The Microcosm Principle and Concurrency in Coalgebras

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Microcosm principle

(Baez & Dolan)
Microcosm principle

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Microcosm principle

(Baez & Dolan)

“A monoid object lives in a monoidal category which is itself a kind of monoid object.”
Coalgebras

are an elegant generalization of transition systems with

states + transitions
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are an elegant generalization of transition systems with
states + transitions

as pairs

$$\langle S, \alpha : S \rightarrow \mathcal{F}S \rangle$$, for \( \mathcal{F} \) a functor
Coalgebras

are an elegant generalization of transition systems with states + transitions as pairs

\[ \langle S, \alpha : S \rightarrow SF \rangle, \text{ for } F \text{ a functor} \]

- a uniform way for treating transition systems
- general notions and results, e.g. generic notion of bisimulation
Examples of Coalgebras

\[ X \xrightarrow{c} \mathcal{P}(A \times X) \]
Examples of Coalgebras

LTS

\[ X \overset{c}{\to} \mathcal{P}(A \times X) \]

Example:

\[ c(x) = \{ \langle a, x \rangle, \langle a, y \rangle \}, \ldots \]
Examples of Coalgebras

(Generative) Probabilistic systems

\[ X \xrightarrow{c} \mathcal{D}(A \times X) \]
Examples of Coalgebras

(Generative) Probabilistic systems

\[ X \xrightarrow{c} \mathcal{D}(A \times X) \]

Example:

\[
\begin{aligned}
c(x) &= \left\{ \langle b, y \rangle \mapsto \frac{1}{3}, \langle a, z \rangle \mapsto \frac{1}{3} \right\}, \\
c(y) &= \left\{ \langle a, y \rangle \mapsto 1 \right\}, \ldots
\end{aligned}
\]
Concurrency in coalgebras

Aim: well-behaved concurrency operations on coalgebras

Solution: via nested algebraic structure, microcosm models
Concurrency in coalgebras

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Solution: via nested algebraic structure, microcosm models

Well-behaved:

* compositional
* associative, commutative, ...
for instance

LTS, synchronous parallel |, with $A(\cdot)$ comm., assoc. partial

$$x \mid y \xrightarrow{a} x' \mid y' \iff x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c$$
for instance

LTS, synchronous parallel \( | \), with \( A(\cdot) \) comm., assoc. partial

\[ x \mid y \xrightarrow{a} x' \mid y' \iff x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c \]

Compositionality: \( x \sim x', y \sim y' \implies x \mid y \sim x' \mid y' \)
for instance

LTS, synchronous parallel |, with \( A(\cdot) \) comm., assoc. partial

\[
x | y \xrightarrow{a} x' | y' \iff x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c
\]

? Compositionality: \( x \sim x', y \sim y' \implies x | y \sim x' | y' \)

? Commutativity: \( x | y \sim y | x \)
for instance

LTS, synchronous parallel $|$ , with $A(\cdot)$ comm., assoc. partial

\[
x | y \xrightarrow{a} x' | y' \iff x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c
\]

? Compositionality: $x \sim x', y \sim y' \implies x | y \sim x' | y'$

? Commutativity: $x | y \sim y | x$

? Associativity: $(x | y) | z \sim x | (y | z)$
\textit{-category}

Ingredients:

category $\mathcal{C}$

$\otimes$  \hspace{1cm} $\alpha$

Example: Sets with cartesian product and $\otimes (h x; h y; z) = hh x; y i; z i$. ... is moreover symmetric monoidal.
⊗-category

Ingredients:

- category $\mathcal{C}$
- bifunctor on $\mathcal{C}$
- natural iso. $\alpha$

Example:

Sets with cartesian product and $\otimes (h x; h y; z) = hh x; y i; z i$... is moreover symmetric monoidal.
⊗-category

Ingredients:

- category \( \mathbb{C} \)
- bifunctor on \( \mathbb{C} \)
- nat. iso. \( \alpha : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z \)
\textbf{⊗-category}

**Ingredients:**

- category $\mathbb{C}$
- bifunctor on $\mathbb{C}$
- natural isomorphism $\alpha : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$

**Example:** Sets with cartesian product and

$$\alpha(\langle x, \langle y, z \rangle \rangle) = \langle \langle x, y \rangle, z \rangle$$

