Coalgebra
Love and Beauty in Science

Ana Sokolova

UNIVERSITY of SALZBURG

Logic Mentoring Workshop 2019

Do you know any coalgebra?

Do you know any coalgebra?

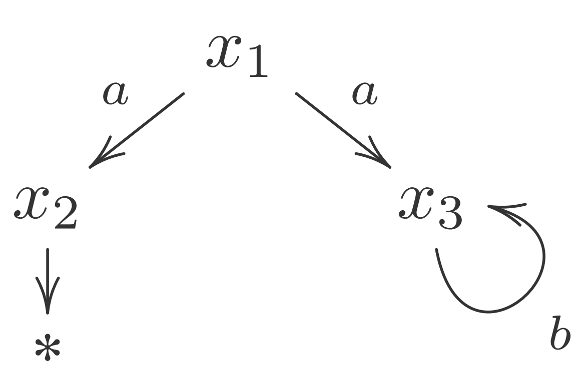
Yes, you know many coalgebras !

Some coalgebras

Some coalgebras

NFA

$X \to 2 \times (\mathcal{P}X)^A$



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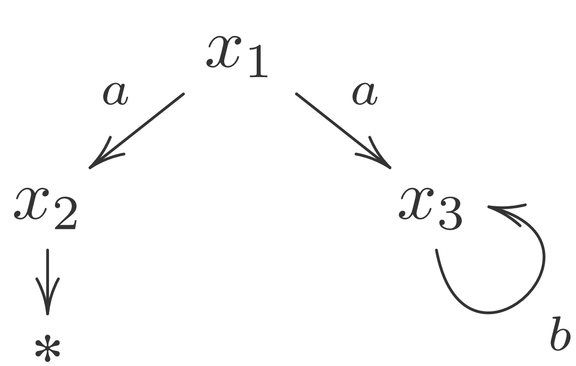
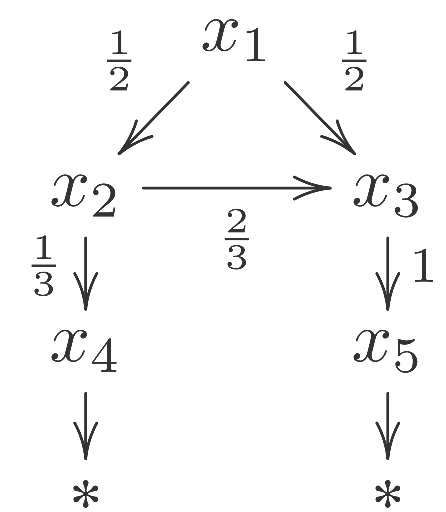


MC

$X \to \mathcal{D}X + 1$



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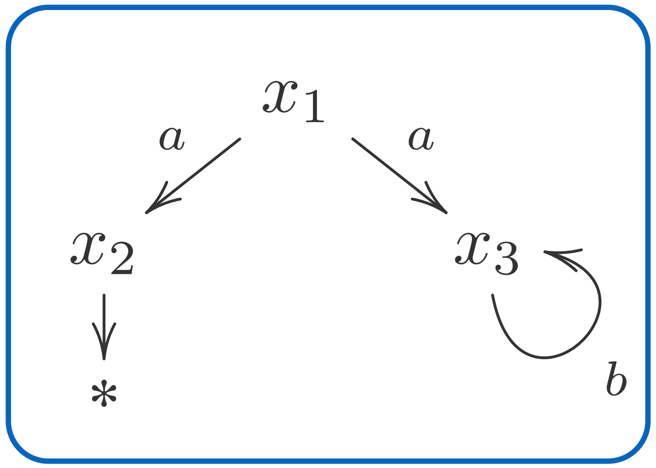

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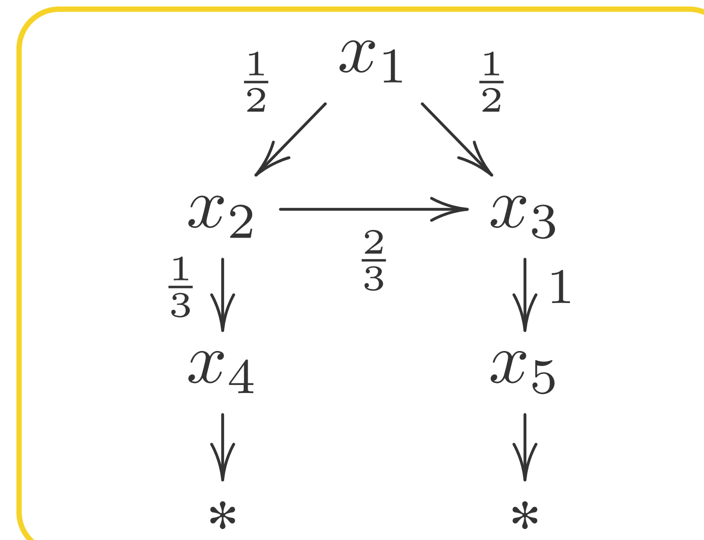
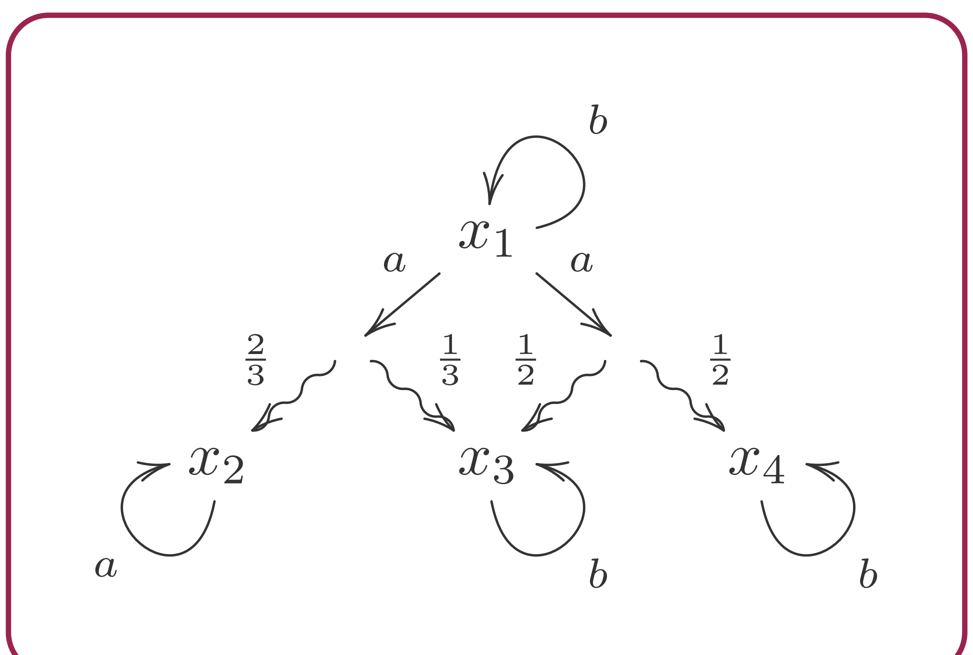


PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



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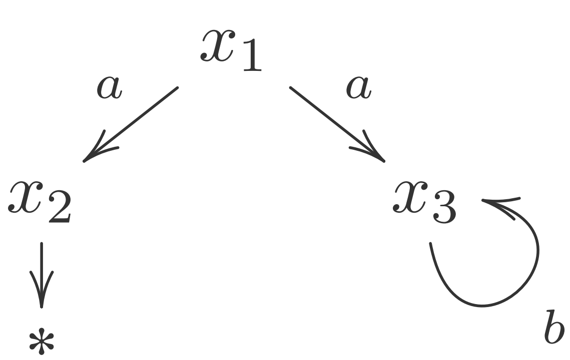

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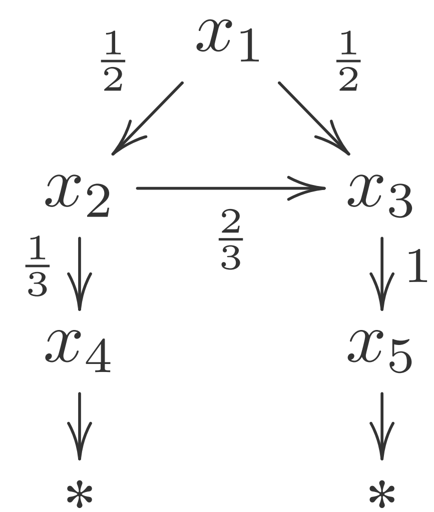

