Coalgebraic behaviour via coinduction

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Joint work with Ichiro Hasuo RIMS, KU, JP and Bart Jacobs RUN, NL

Outline

- introduction - formal methods, models and semantics
- from LTS to coalgebras
- Bisimilarity can’t be traced, BUT
  - * bisimilarity via coinduction in $\mathbf{Sets}$
  - * trace semantics also via coinduction...

Formal methods

are mathematically based techniques for

- specification
- development
- verification

of software and hardware systems

Formal methods

In general:

- models - transition systems, automata, terms,...
  with a clear semantics
- analysis - model checking,
  theorem proving,
  process algebra,...

Formal methods

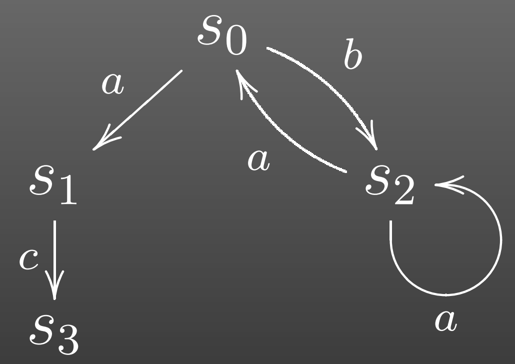
Here:

- models - transition systems, coalgebras
- analysis - via behavior semantics

Aim: One framework for many models and semantics !

Standard model - LTS

labelled transition systems A - labels



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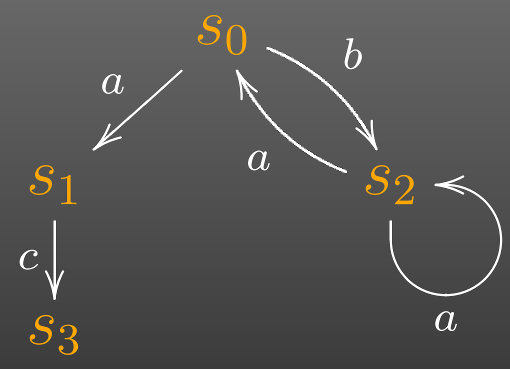

states S $+$ transitions $\alpha : S \to \mathcal{P}(A \times S)$

$$\alpha(s_0) = \{\langle a, s_1\rangle, \langle b, s_2\rangle\},\ \alpha(s_1) = \{\langle c, s_3\rangle\},\ \ldots$$

Behavior semantics

are used for verification

- behavior equivalence ($\equiv$) identifies states with same behavior
- behavior preorder ($\sqsubseteq$) orders states according to behavior

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there are many of them: bisimilarity, trace, ...

Behavior semantics

verification amounts to:

- given
  - * Sys - model of the system, LTS
  - * Spec - specification, LTS

Behavior semantics

verification amounts to:

- given
  - * Sys - model of the system, LTS
  - * Spec - specification, LTS
- verify if

  Sys $\equiv$ Spec or Sys $\sqsubseteq$ Spec

Bisimulation - LTS

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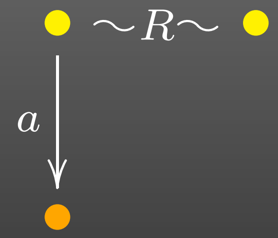

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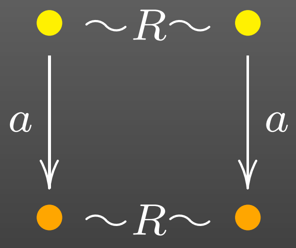

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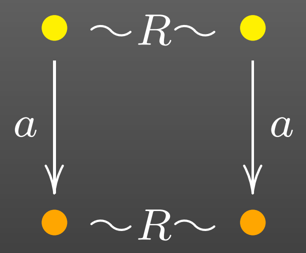

Transfer condition: $\langle s, t\rangle \in R \quad \Longrightarrow$

$$s \xrightarrow{a} s' \Rightarrow (\exists t')\ t \xrightarrow{a} t',\ \langle s', t'\rangle \in R$$

Bisimulation - LTS

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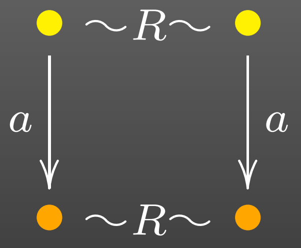

