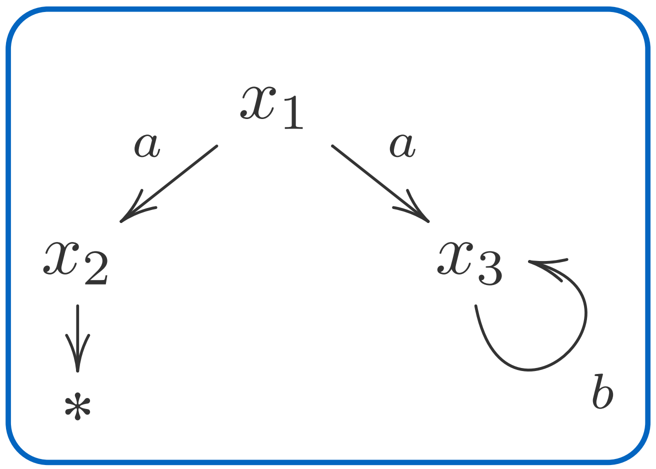


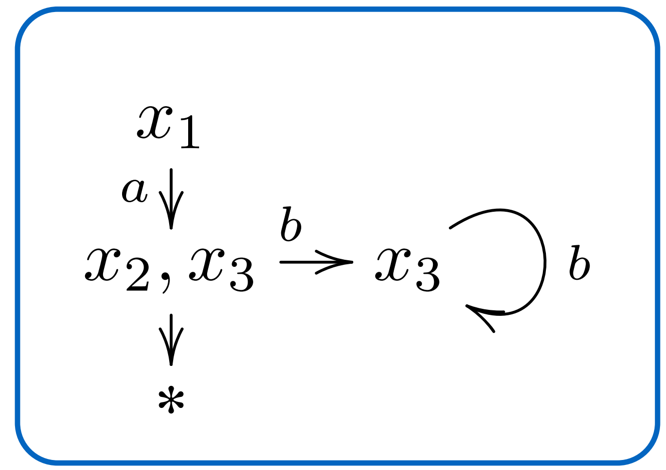



# Determinisations

## NFA

$X \to 2 \times (\mathcal{P}X)^A$




[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

# Determinisations

# Determinisations

## Generative PTS

$$X \to \mathcal{D}(1 + A \times X)$$

$$\begin{tikzcd}
 & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{2}}"] & \\
x_2 \arrow[d, "{b,1}"'] & & x_3 \arrow[d, "{c,1}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

# Determinisations

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$X \to \mathcal{D}(1 + A \times X)$

$$\begin{tikzcd}[column sep=small]
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{2}}"] & \\
x_2 \arrow[d, "{b,1}"'] & & x_3 \arrow[d, "{c,1}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

# Determinisations

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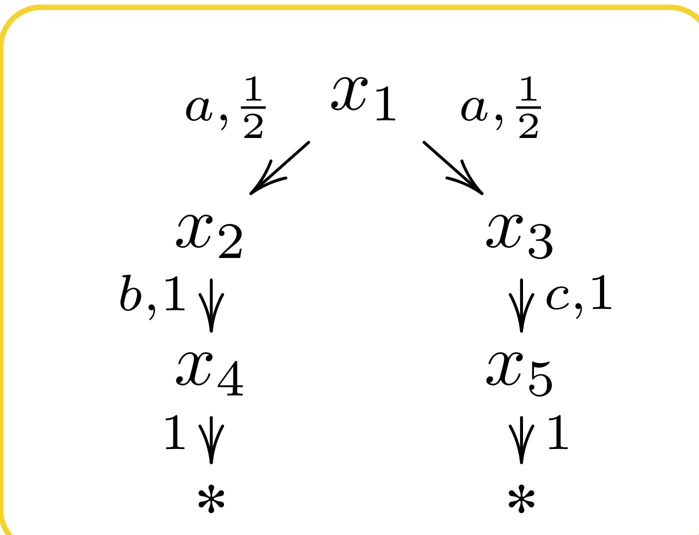
$$\begin{tikzcd}[column sep=small]
& x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{2}}"] & \\
x_2 \arrow[d, "{b,1}"'] & & x_3 \arrow[d, "{c,1}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