... is moreover symmetric monoidal
Microcosm phenomenon

“a ⊗-object lives in a ⊗-category"
Microcosm phenomenon

A \( \otimes \)-object or a semigroup in a \( \otimes \)-category \( \mathbb{C} \) is

- an object \( S \in \mathbb{C} \)
- with a binary operation \( m : S \otimes S \rightarrow S \)
- which is associative
Microcosm phenomenon

A $\otimes$-object or a semigroup in a $\otimes$-category $\mathbb{C}$ is

- an object $S \in \mathbb{C}$
- with a binary operation $m : S \otimes S \to S$
- which is associative

$$
\begin{align*}
S \otimes (S \otimes S) & \xrightarrow{\alpha} (S \otimes S) \otimes S \xrightarrow{m \otimes \text{id}} S \otimes S \\
\text{id} \otimes m & \downarrow \quad \text{id} \otimes m \\
S \otimes S & \xrightarrow{m} S
\end{align*}
$$
Compatible functors

A functor $F : \mathcal{C} \to \mathcal{D}$ is a $\otimes$-functor between the $\otimes$-categories, if there is a natural transformation that commutes with the associativity
Compatible functors

A functor $F : \mathcal{C} \to \mathcal{D}$ is a $\otimes$-functor between the $\otimes$-categories, if there is a natural transformation

$$\text{sync} : FX \otimes FY \to F(X \otimes Y)$$
Compatible functors

A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is a $\otimes$-functor between the $\otimes$-categories, if there is a natural transformation

$$\text{sync} : FX \otimes FY \rightarrow F(X \otimes Y)$$

that commutes with the associativity $\alpha$
Compatible functors

A functor \( F : \mathcal{C} \to \mathcal{D} \) is a \( \otimes \)-functor between the \( \otimes \)-categories, if there is a natural transformation

\[
\text{sync} : FX \otimes FY \to F(X \otimes Y)
\]

that commutes with the associativity \( \alpha \)

\[
\begin{aligned}
FX \otimes (FY \otimes FZ) & \xrightarrow{\alpha} (FX \otimes FY) \otimes FZ \\
F(X \otimes (Y \otimes Z)) & \xrightarrow{F\alpha} F((X \otimes Y) \otimes Z)
\end{aligned}
\]

\[
\text{sync} \circ (id \otimes \text{sync}) \downarrow & \quad \text{sync} \circ (\text{sync} \otimes id) \\
\downarrow & \quad \downarrow
\]

12-2-2007, ICIS colloquium Nijmegen – p.9/24
Structure lifts to coalgebras

given a \( \otimes \)-category \( \mathcal{C} \) and a \( \otimes \)-functor \( F \)
Structure lifts to coalgebras

given a \(\otimes\)-category \( \mathcal{C} \) and a \(\otimes\)-functor \( F \)

the category of \( F \)-coalgebras is a \(\otimes\)-category
Structure lifts to coalgebras

given a $\otimes$-category $\mathbb{C}$ and a $\otimes$-functor $F$

the category of $F$-coalgebras is a $\otimes$-category

... with

\[
\begin{align*}
F(X) & \otimes F(Y) \\
X & \otimes Y
\end{align*}
\]

\[
\begin{align*}
FX & \otimes FY \\
X & \otimes Y
\end{align*}
\]
Structure lifts to coalgebras

given a $\otimes$-category $\mathbb{C}$ and a $\otimes$-functor $F$

the category of $F$-coalgebras is a $\otimes$-category

... with

\[
\begin{array}{ccc}
FX & \otimes & FY \\
\downarrow c & \otimes & \downarrow d \\
X & = & \uparrow \text{sync} \uparrow c \otimes d \\
& & \uparrow \text{sync} \uparrow FX \otimes FY \\
& & \uparrow \text{sync} \uparrow X \otimes Y
\end{array}
\]