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Bisimulation - LTS

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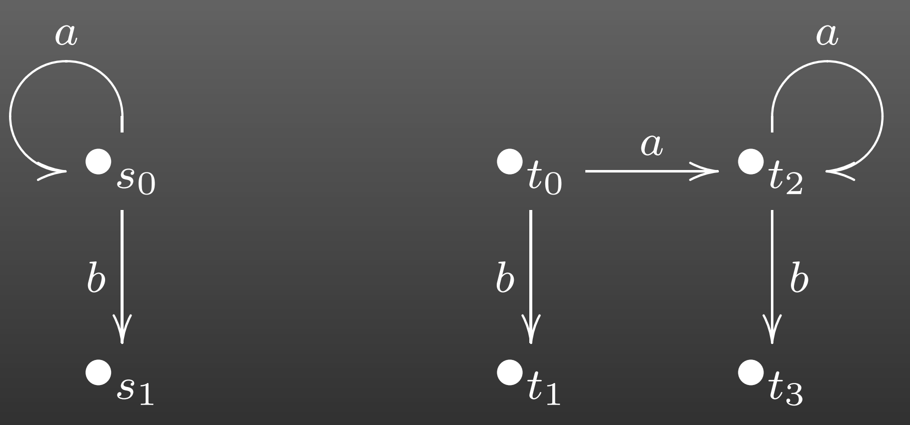

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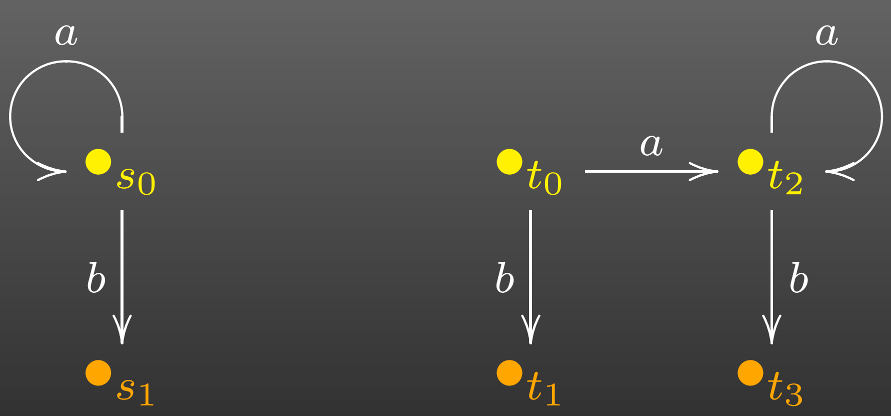


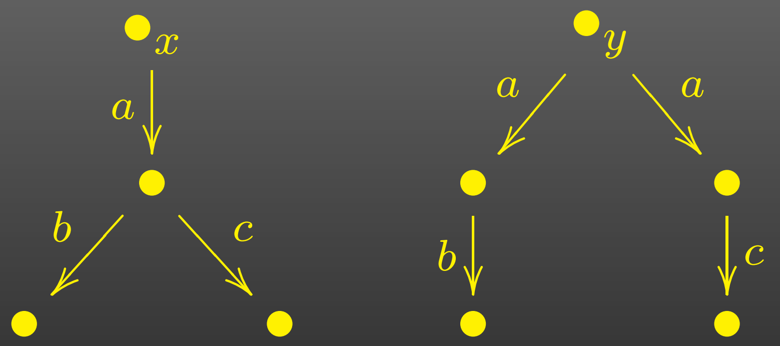
the coloring is a bisimulation, so s_0 and t_0 are bisimilar

LT/BT spectrum

Bisimilarity is not the only semantics

LT/BT spectrum

Are these LTSs equivalent?