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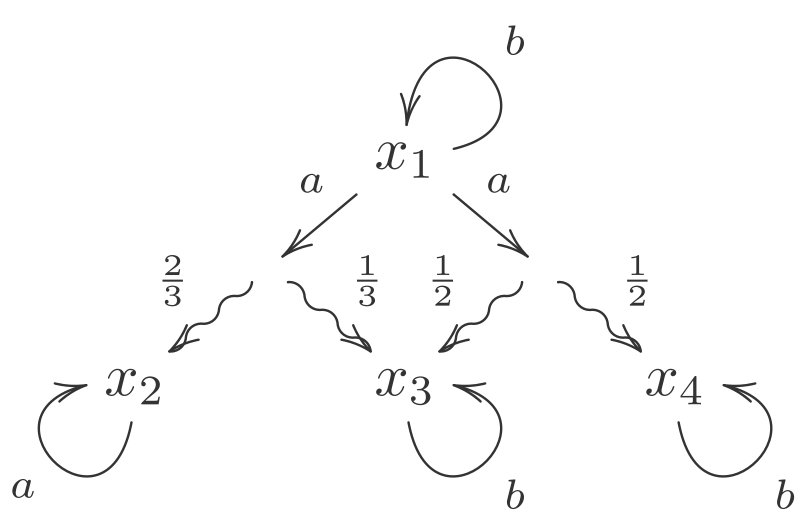
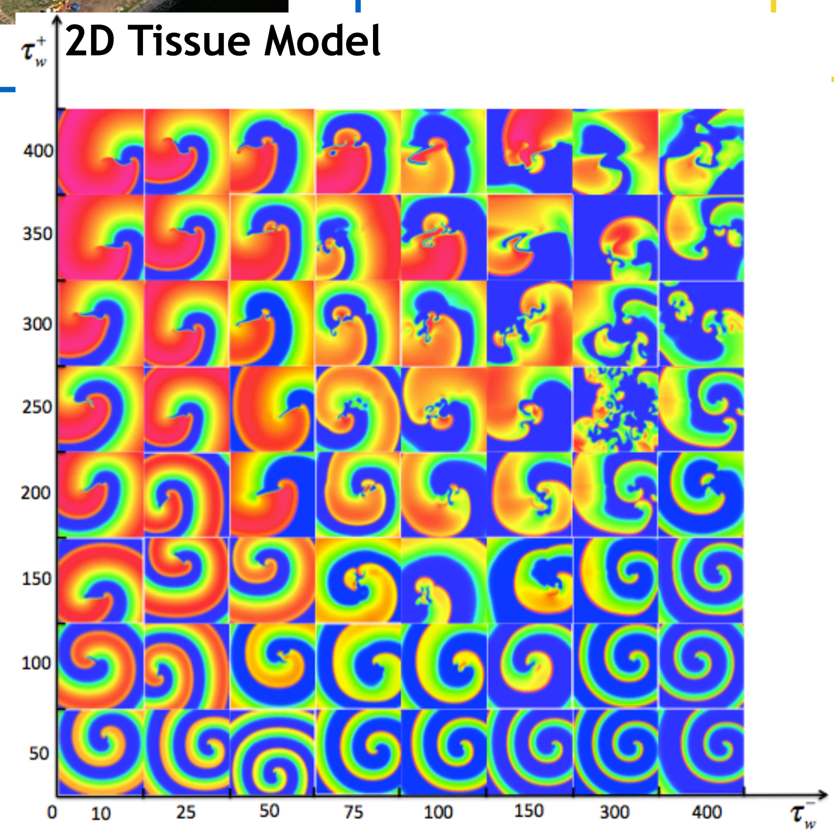


Various transitions systems / automata are coalgebras

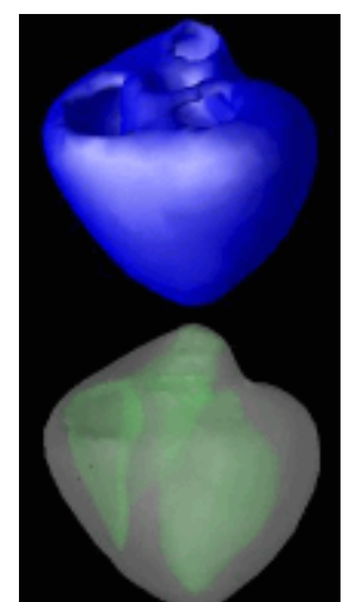
Where do they live ?

Where do they live ?

2D Tissue Model



3D Organ Model



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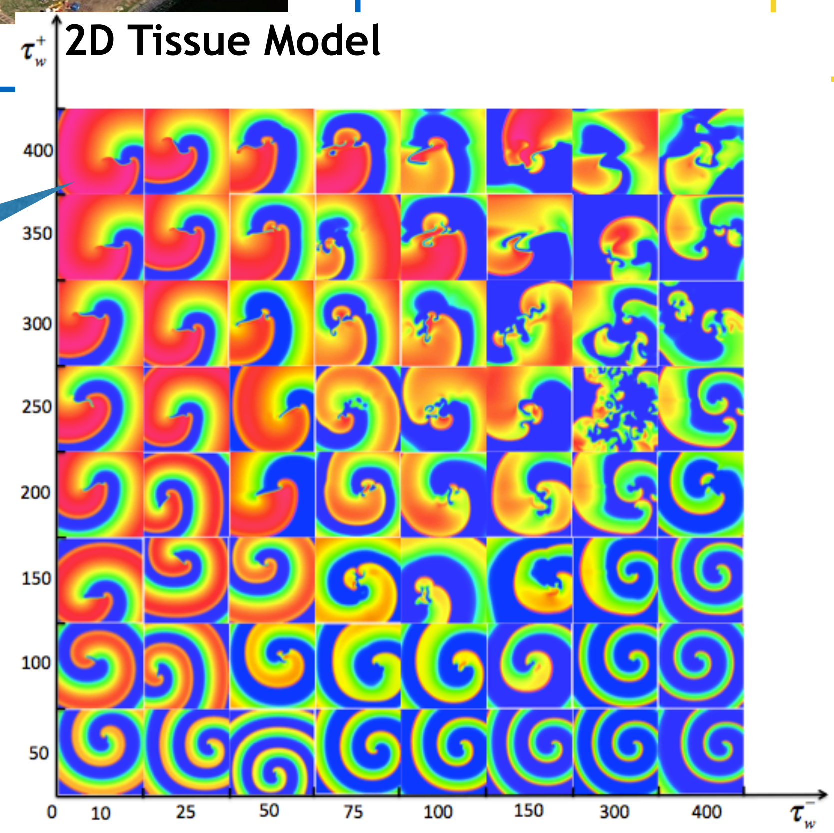

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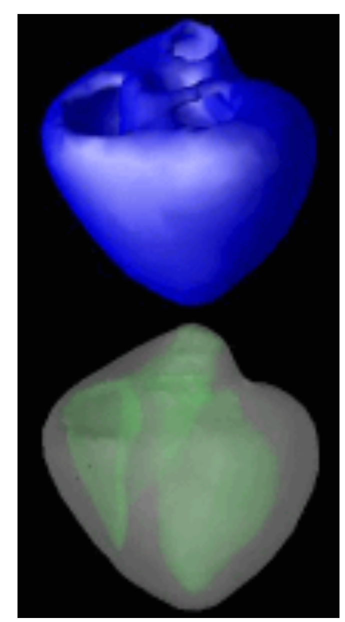

Bartocci et al.
TCS09, CAV11

Where do they live ?

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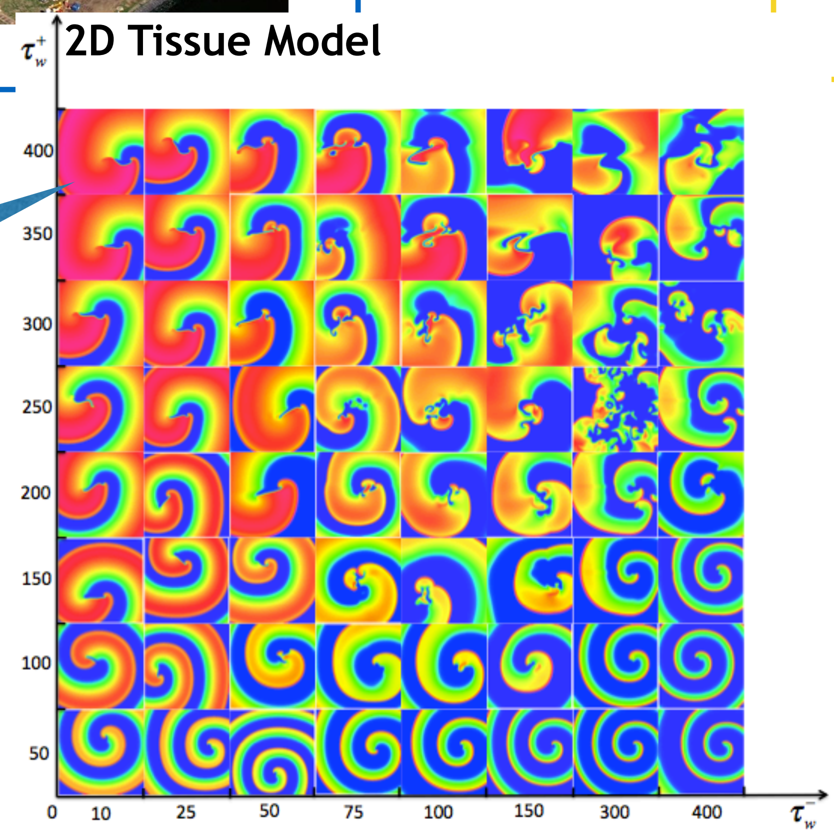

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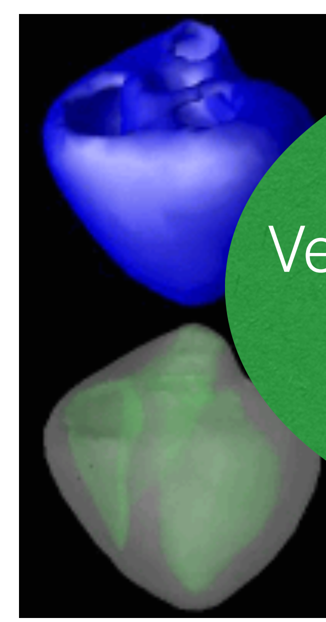

Bartocci et al. TCS09, CAV11

Verification requires clear semantics

Where do they live ?