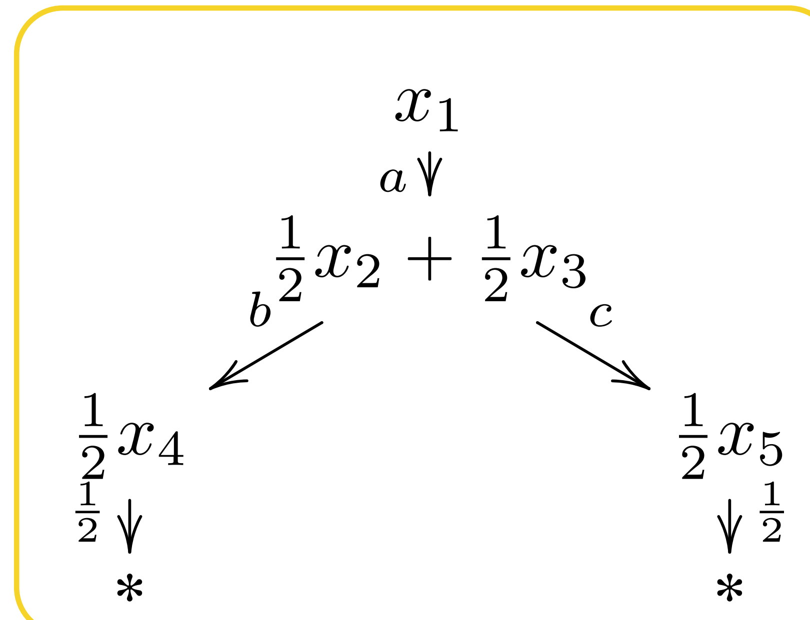
$$\begin{tikzcd}[column sep=small]
& x_1 \arrow[d, "a"'] & \\
& \frac{1}{2}x_2 + \frac{1}{2}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\frac{1}{2}x_4 \arrow[d, "\frac{1}{2}"'] & & \frac{1}{2}x_5 \arrow[d, "\frac{1}{2}"] \\
* & & *
\end{tikzcd}$$

# Determinisations

## Generative PTS

X → $\mathcal{D}$ (1 + A x X)




[Silva, S. MFPS'11]

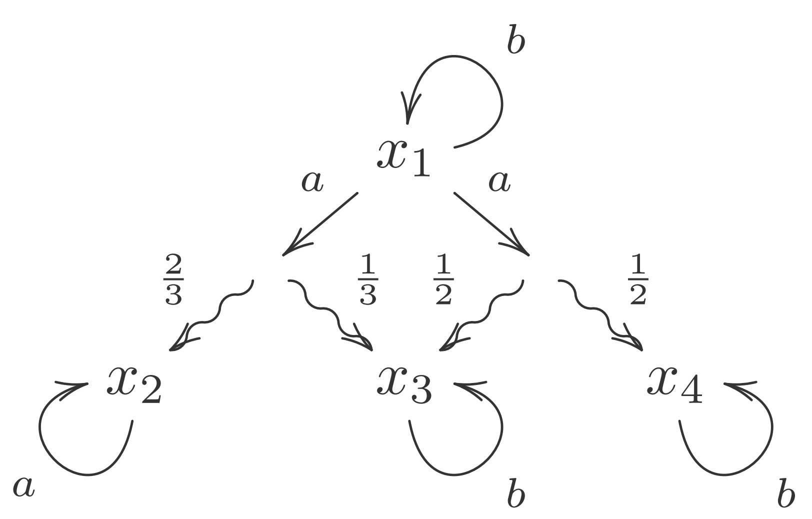
[Jacobs, Silva, S. JCSS'15]