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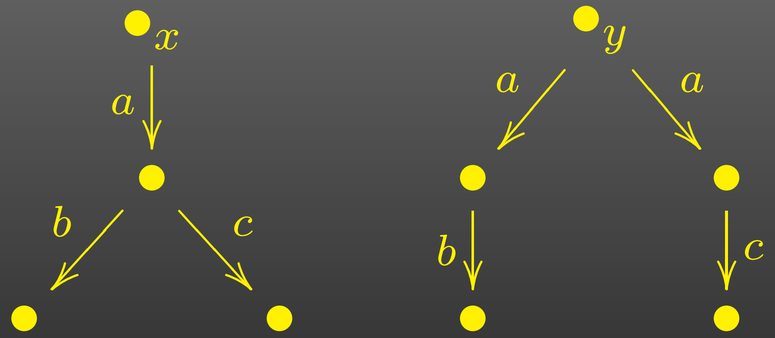

x and y are:

- different wrt. bisimilarity

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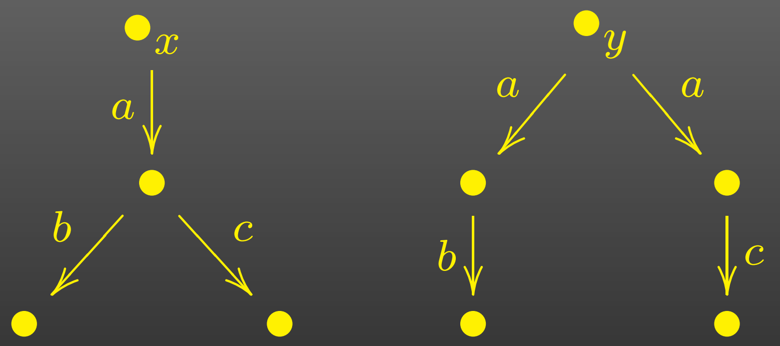

x and y are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

Traces - LTS with $\checkmark$

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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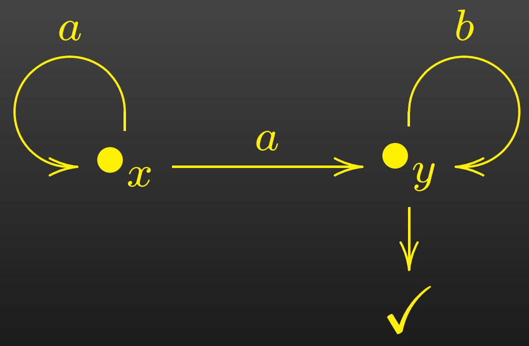
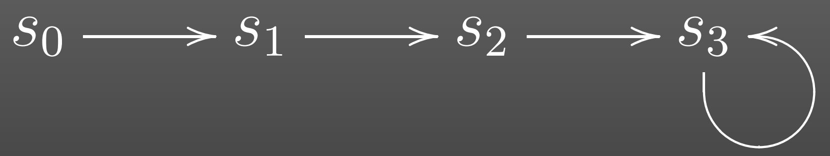
Example:



$$\mathrm{tr}(y) \;=\; b^*, \qquad \mathrm{tr}(x) \;=\; a^+ \cdot \mathrm{tr}(y) \;=\; a^+ \cdot b^*$$

Other models

deterministic systems



Other models

deterministic systems

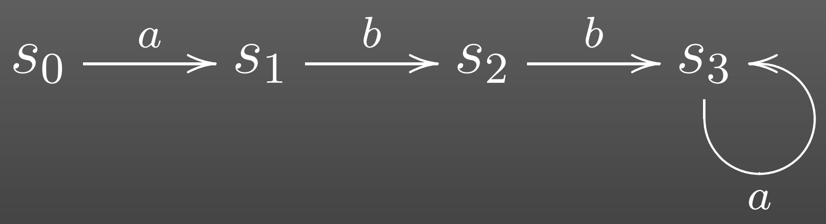
$$s_0 \longrightarrow s_1 \longrightarrow s_2 \longrightarrow s_3 \circlearrowleft$$