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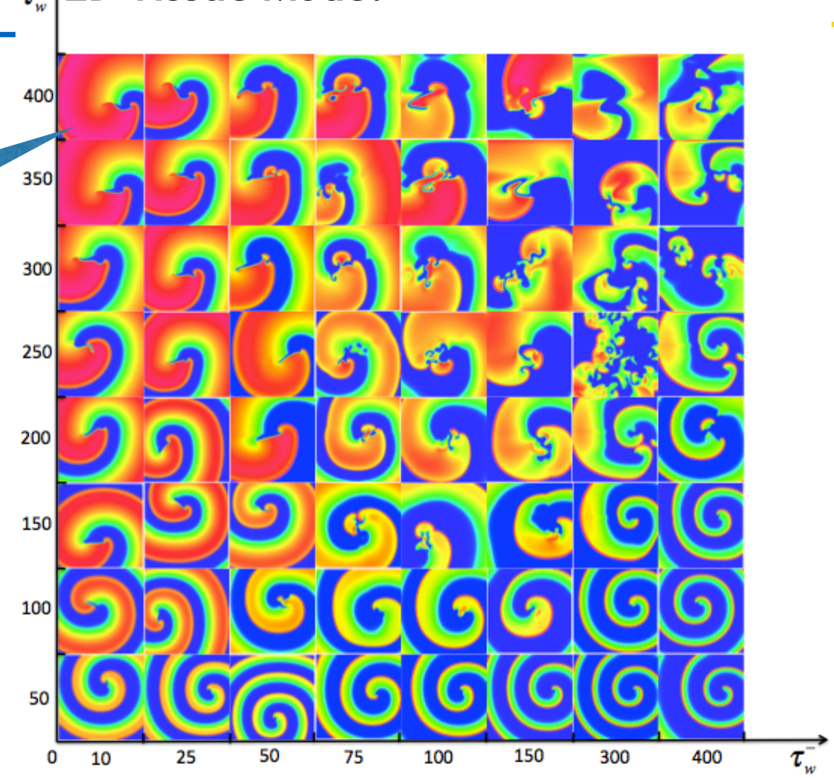

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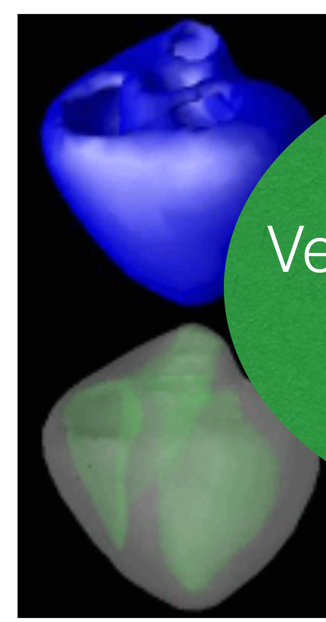

Verification requires clear semantics

and suffers from state-space explosion

Behavioural Equivalences

NFA

$X \to 2 \times (\mathcal{P}X)^A$

Behavioural Equivalences

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$X \to 2 \times (\mathcal{P}X)^A$

language equivalence

Behavioural Equivalences

NFA

$X \rightarrow 2 \times (\mathcal{P}X)^A$

Two states are equivalent iff the languages recognised from these two states are the same.

language equivalence

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bisimilarity

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language equivalence

An equivalence relation $R \subseteq X \times X$ is a bisimulation of the NFA $(o, n) \colon X \to 2 \times (\mathcal{P}X)^A$ iff whenever $(x, y) \in R$, we have $o(x) = o(y)$ and for all $a \in A$

$$x \xrightarrow{a} x' \quad \Rightarrow \quad \exists y'. y \xrightarrow{a} y' \wedge (x', y') \in R.$$

Bisimilarity, denoted by $\sim$, is the largest bisimulation.

bisimilarity

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R

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$\bullet \quad R \quad \bullet$

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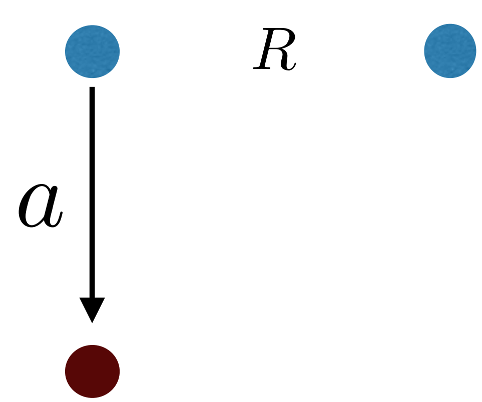

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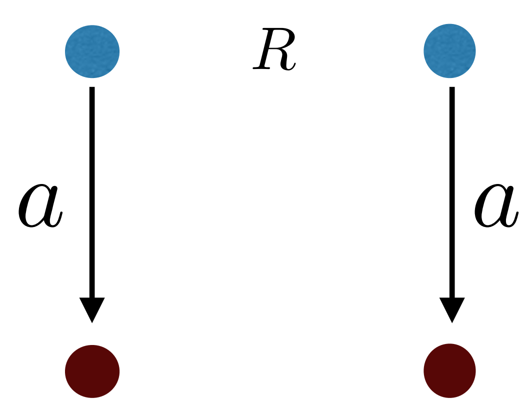

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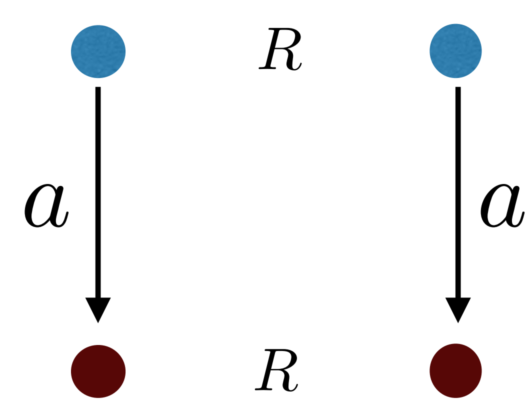

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bisimulation

$$\begin{tikzcd} \bullet \arrow[r, phantom, "R"] \arrow[d, "a"'] & \bullet \arrow[d, "a"] \\ \bullet \arrow[r, phantom, "R"] & \bullet \end{tikzcd}$$

largest bisimulation

bisimilarity

Behavioural Equivalences

MC

$$X \to \mathcal{D}X + 1$$

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bisimilarity

Behavioural Equivalences

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An equivalence relation $R \subseteq X \times X$ is a bisimulation of the MC $c \colon X \to \mathcal{D}X + 1$ iff whenever $(x, y) \in R$, then either $c(x) = c(y) = *$ or for all R-equivalence classes C we have

$$\sum_{z \in C} c(x)(z) = \sum_{z \in C} c(y)(z).$$

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Why are they both called bisimilarity ?

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bisimilarity

What do they have in common ?

Behavioural Equivalences

MC

$X \to \mathcal{D}X + 1$

bisimilarity

Behavioural Equivalences

MC

$X \to \mathcal{D}X + 1$

bisimulation

R

bisimilarity

Behavioural Equivalences

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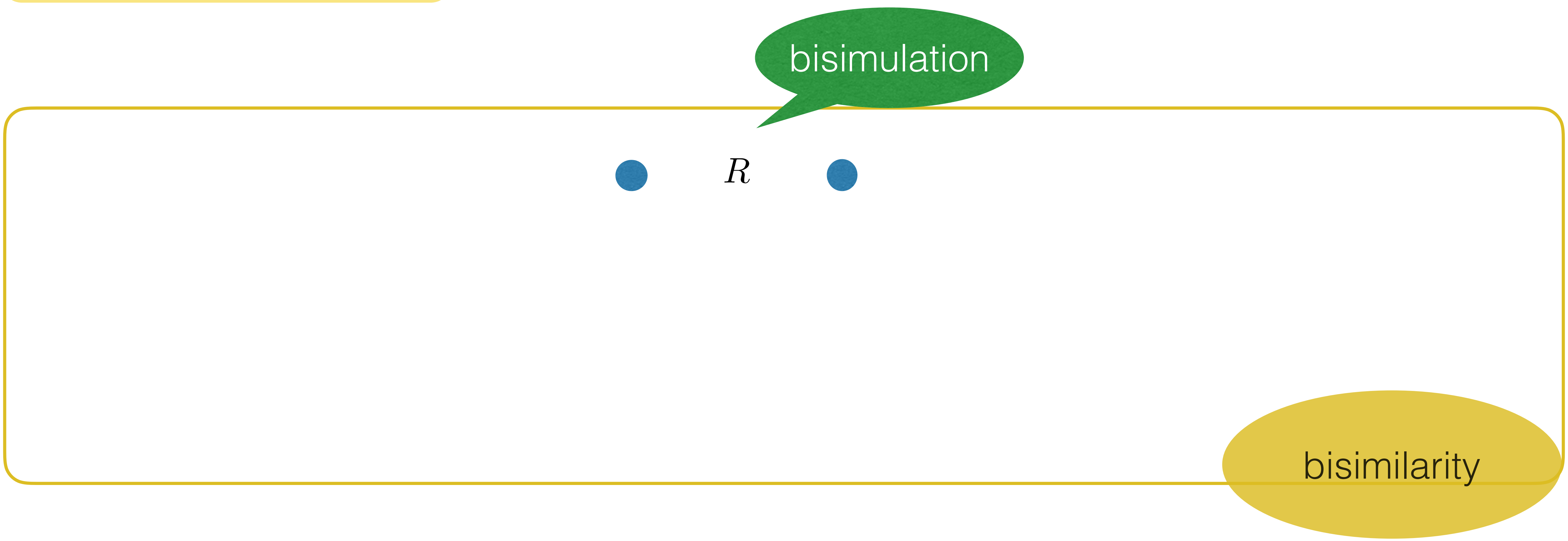