Hence - process operations on coalgebras!
Well-behaved operations

assume final F-coalgebra $\zeta : Z \xrightarrow{\cong} FZ$ exists

... by finality unique homomorphism $\| : Z \otimes Z \to Z$
Well-behaved operations

assume final F-coalgebra $\zeta : Z \xrightarrow{\cong} FZ$ exists
... by finality unique homomorphism $\parallel : Z \otimes Z \rightarrow Z$

associativity: $\zeta$ with $\parallel$ is a semigroup in Coalg$_F$
Well-behaved operations

assume final F-coalgebra $\zeta : Z \to FZ$ exists
... by finality unique homomorphism $\parallel : Z \otimes Z \to Z$

associativity: $\zeta$ with $\parallel$ is a semigroup in $\text{Coalg}_F$

compositionality: $\text{beh}(c \otimes d) = \parallel \circ (\text{beh}(c) \otimes \text{beh}(d))$
... $\text{beh}(c)$ is obtained by finality ...
Another $\otimes$-category

the slice category $\mathcal{C}/Z$

objects: arrows $X \to Z$ in $\mathcal{C}$

arrows: $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \downarrow & \downarrow \\ Z & & Z \end{array}$ commuting triangles
Another $\otimes$-category

the slice category $\mathcal{C}/Z$

objects: arrows $X \to Z$ in $\mathcal{C}$

arrows: commuting triangles

is a $\otimes$-category with
Another $\otimes$-category

the slice category $\mathcal{C}/Z$

objects: arrows $X \to Z$ in $\mathcal{C}$

arrows: $\begin{array}{c} X \overset{f}{\longrightarrow} Y \\ \downarrow \hspace{1cm} \downarrow \hspace{1cm} \downarrow \\ Z \end{array}$ commuting triangles

is a $\otimes$-category with

$$(X \overset{f}{\to} Z) \otimes (Y \overset{g}{\to} X) \overset{\text{def}}{=} (X \otimes Y \overset{f \otimes g}{\to} Z \otimes Z \overset{\parallel}{\to} Z)$$
Another $\otimes$-category

the slice category $\mathcal{C}/Z$

**objects:** arrows $X \rightarrow Z$ in $\mathcal{C}$

**arrows:** $\begin{array}{c}
\xymatrix{
X \ar[r]^f & Y \\
& Z \\
Z \ar[ur] &}
\end{array}$ commuting triangles

is a $\otimes$-category with

$$(X \xrightarrow{f} Z) \otimes (Y \xrightarrow{g} X) \overset{\text{def}}{=} (X \otimes Y \xrightarrow{f \otimes g} Z \otimes Z \xrightarrow{\|} Z)$$

compositionality is direct here!
All together...
All together...

we have a commuting diagram of $\boxtimes$-functors between the $\boxtimes$-categories

$$\text{Coalg}_\mathcal{F} \xrightarrow{\text{beh}} \mathcal{C}/\mathcal{Z}$$

$\mathcal{C}$

all with identity as sync-map
In Sets...

bisimilarity is the final coalgebra semantics!
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\[ x \sim y \iff \text{beh}(c)(x) = \text{beh}(d)(y) \]
In Sets...

bisimilarity is the final coalgebra semantics!

\[ x \sim y \iff \text{beh}(c)(x) = \text{beh}(d)(y) \]

compositionality:

\[ x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 | y_1 \sim x_2 | y_2 \]

bisimilarity is a congruence!
In Sets...

bisimilarity is the final coalgebra semantics!

\[ x \sim y \iff \text{beh}(c)(x) = \text{beh}(d)(y) \]

compositionality:

\[ x_1 \sim x_2 \text{ and } y_1 \sim y_2 \implies x_1 \mid y_1 \sim x_2 \mid y_2 \]

bisimilarity is a congruence!

associativity:

\[ x \mid (y \mid z) \sim (x \mid y) \mid z \]
LTS

the parallel composition

\[ x \mid y \xrightarrow{a} x' \mid y' \iff x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c \]

on LTS with finite branching \( X \rightarrow \mathcal{P}_\omega(A \times X) \) is

compositional and associative
LTS

the parallel composition

\[ \frac{x \mid y \xrightarrow{a} x' \mid y'}{x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c} \]

on LTS with finite branching \( X \xrightarrow{c} \mathcal{P}_\omega(A \times X) \) is

compositional and associative

since the LTS functor is a \( \otimes \)-functor with

\[ \text{sync}_{\text{LTS}}(U, V) = \{ \langle l, \langle u, v \rangle \rangle \mid \langle a, u \rangle \in U, \langle b, v \rangle \in V, l = a \cdot b \} \]
PTS