states S + transitions $\alpha : S \to S$

$$\alpha(s_0) = s_1,\ \alpha(s_1) = s_2,\ \ldots$$

Other models

labelled deterministic systems A - labels



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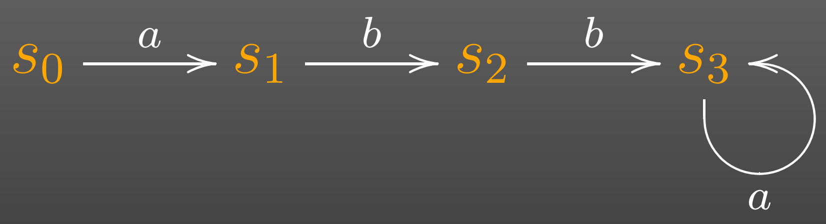
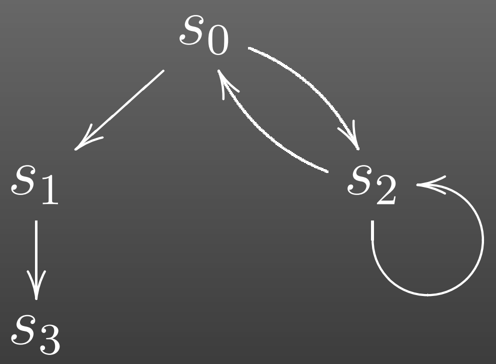


states S $+$ transitions $\alpha : S \to A \times S$

$$\alpha(s_0) = \langle a, s_1 \rangle,\ \alpha(s_1) = \langle b, s_2 \rangle,\ \ldots$$

Other models

transition systems



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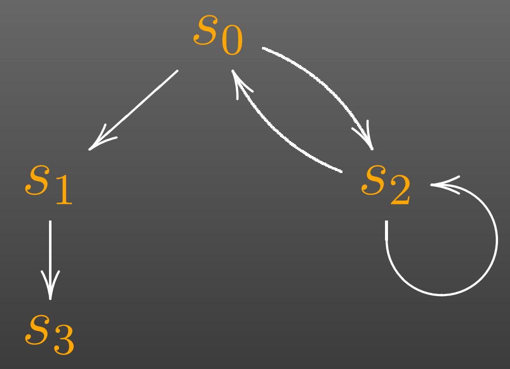

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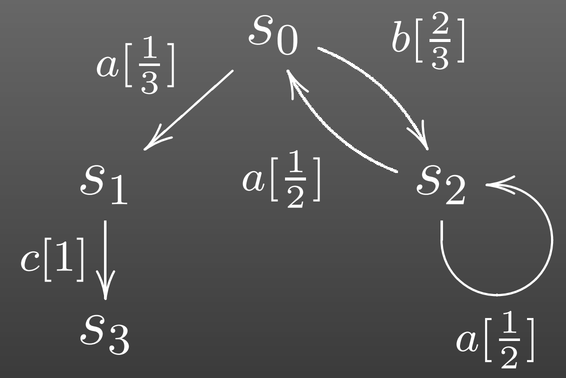
generative probabilistic systems A - labels



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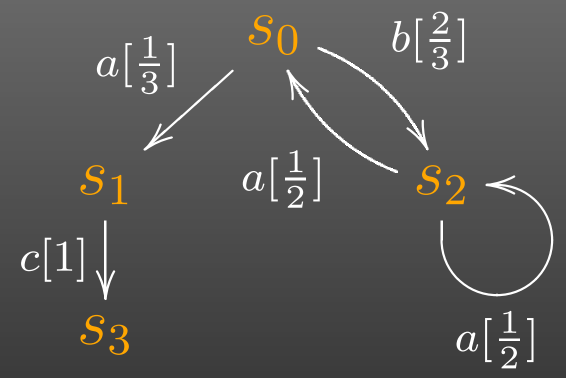

states S $+$ transitions $\alpha : S \to \mathcal{D}(A \times S) + 1$

$$\alpha(s_0) = \left(\langle a, s_1\rangle \mapsto \tfrac{1}{3}, \langle b, s_2\rangle \mapsto \tfrac{2}{3}\right),$$
$$\alpha(s_1) = \left(\langle c, s_3\rangle \mapsto 1\right), \ \ldots$$

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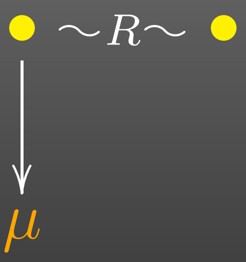

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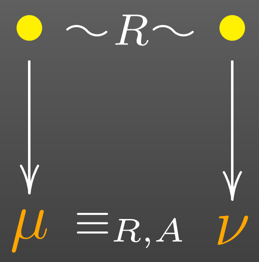

$\equiv_{R,A}$ relates distributions that assign the same probability to each label and each R-class

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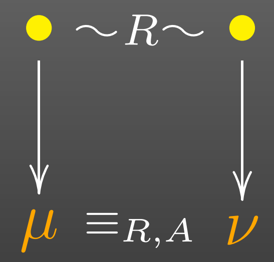