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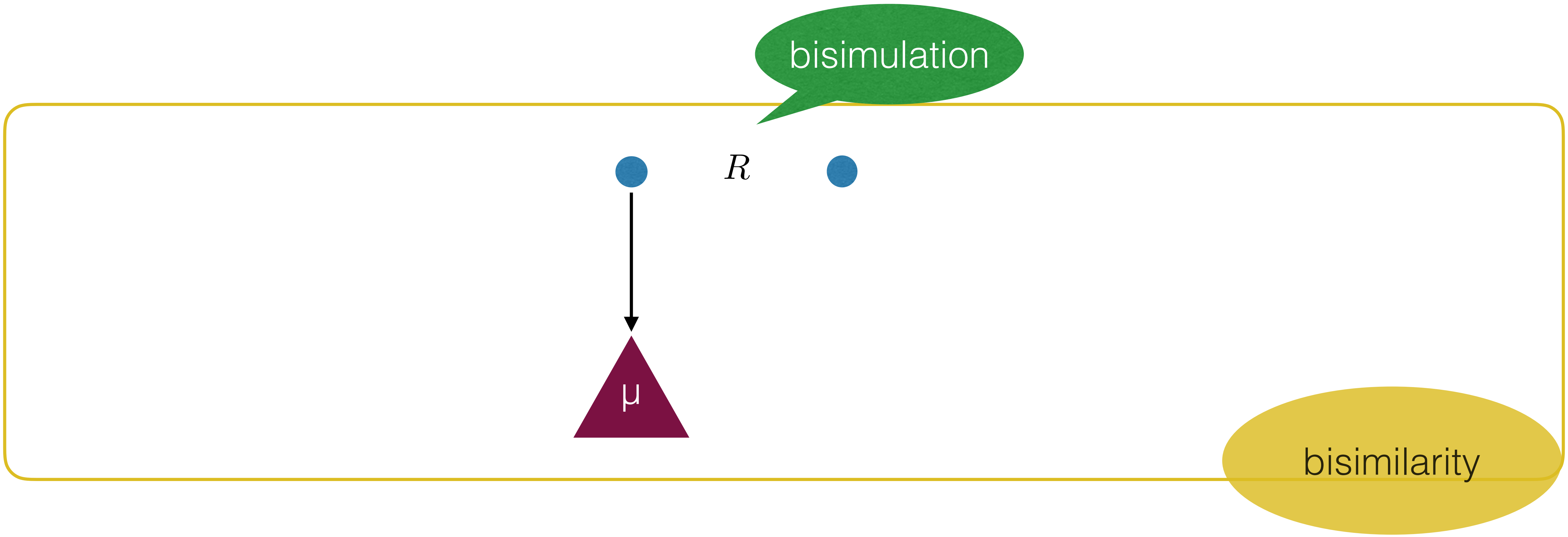

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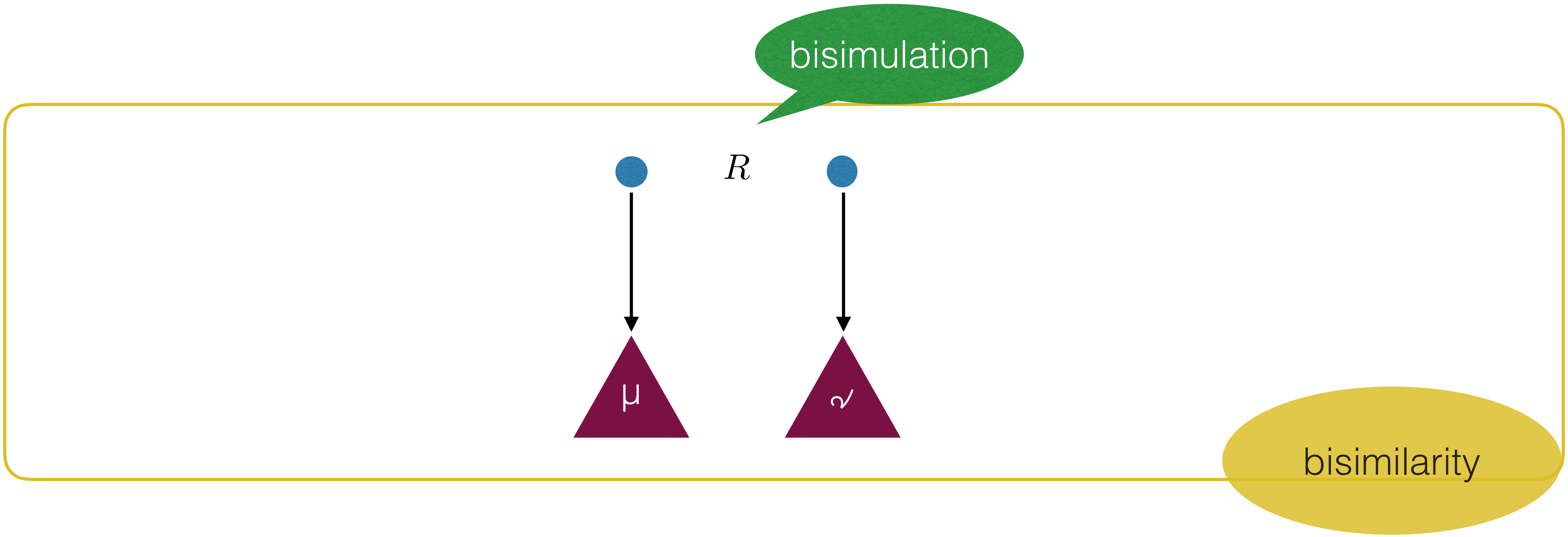

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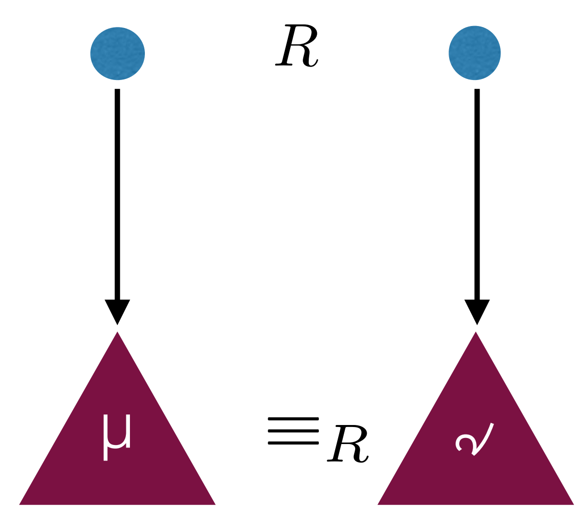

bisimilarity

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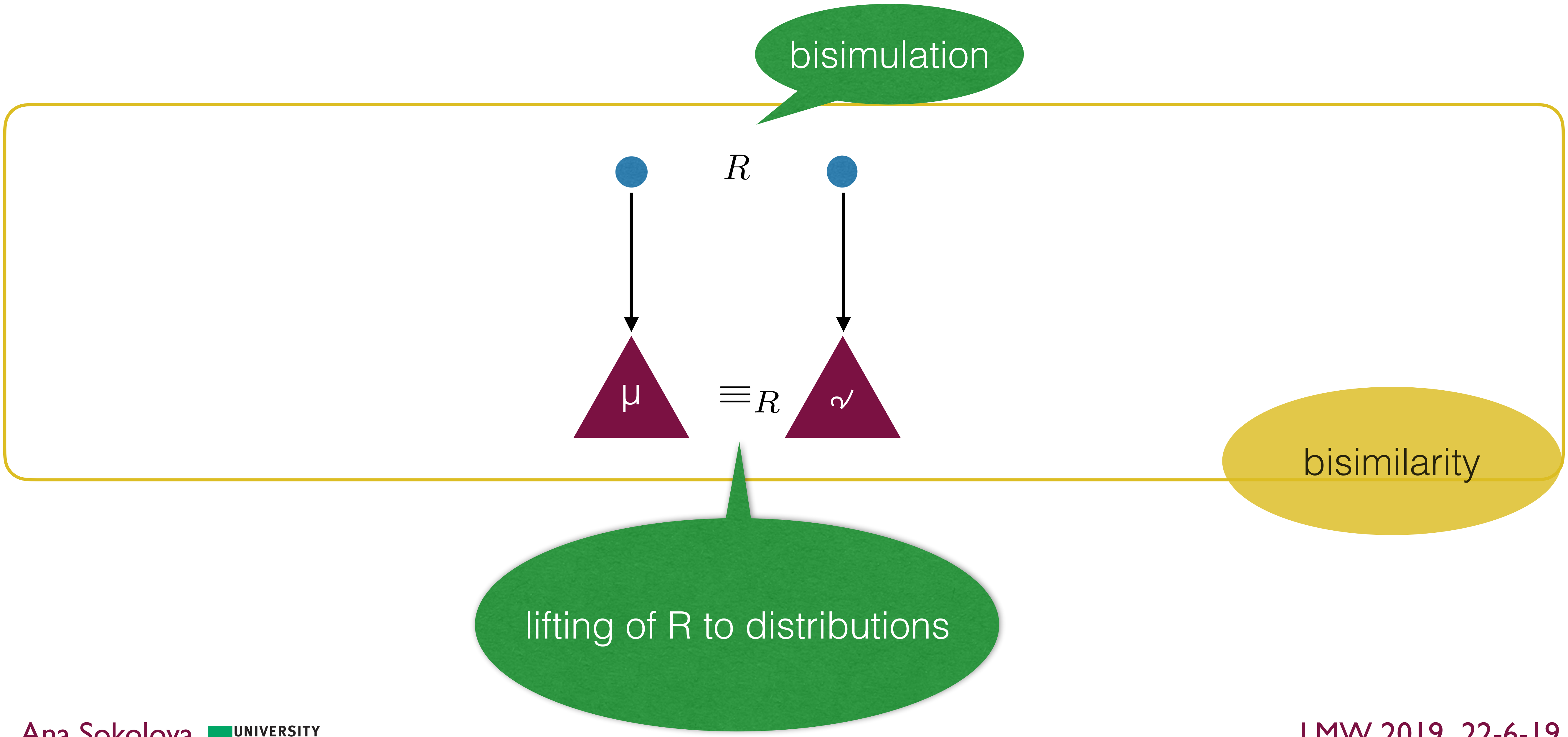