the parallel composition

\[ x \mid y \xrightarrow{p} x' \mid y' \iff p = \sum_{q,r : l=a \cdot b, x \xrightarrow{a[q]} x', y \xrightarrow{b[r]} y'} q \cdot r \]

on PTS with finite branching \( X \xrightarrow{c} \mathcal{D}_\omega(A \times X) \) is

compositional and associative
PTS

the parallel composition

\[ x | y \xrightarrow{\{p\}} x' | y' \quad \iff \quad p = \sum_{q,r: l=a \cdot b} x^a \xrightarrow{\{q\}} x', y^b \xrightarrow{\{r\}} y' \quad q \cdot r \]

on PTS with finite branching \( X \xrightarrow{c} D_\omega(A \times X) \) is

compositional and associative

since the PTS functor is a \( \otimes \)-functor with

\[
\text{sync}_{PTS}(\xi, \psi)(l, \langle x, y \rangle) = \sum_{a,b: l=a \cdot b} \xi(a, x) \cdot \psi(b, y)
\]
In Kleisli category...

trace semantics is the final coalgebra semantics!

[Hasuo, Jacobs, Sokolova - CMCS’06]
In Kleisli category...

traces for $TF$-coalgebras

- monad $T$ - branching type
- functor $F$ - transition type
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traces for $TF$-coalgebras

- monad $T$ - branching type
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works for LTS and PTS (with termination)
In Kleisli category...

traces for $TF$-coalgebras

- monad $T$ - branching type
- functor $F$ - transition type

works for LTS and PTS (with termination)

now: $\mathcal{Kl}(T)$ is a $\otimes$-category, if $T$ is a $\otimes$-monad.
In Kleisli category...

traces for $TF$-coalgebras

- monad $T$ - branching type
- functor $F$ - transition type

works for LTS and PTS (with termination)

now: $\mathcal{K}l(T)$ is a $\boxtimes$-category, if $T$ is a $\boxtimes$-monad.

hence: compositionality of trace semantics!
Let’s generalize
Let’s generalize

tensor $\otimes$ $\implies$ signature $\Sigma$

associativity $\implies$ a set of equations $E$
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tensor $\otimes$ $\implies$ signature $\Sigma$

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$\otimes$-category $\implies$ $(\Sigma, E)$ -category
Let’s generalize

tensor $\otimes$ $\implies$ signature $\Sigma$

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$\otimes$-category $\implies$ $(\Sigma, E)$ -category

$\otimes$-object $\implies$ $(\Sigma, E)$ -object
Let’s generalize

tensor $\otimes$ $\implies$ signature $\Sigma$

associativity $\implies$ a set of equations $E$

$\otimes$-category $\implies$ $(\Sigma, E)$ -category

$\otimes$-object $\implies$ $(\Sigma, E)$ -object

$\otimes$-microcosm model $\implies$ $(\Sigma, E)$ -microcosm model
Let’s generalize

tensor $\otimes \quad \implies \quad$ signature $\Sigma$

associativity $\implies$ a set of equations $E$

$\otimes$-category $\implies (\Sigma, E)$-category

$\otimes$-object $\implies (\Sigma, E)$-object

$\otimes$-microcosm model $\implies (\Sigma, E)$-microcosm model

$\otimes$-functor $\implies (\Sigma, E)$-functor
(Σ, E)-categories

signature Σ, endofunctor Σ = \( \prod_{f \in \Sigma} (\_)^{|f|} \)
$(\Sigma, E)$-categories

signature $\Sigma$, endofunctor $\Sigma = \coprod_{f \in \Sigma}(-)^{|f|}$

$\Sigma$-algebra: $\Sigma A \rightarrow A$, for $f \in \Sigma$, $[f]: A^{|f|} \rightarrow A$, also $[t]$
$(\Sigma, E)$-categories

signature $\Sigma$, endofunctor $\Sigma = \coprod_{f \in \Sigma} (\_)^{|f|}$

$\Sigma$-algebra: $\Sigma A \to A$, for $f \in \Sigma$, $\llbracket f \rrbracket: A^{|f|} \to A$, also $\llbracket t \rrbracket$