Transfer condition: $\langle s,t\rangle \in R \quad \Longrightarrow$

$$s \to \mu \Rightarrow \; t \to \nu, \; \mu \equiv_{R,A} \nu$$

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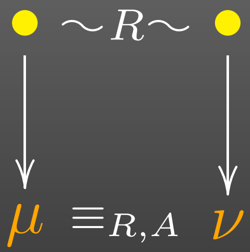

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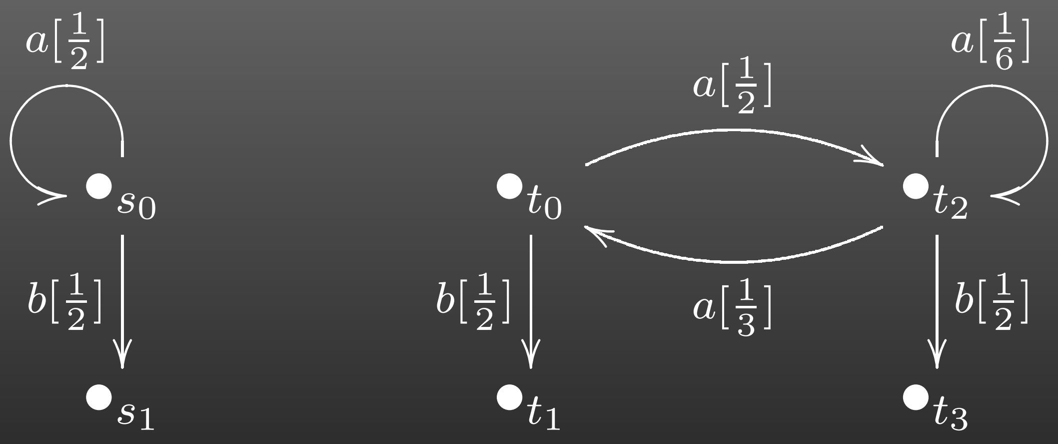
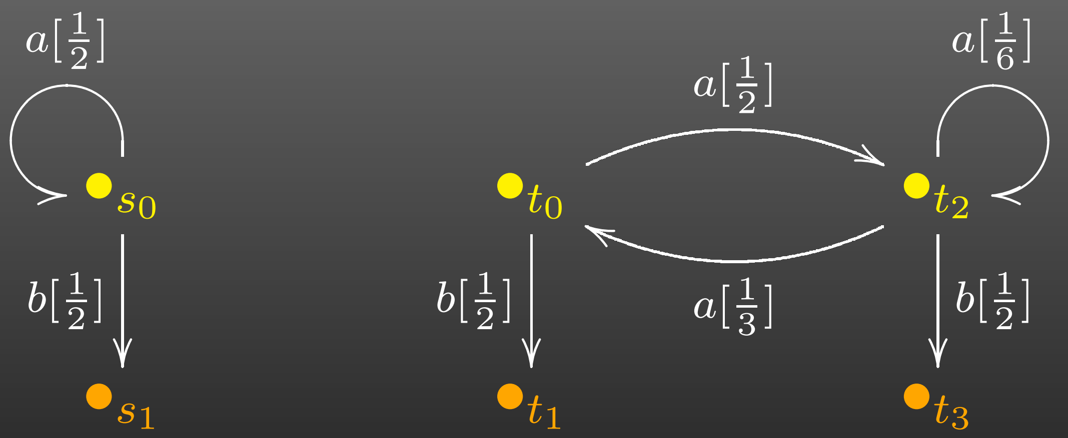


Bisimulation - generative

Example: Consider the generative systems



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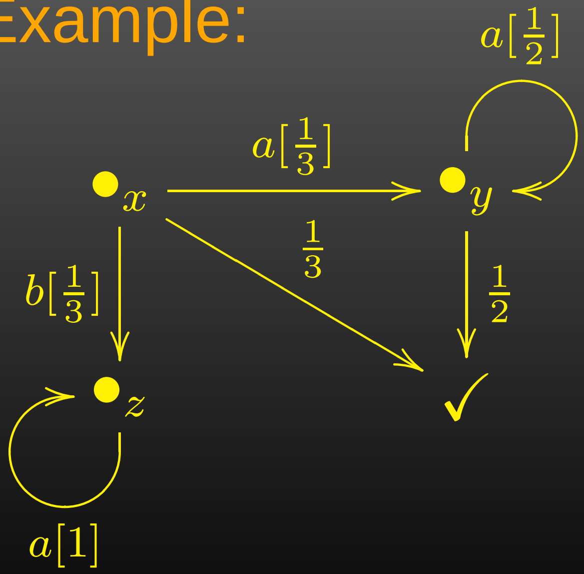

$$\mathrm{tr}(x): \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3} \cdot \frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\cdots$$