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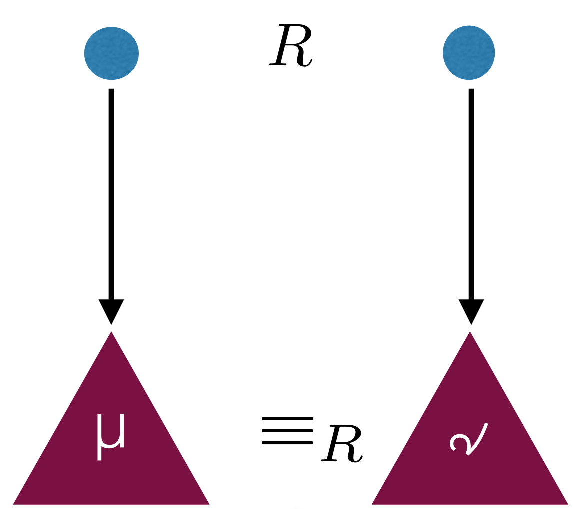

bisimilarity

lifting of R to distributions

assign the same probability to "R-classes"

Behavioural Equivalences

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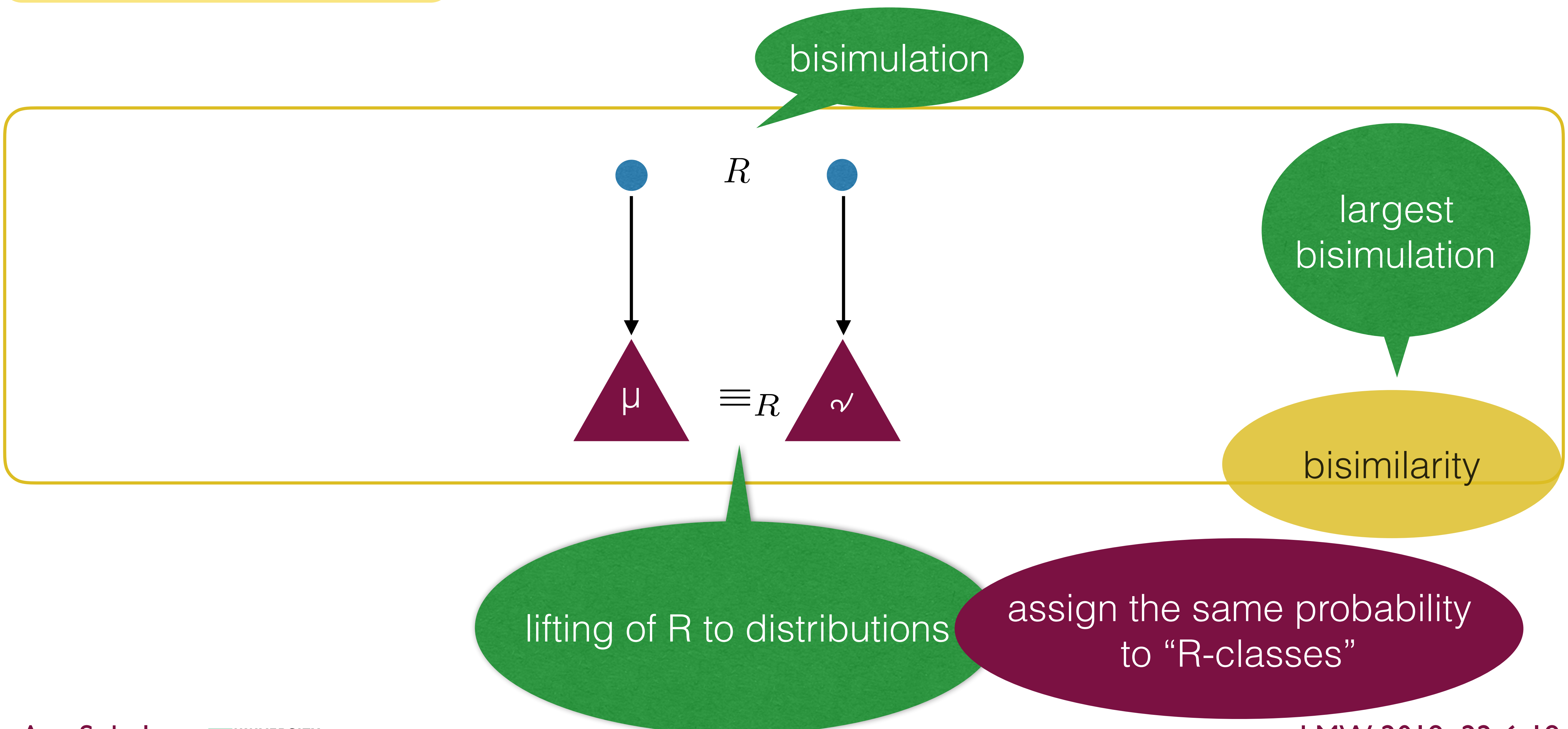

Coalgebra

Uniform framework for dynamic transition systems, based on category theory.

A coalgebra is generic transition system:

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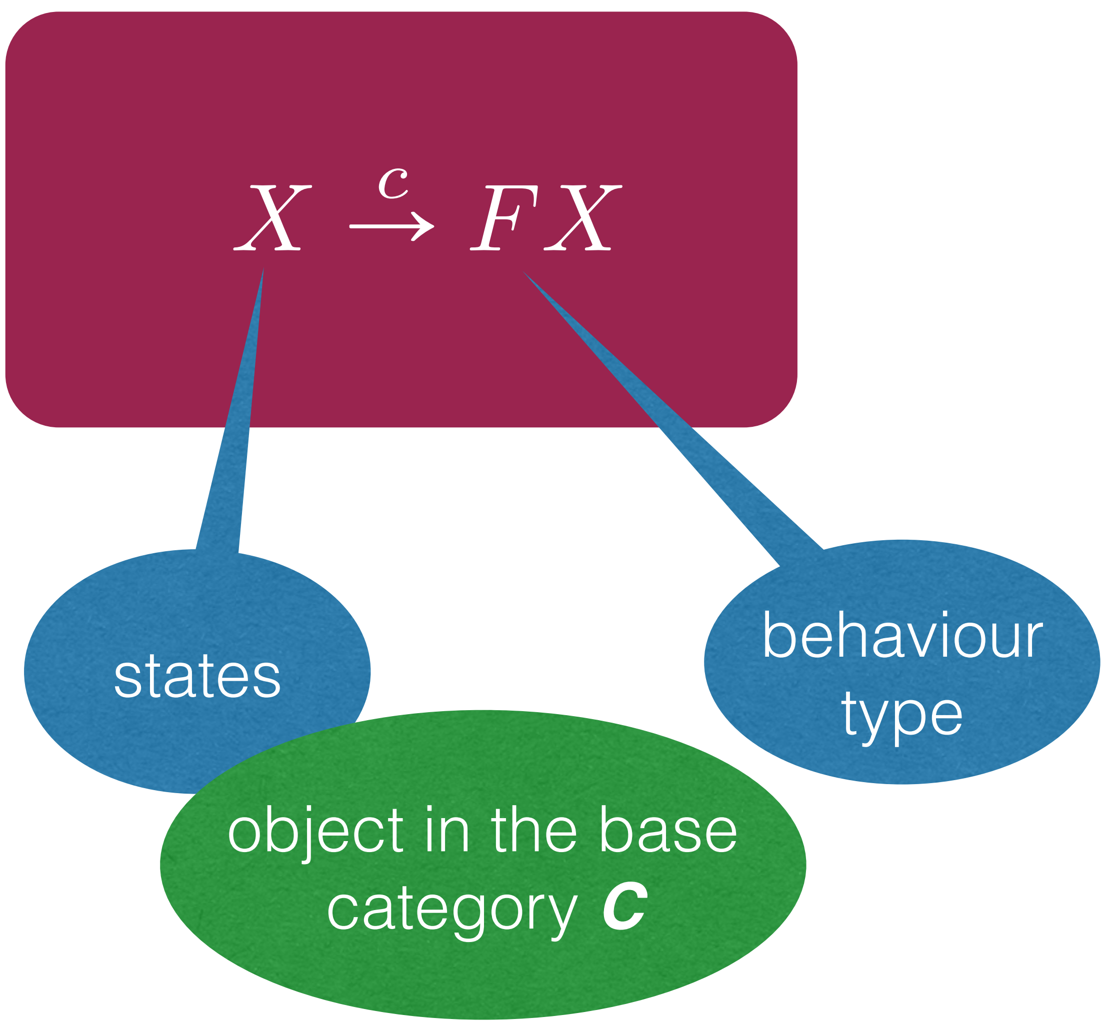