# Determinisations

# Determinisations

PA

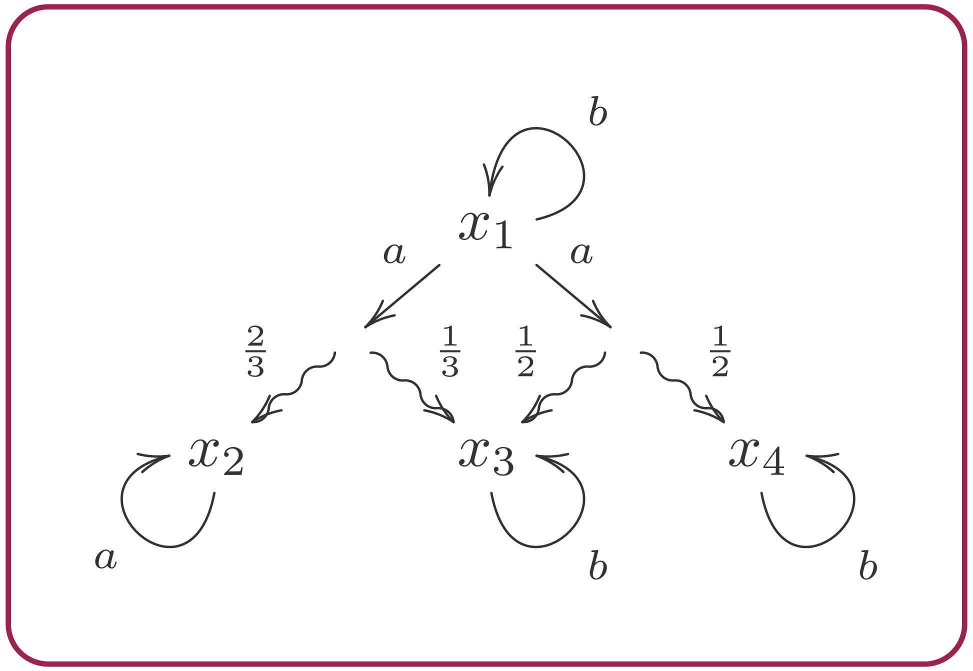
$X \to (\mathcal{P}\mathcal{D}X)^A$



# Determinisations

PA

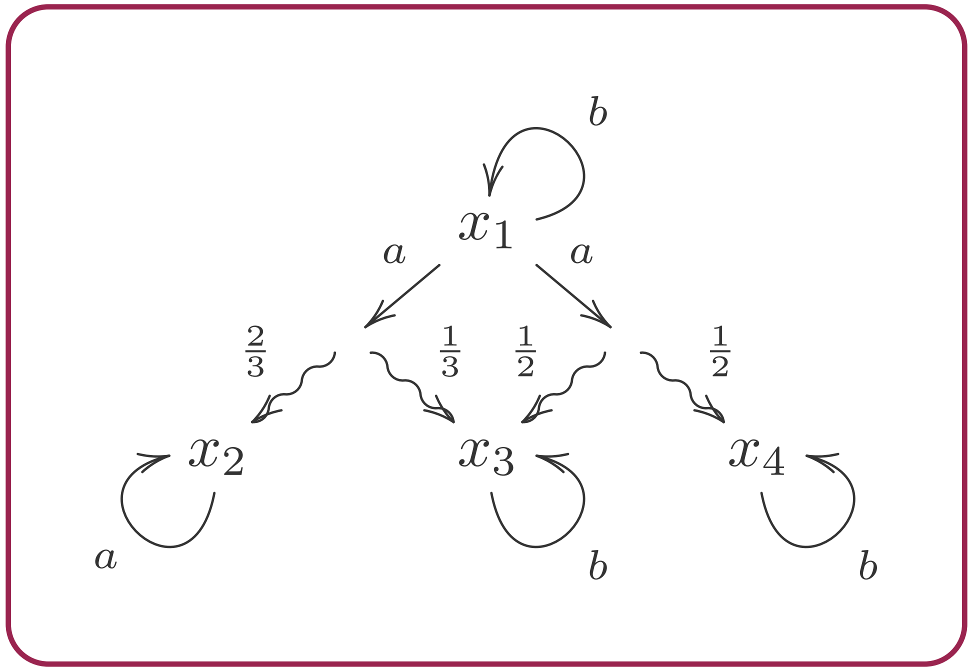
$X \to (\mathcal{P}\mathcal{D}X)^A$



# Determinisations

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