$(\Sigma, E)$-cat.: $\Sigma$-algebra in $\text{Cat}$ s.t. for $(s = t) \in E$

there is a nat. iso. $\alpha$:

\[ \begin{array}{c}
\mathbb{C}^n \\
\cong \downarrow \alpha \\
\mathbb{C}
\end{array} \]
$(\Sigma, E)$-objects

on a $(\Sigma, E)$-category $\mathcal{C}$ consider the endofunctor

$$\hat{\Sigma} = \bigsqcup_{f \in \Sigma} [f] \circ \Delta|f|$$
$(\Sigma, E)$-objects

on a $(\Sigma, E)$-category $C$ consider the endofunctor

$$\hat{\Sigma} = \bigsqcup_{f \in \Sigma} [f] \circ \Delta_{|f|}$$

example: $\Sigma = \{\otimes\}$, $\hat{\Sigma}X = X \otimes X$
\((\Sigma, E)\)-objects

on a \((\Sigma, E)\)-category \(C\) consider the endofunctor

\[
\hat{\Sigma} = \bigsqcup_{f \in \Sigma} [f] \circ \Delta_{|f|}
\]

example:

\[
\Sigma = \{\otimes\}, \quad \hat{\Sigma}X = X \otimes X
\]

\((\Sigma, E)\)-object: \(\hat{\Sigma}\)-algebra in \(C\) s.t. for \((s = t) \in E\)

\[
[s](A, \ldots, A) \xrightarrow{\alpha} [t](A, \ldots, A)
\]

\[
[s] \quad \Rightarrow \quad A \quad \Rightarrow \quad [t]
\]
\((\Sigma, E)\)-objects

on a \((\Sigma, E)\)-category \(\mathcal{C}\) consider the endofunctor

\[
\hat{\Sigma} = \bigsqcup_{f \in \Sigma} [f] \circ \Delta_{|f|}
\]

event:

\(\Sigma = \{\otimes\}, \quad \hat{\Sigma}X = X \otimes X\)

\((\Sigma, E)\)-object:

\(\hat{\Sigma}\)-algebra in \(\mathcal{C}\) s.t. for \((s = t) \in E\)

\[
\begin{align*}
[s](A, \ldots, A) & \xrightarrow{\alpha} \Gamma (A, \ldots, A) \\
[s] & \cong A
\end{align*}
\]

microcosm model of \((\Sigma, E)\):

\((\Sigma, E)\)-object in a \((\Sigma, E)\)-category
\((\Sigma, E)\)-functor

is \(F : \mathcal{C} \rightarrow \mathcal{D}\) that forms lax coalgebra homomorphism \(\varphi\):

\[
\begin{array}{c}
\prod_{f \in \Sigma} \mathcal{C}|f| \\
\downarrow
\end{array}
\xrightarrow{\prod_{f \in \Sigma} F|f|}
\begin{array}{c}
\prod_{f \in \Sigma} \mathcal{D}|f| \\
\downarrow
\end{array}

\text{C} \xrightarrow{\varphi} \text{D}
\]

\(F : \mathcal{C} \rightarrow \mathcal{D}\)
\((\Sigma, E)\)-functor

is \(F : \mathcal{C} \rightarrow \mathcal{D}\) that forms lax coalgebra homomorphism \(\varphi\):

\[
\begin{array}{ccc}
\Pi_{f \in \Sigma} \mathcal{C}|f| & \overset{\Pi_{f \in \Sigma} F|f|}{\longrightarrow} & \Pi_{f \in \Sigma} \mathcal{D}|f| \\
\downarrow & \varphi \searrow & \downarrow \\
\mathcal{C} & \underset{F}{\longrightarrow} & \mathcal{D}
\end{array}
\]

i.e. a family of natural transformations, for \(f \in \Sigma\)

\[\varphi^f : \llbracket f \rrbracket F|f| \Rightarrow F\llbracket f \rrbracket\]
\((\Sigma, E)\)-functor

is \(F : \mathbb{C} \to \mathbb{D}\) that forms lax coalgebra homomorphism \(\varphi\):

\[
\begin{array}{ccc}
\coprod_{f \in \Sigma} \mathbb{C} |f| & \xrightarrow{\coprod_{f \in \Sigma} F|f|} & \coprod_{f \in \Sigma} \mathbb{D} |f| \\
\downarrow & \varphi & \downarrow \\
\mathbb{C} & \xrightarrow{F} & \mathbb{D}
\end{array}
\]