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behaviour type

functor on the base category ***C***

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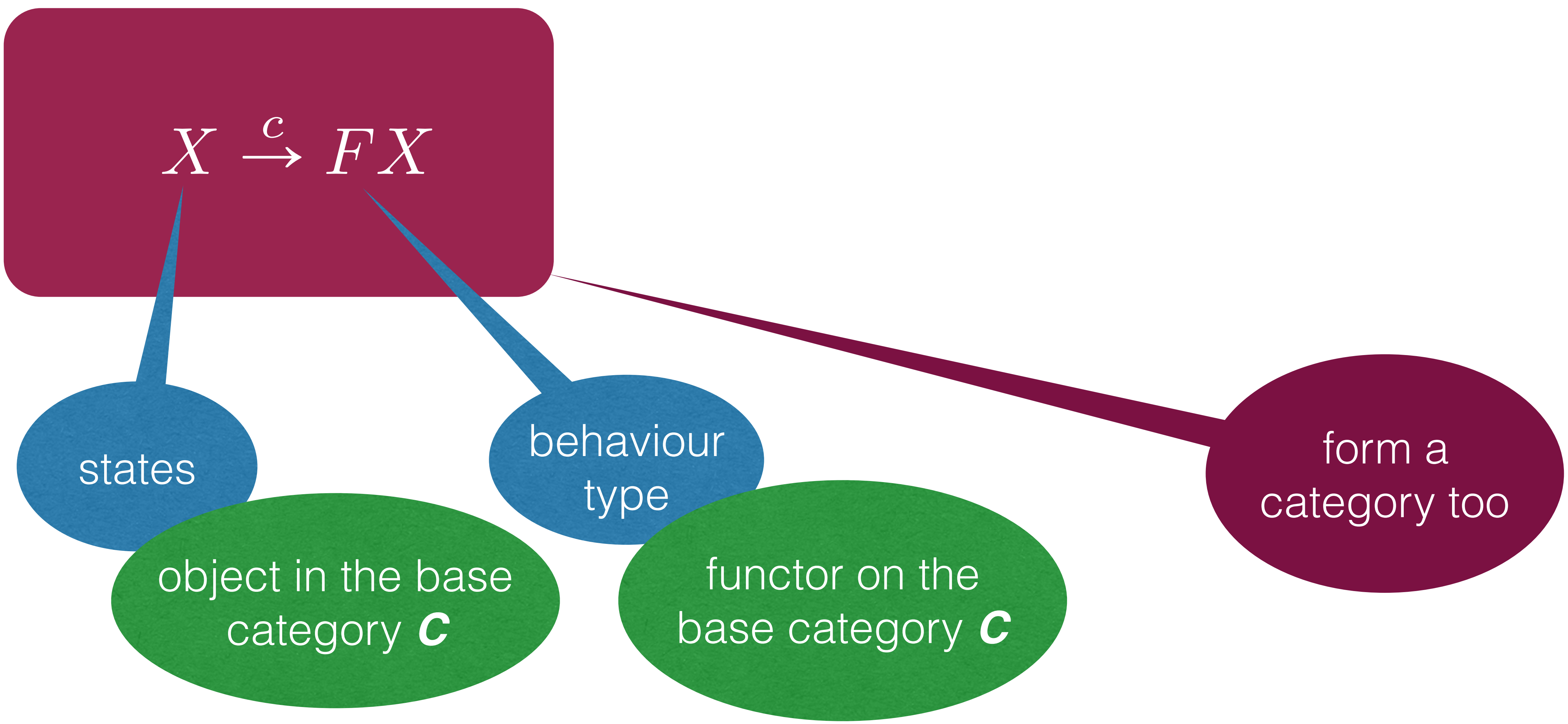

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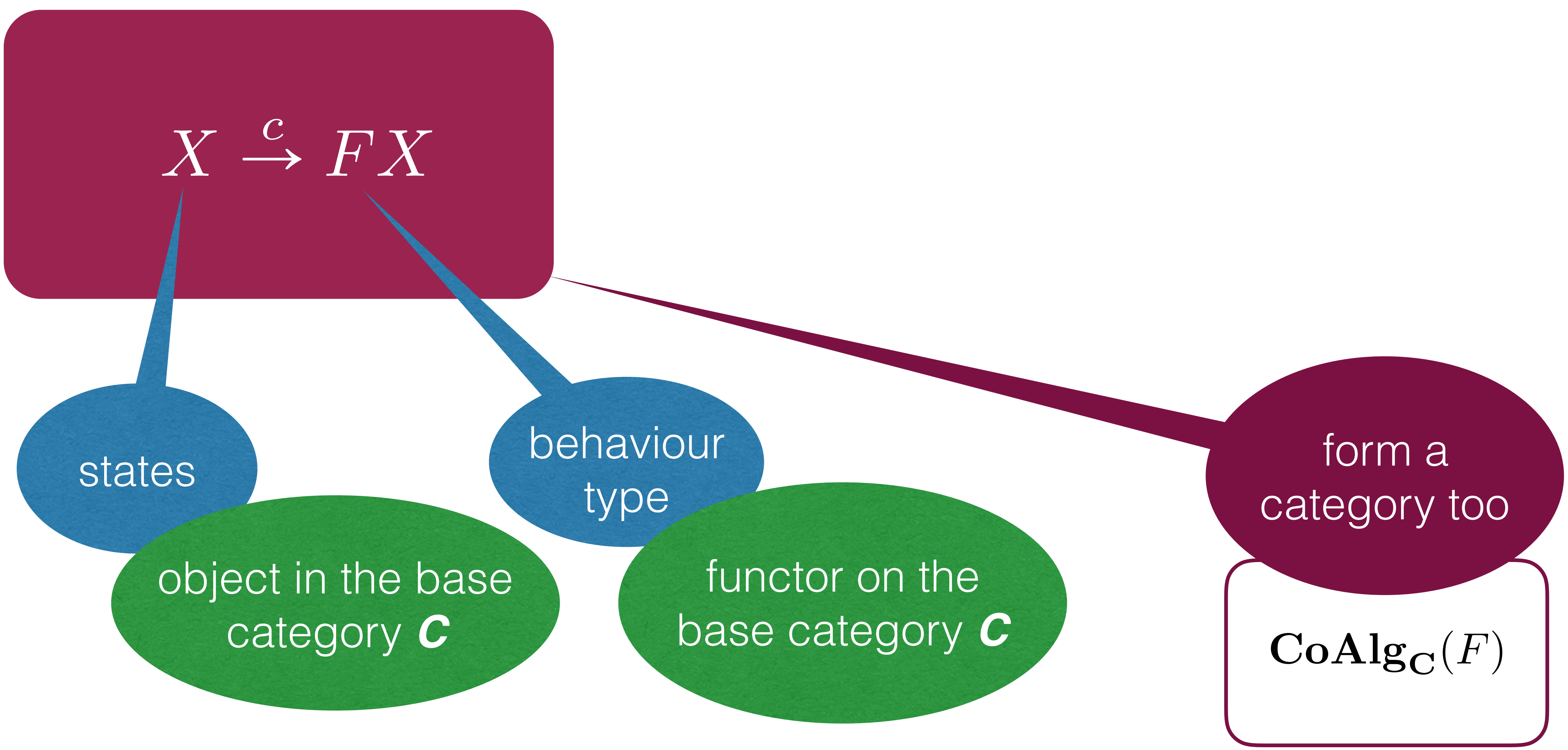

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states

behaviour type

object in the base category ***C***

functor on the base category ***C***

form a category too

$$\mathbf{CoAlg}_{\mathbf{C}}(F)$$

generic notion of behavioural equivalence

$\approx$

The category of F-coalgebras

$\mathbf{CoAlg_C}(F)$

Objects = coalgebras

Arrows = coalgebra homomorphisms

Two states $x, y \in X$ are behaviourally equivalent, notation $x \approx y$ iff there exists a coalgebra homomorphism $h \colon X \to Y$ from $c \colon X \to FX$ to some coalgebra $d \colon Y \to FY$ such that $h(x) = h(y)$.