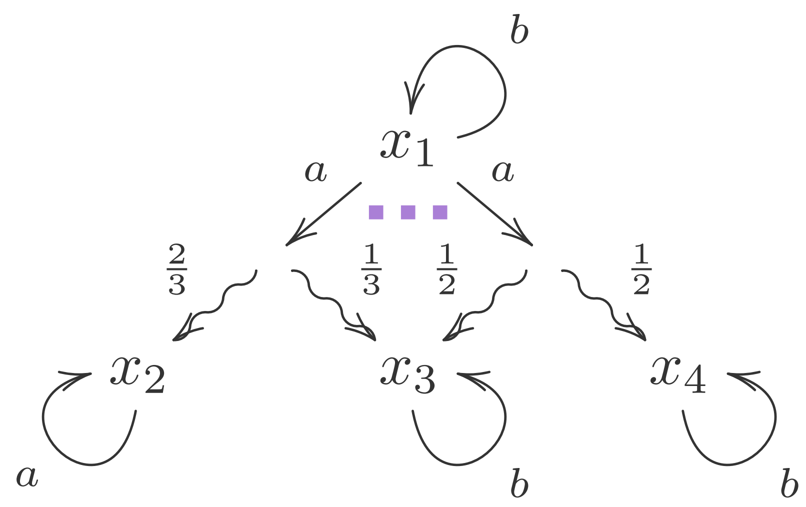

belief-state transformer

$$\begin{array}{ccccc} & & \frac{1}{3}x_1 + \frac{2}{3}x_2 & & \dots \\ & \stackrel{a}{\swarrow} & & \stackrel{a}{\searrow} & \\ \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 & & \dots & & \frac{8}{9}x_2 + \frac{1}{9}x_3 \end{array}$$