Coalgebras

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

- rich mathematical structure
- a uniform way for treating transition systems
- general notions and results, generic notion of bisimulation

Coalgebras

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

$\mathcal{F}$-coalgebras together with coalgebra homomorphisms

$$\begin{array}{ccc} \mathcal{F}S & \overset{\mathcal{F}(h)}{\dashrightarrow} & \mathcal{F}T \\ \alpha\uparrow & & \uparrow\beta \\ S & \overset{h}{\dashrightarrow} & T \end{array}$$

form a category $\mathsf{Coalg}_{\mathcal{F}}$

Coalgebraic bisimulation

A bisimulation on

$$\langle S, \alpha : S \to \mathcal{F}S \rangle$$

is (an equivalence) $R \subseteq S \times S$ such that

Coalgebraic bisimulation

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$$\begin{array}{ccccc} S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & S \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\ \mathcal{F}S & \xleftarrow[\mathcal{F}\pi_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}\pi_2]{} & \mathcal{F}S \end{array}$$

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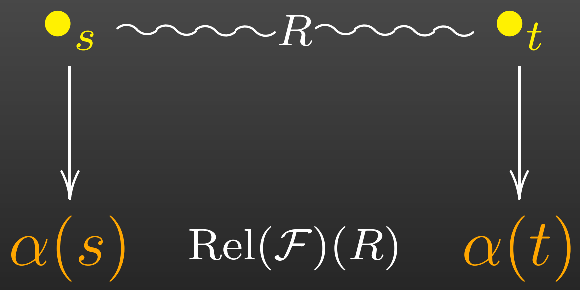

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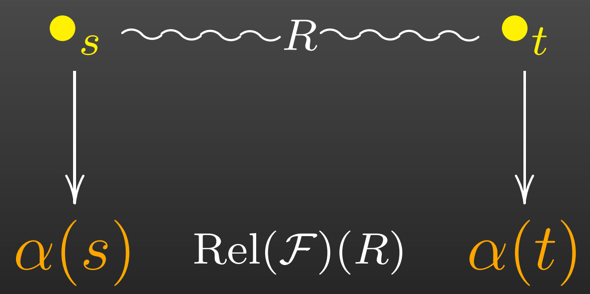

Transfer condition: $\langle s, t \rangle \in R \implies$

$$\langle \alpha(s), \alpha(t) \rangle \in \mathrm{Rel}(\mathcal{F})(R)$$

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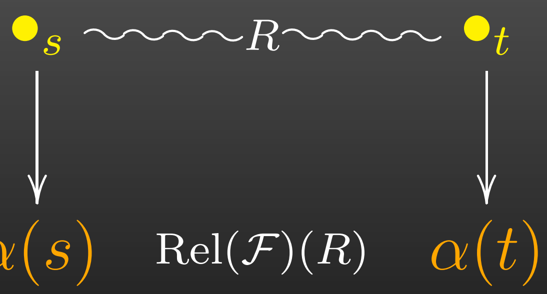

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is (an equivalence) $R \subseteq S \times S$ such that

$$\begin{array}{ccc} \bullet_s & \sim\sim\sim R \sim\sim\sim & \bullet_t \\ \downarrow & & \downarrow \\ \alpha(s) & \mathrm{Rel}(\mathcal{F})(R) & \alpha(t) \end{array}$$

Theorem: Coalgebraic and concrete bisimilarity coincide (in all known cases)

Trace of a coalgebra ?

Trace of a coalgebra ?