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$$h\colon X \to Y$$

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "c_X"'] & Y \arrow[d, "c_Y"] \\ FX \arrow[r, "Fh"'] & FY \end{tikzcd}$$

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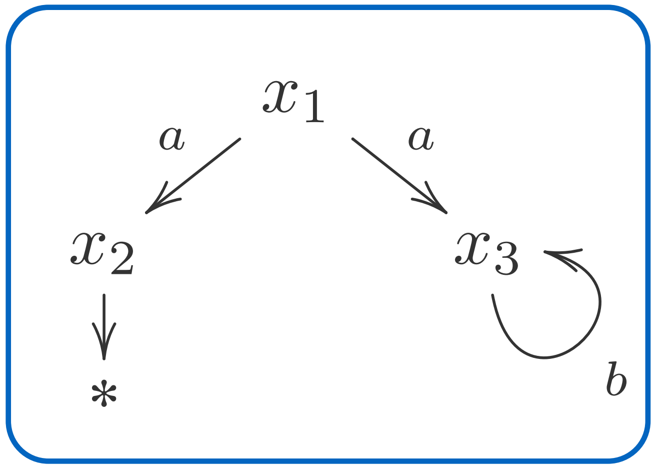
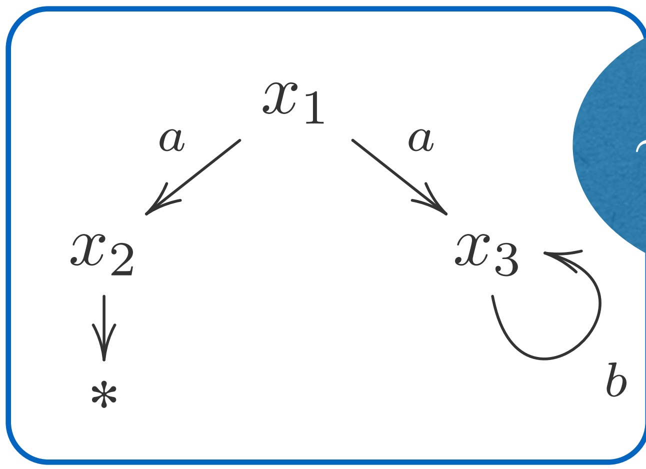


Behavioural equivalence is bisimilarity

NFA

X → 2 x ($\mathcal{P}$X)^A



$\sim \; = \; \approx$

Behavioural equivalence is bisimilarity

NFA

$$X \to 2 \times (\mathcal{P}X)^A$$

$$\begin{tikzcd}
& x_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
x_2 \arrow[d] & & x_3 \arrow[loop right, "b"] \\
* & &
\end{tikzcd}$$

$$\sim \; = \; \approx$$

MC

$$X \to \mathcal{D}X + 1$$

$$\begin{tikzcd}
& x_1 \arrow[dl, "\frac{1}{2}"'] \arrow[dr, "\frac{1}{2}"] & \\
x_2 \arrow[rr, "\frac{2}{3}"'] \arrow[d, "\frac{1}{3}"'] & & x_3 \arrow[d, "1"] \\
x_4 \arrow[d] & & x_5 \arrow[d] \\
* & & *
\end{tikzcd}$$

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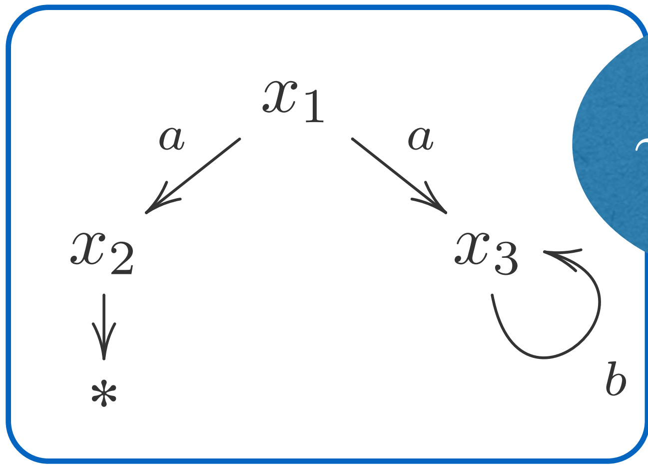

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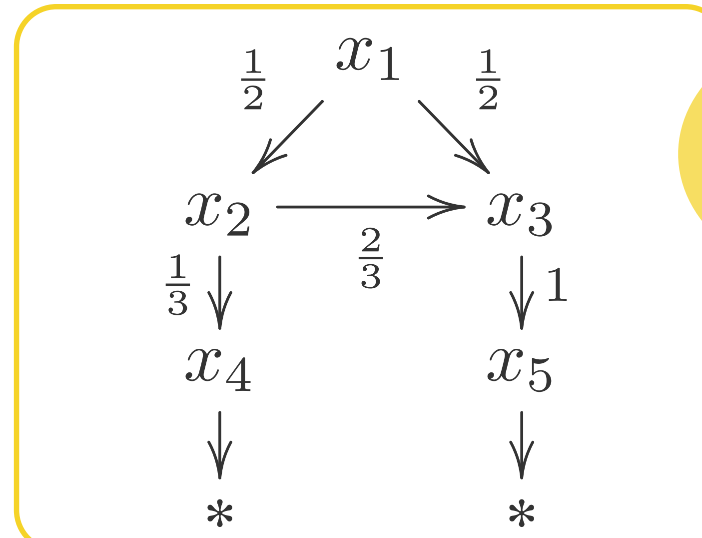

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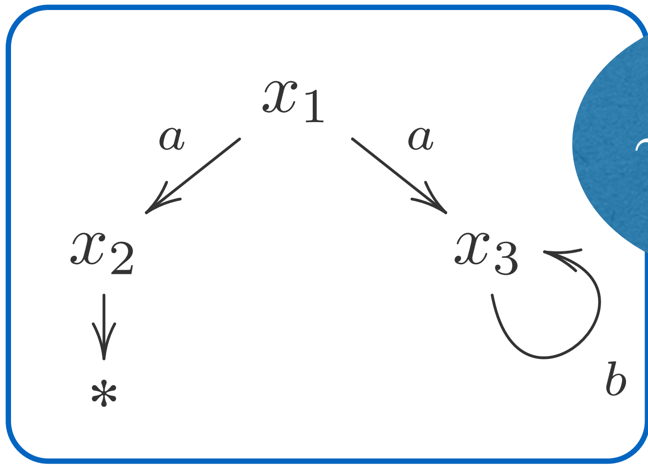

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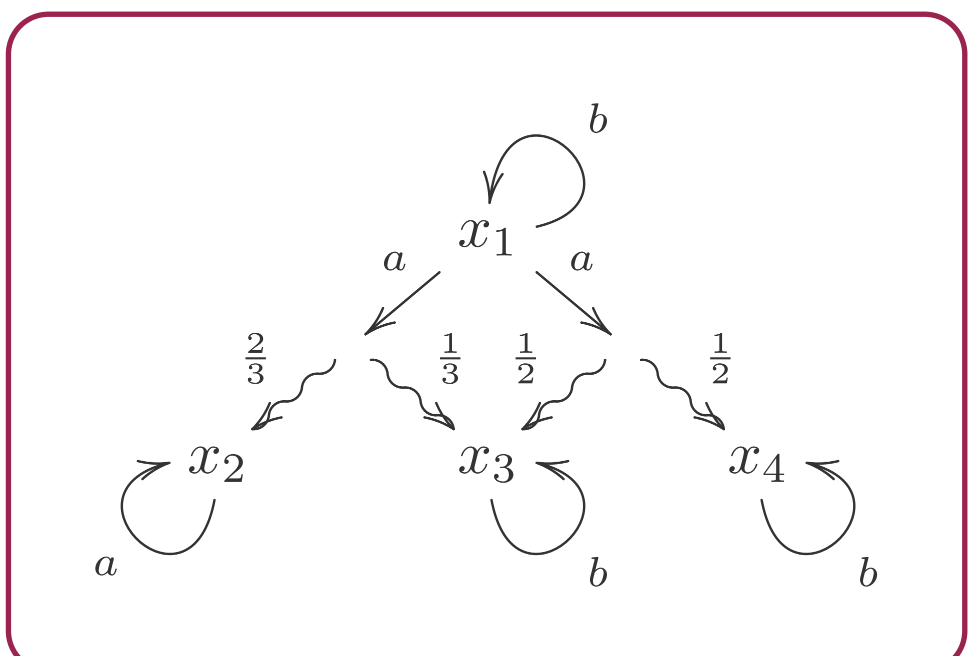

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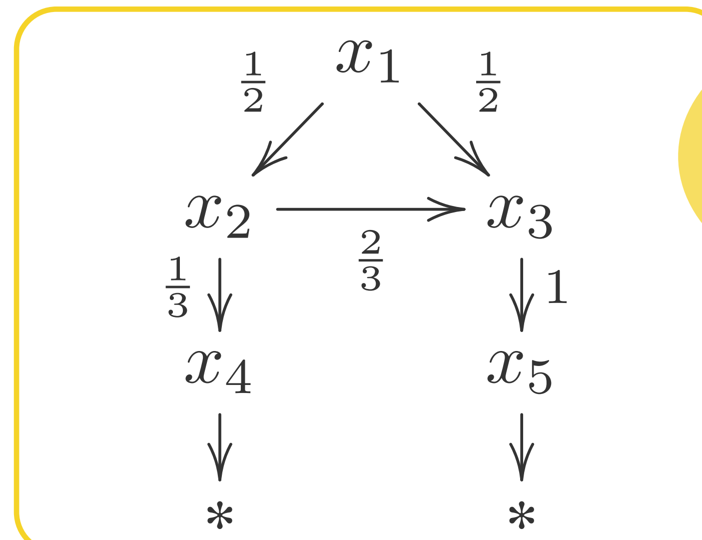