# Determinisations

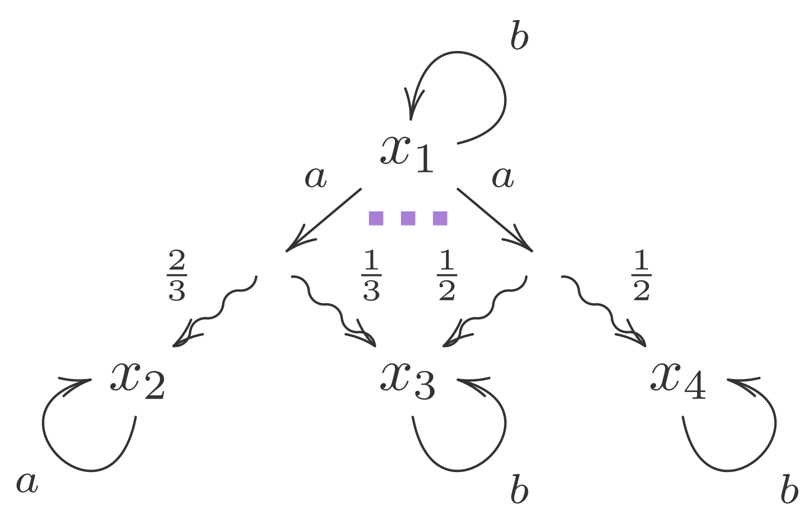
# Determinisations

$X \to (\mathcal{C}X)^A$



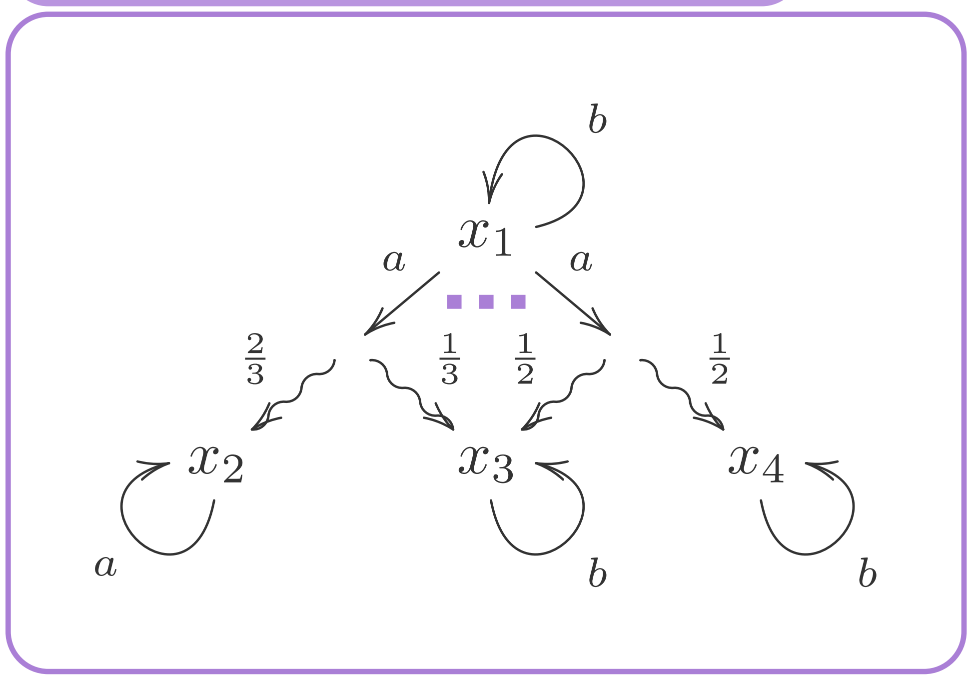
# Determinisations

$X \to (\mathscr{C}X)^A$



# Determinisations

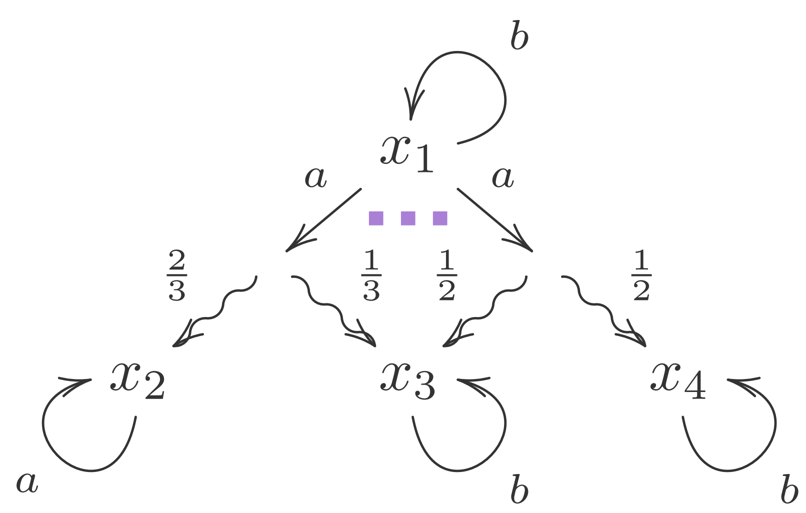
$X \to (\mathcal{C}X)^A$




$$
\begin{array}{c}
x_1 \\
a \big\downarrow \\
\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) \oplus \left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right)
\end{array}
$$