i.e. a family of natural transformations, for \(f \in \Sigma\)

\[
\varphi^f : [f] F|f| \Rightarrow F[f]
\]

that commutes with the equations for \((s = t) \in E, \alpha,\)

\[
F \alpha \circ \varphi^s = \varphi^t \circ \alpha
\]
Generalized results
Generalized results

given a \((\Sigma, E)\)-category \(C\) and a \((\Sigma, E)\)-functor \(F\)

the category of \(F\)-coalgebras is a \((\Sigma, E)\)-category

with

\[
[f](\vec{c}_i) \overset{\text{def}}{=} \varphi^f \circ [f]_C(\vec{c}_i)
\]
Generalized results

assume final F-coalgebra \( \zeta : Z \xrightarrow{\cong} FZ \) exists

by finality \([f]_\zeta : [f](Z, \ldots, Z) \to Z\)

equations: \( \zeta \) is a microcosm model in \( \text{Coalg}_{\mathcal{F}} \)

compositionality: \( \text{beh} ([f](\bar{c}_i)) = [f]_\zeta \circ [f]_C (\text{beh}(c_i)) \)
Generalized results

All together we have a commuting diagram of \((\Sigma, E)\)-functors between the \((\Sigma, E)\)-categories

\[ \text{Coalg}_{\mathcal{F}} \xrightarrow{\text{beh}} \mathcal{C}/Z \]

\[ U \quad \text{dom} \]

all with identity natural transformations
Generalized results

For compositionality of trace semantics:

If $T$ is a $(\Sigma, E)$-monad, then $\mathcal{K}(T)$ is a $(\Sigma, E)$-category
Let’s generalize further

$$\Sigma, E \implies \text{Lawvere theory } L$$
Let’s generalize further

\((\Sigma, E)\)  \implies \text{ Lawvere theory } \mathbb{L}

a small FP-category \(\mathbb{L}\) with an FP-functor

\(H : \text{Nat}^{\text{op}} \rightarrow \mathbb{L}\)
Let’s generalize further

\[(\Sigma, E) \implies \text{Lawvere theory } \mathbb{L}\]

a small FP-category \(\mathbb{L}\) with an FP-functor

\[H : \textbf{Nat}^{\text{op}} \rightarrow \mathbb{L}\]

\(\mathbb{L}\)-category: a (pseudo) functor \(\mathbb{L} \xrightarrow{\mathcal{C}} \textbf{Cat}\)
Let’s generalize further

\[(\Sigma, E) \implies \text{Lawvere theory } \mathbb{L}\]

a small FP-category \( \mathbb{L} \) with an FP-functor

\[H : \text{Nat}^{\text{op}} \to \mathbb{L}\]

- \( \mathbb{L} \)-category: a (pseudo) functor \( \mathbb{L} \overset{C}{\to} \text{Cat} \)
- \( \mathbb{L} \)-object: a lax natural transformation

\[
\begin{array}{ccc}
\mathbb{L} & \xleftarrow{1} & \mathbb{L} \\
\downarrow & & \downarrow \\
C & \xrightarrow{X} & \text{Cat}
\end{array}
\]
Let’s generalize further

\((\Sigma, E) \implies \text{Lawvere theory } L\)

a small FP-category \(L\) with an FP-functor

\(H : \text{Nat}^{\text{op}} \to L\)

\(L\)-category: a (pseudo) functor \(L \xrightarrow{C} \text{Cat}\)

\(L\)-object: a lax natural transformation

\[\begin{array}{ccc}
L & \xrightarrow{1} & \downarrow X \\
\downarrow X & \Rightarrow & \downarrow X \\
C & \xrightarrow{2} & \text{Cat}
\end{array}\]

All results hold - even in greater generality!
Conclusion

- nested structure: microcosm models
Conclusion

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  [on different levels of abstraction]
Conclusion

- nested structure: microcosm models
  [on different levels of abstraction]
- structured base category and compatible functor yields structure on coalgebras
Conclusion

- nested structure: microcosm models
  [on different levels of abstraction]
- structured base category and compatible functor yields structure on coalgebras
  * process operations satisfy equations
Conclusion

- nested structure: microcosm models
  [on different levels of abstraction]
- structured base category and compatible functor yields structure on coalgebras
  - process operations satisfy equations
  - compositionality