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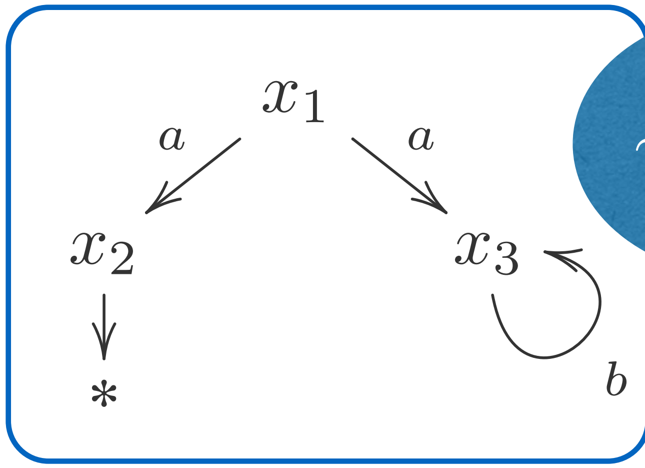

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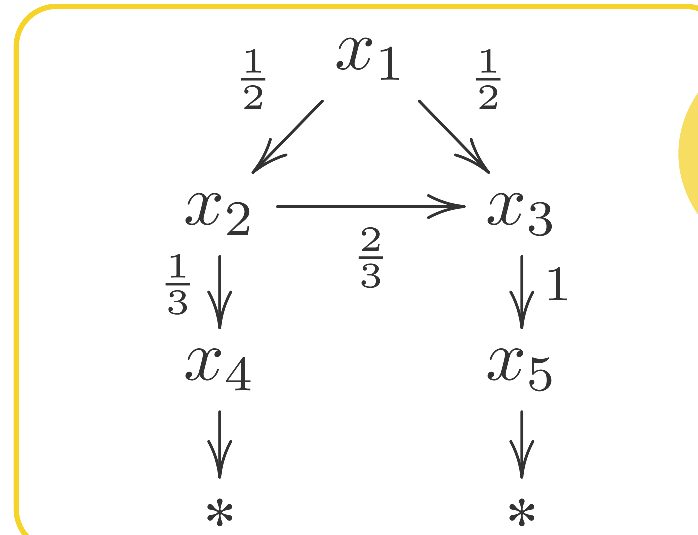

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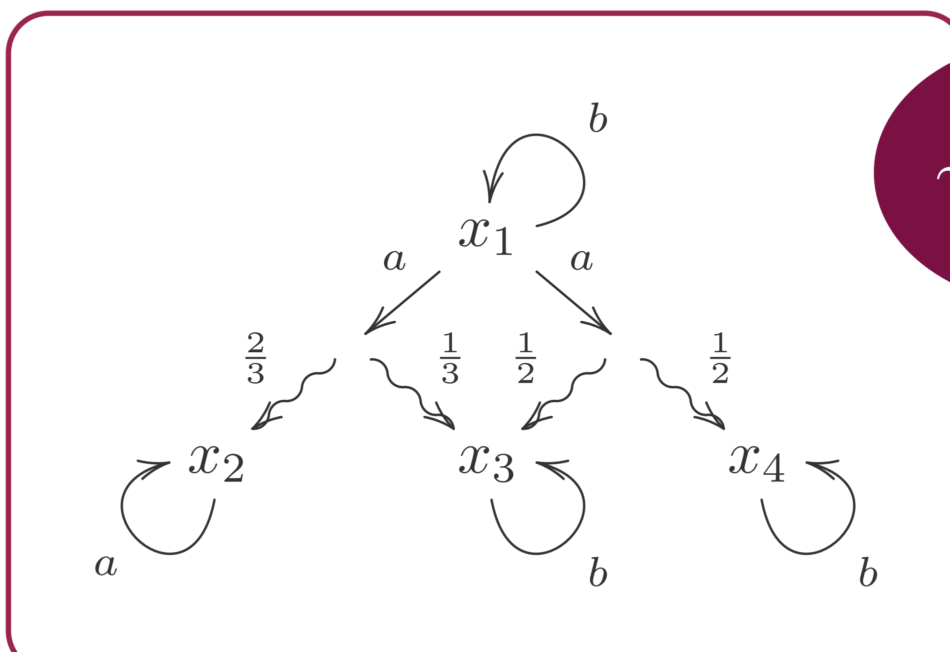

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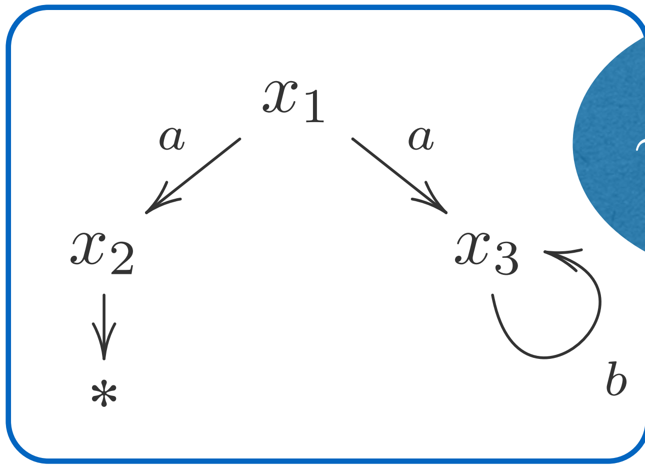
NFA

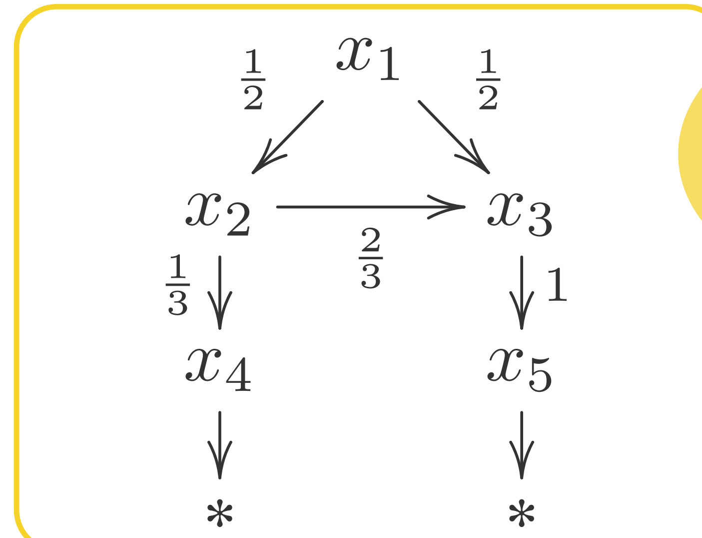
$X \to 2 \times (\mathcal{P}X)^A$



$\sim \; = \; \approx$

MC

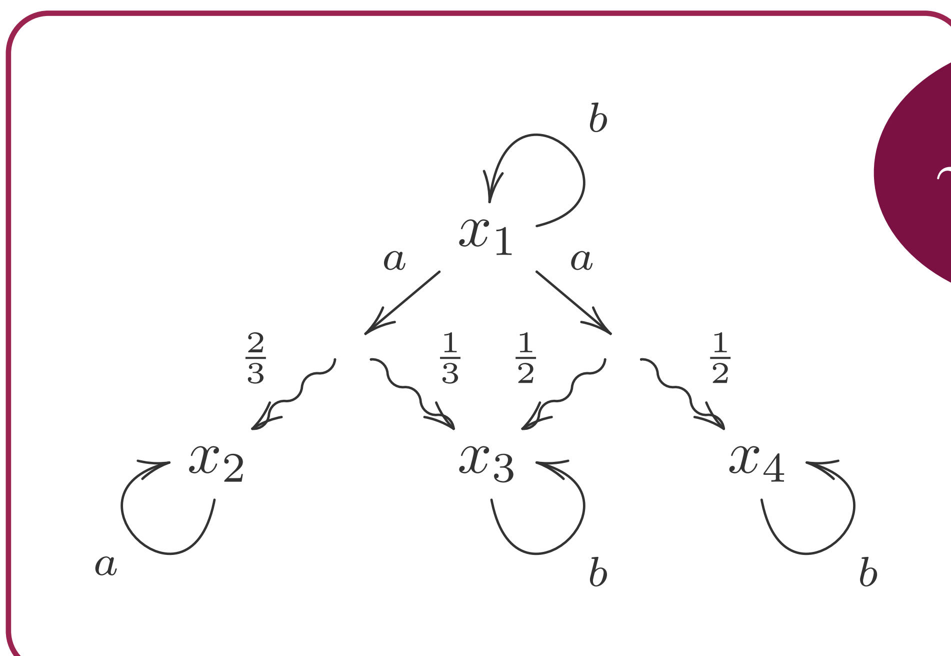
$X \to \mathcal{D}X + 1$



$\sim \; = \; \approx$

PA

$X \to (\mathcal{P}\mathcal{D}X)^A$



$\sim \; = \; \approx$

on **Sets**

Isn't that beautiful ?

Isn't that beautiful ?

if yes, read Rutten and Jacobs!

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and come to my talk tomorrow at WiL

Isn't that beautiful ?

if yes, read Rutten and Jacobs!

and come to my talk tomorrow at Wil

and to the talk of our LICS paper on Wednesday

Beyond coalgebra

Beyond coalgebra

What is the best about doing science ?

What I love about doing science

relevance

it's rewarding

elegance

striving for perfection

communicating

novelty

creativity

community

beauty

open

challenging

meaningful

work with people

explaining

freedom

work alone

discovering

integrity

joy

Let’s bring some order here

communicating			challenging		open
	freedom			with people	
joy		striving for perfection		beauty	relevant
	meaningful		alone		

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How to survive the PhD time and further until tenure ?

- Do what you do best and love
- Choose relevant topics
- Mix topics
- Learn from masters
- Dare to be independent
- Leave a trace
- There is plenty to learn: Exchange roles
- Do not worry much
- Have fun

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