# Determinisations

$X \to (\mathcal{C}X)^A$

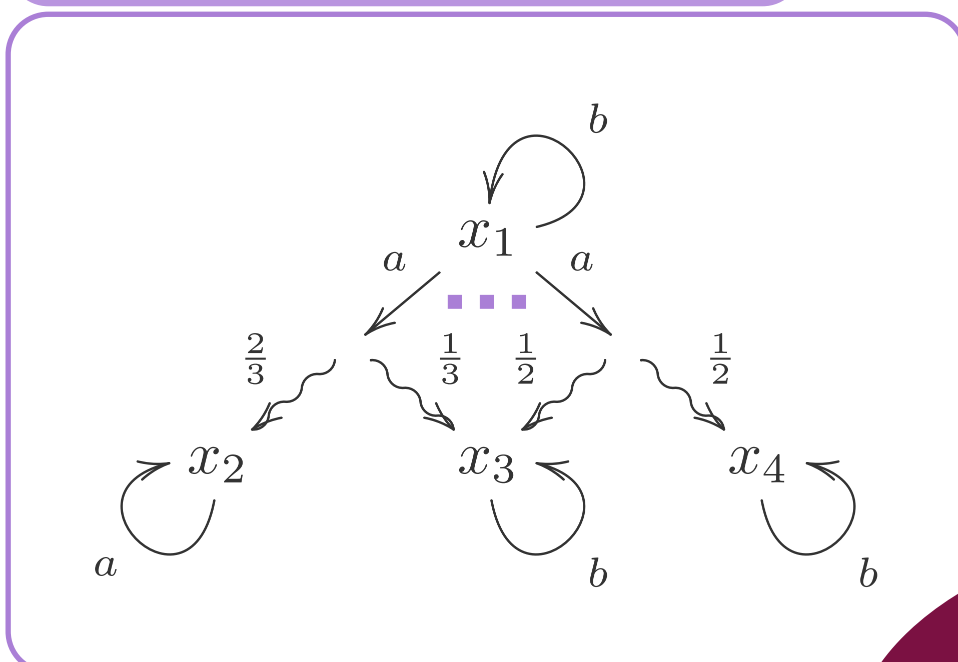



$$
\begin{array}{c}
x_1 \\
a \big\downarrow \\
(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3) \oplus (\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4)
\end{array}
$$

LTS on a convex semilattice

# Determinisations

$X \to (\mathcal{C}X)^A$



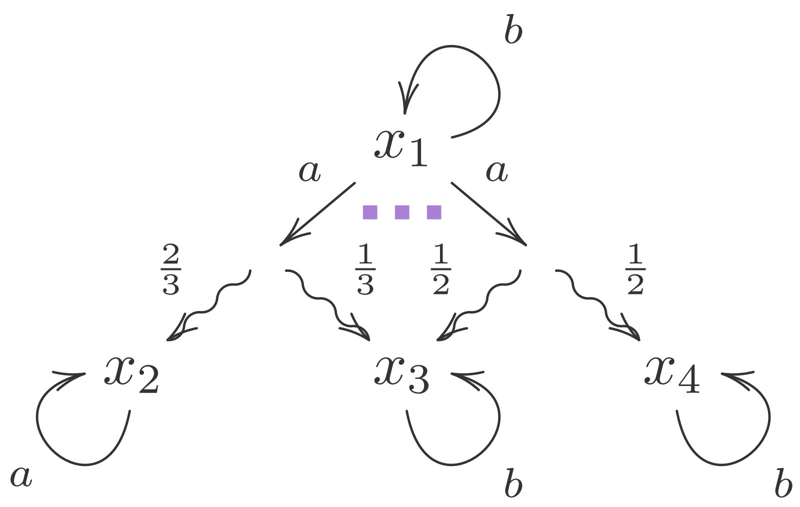

$$\begin{array}{c} x_1 \\ a \big\downarrow \\ (\frac{2}{3}x_2 + \frac{1}{3}x_3) \oplus (\frac{1}{2}x_3 + \frac{1}{2}x_4) \end{array}$$

Theory of traces for PA @LICS

LTS on a convex semilattice

# Determinisations

$X \to (\mathcal{C}X)^A$




$$x_1 \xrightarrow{a} \left(\tfrac{2}{3}x_2 + \tfrac{1}{3}x_3\right) \oplus \left(\tfrac{1}{2}x_3 + \tfrac{1}{2}x_4\right)$$

Theory of traces for PA @LICS

LTS on a convex semilattice

Thank You !