- Power&Turi '99 - $\mathcal{P}(1 + \Sigma \times \_)$
- Jacobs '04 - $\mathcal{P}\mathcal{F}$
- Hasuo&Jacobs CALCO '05, CALCO Jnr '05 - $\mathcal{P}\mathcal{F}$, $\mathcal{D}\mathcal{F}$
- Hasuo&Jacobs&Sokolova CMCS'06, LMCS 3(4:11)'07
  Generic Trace Semantics via Coinduction
  $\mathcal{T}\mathcal{F}$, order-enriched setting

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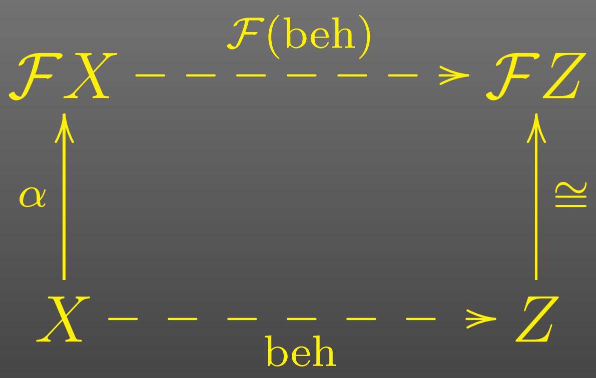
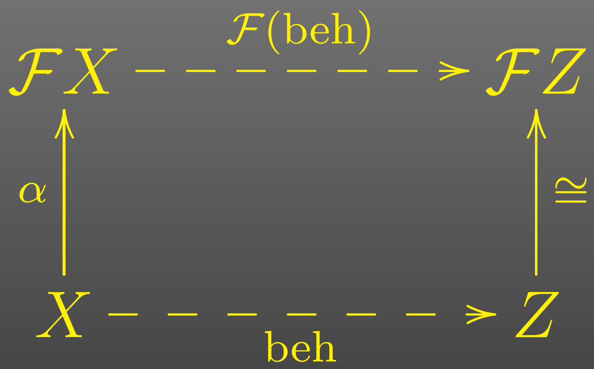
main idea: coinduction in a Kleisli category

Coinduction



system final coalgebra

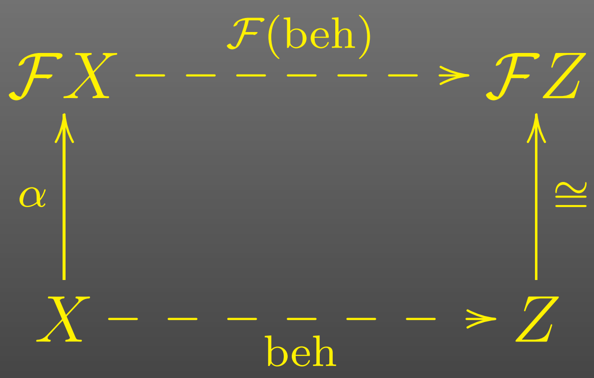
Coinduction



system final coalgebra

- finality = $\exists!$(morphism for any $\mathcal{F}$- coalgebra)
- beh gives the behavior of the system
- this yields final coalgebra semantics

Coinduction



system final coalgebra

- f.c.s. in **Sets** = bisimilarity
- f.c.s. in a Kleisli category = trace semantics

Types of systems

For trace semantics systems are suitably modelled as coalgebras in $\mathbb{S}\mathrm{ets}$

$$X \xrightarrow{c} \mathcal{T}\mathcal{F}X$$

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needed: distributive law $\mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$

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is needed since branching is irrelevant:

LTS with ✓ - $\mathcal{PF} = \mathcal{P}(1 + \Sigma \times \_)$

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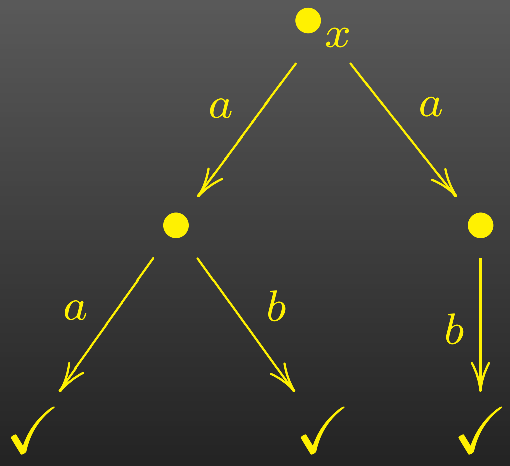

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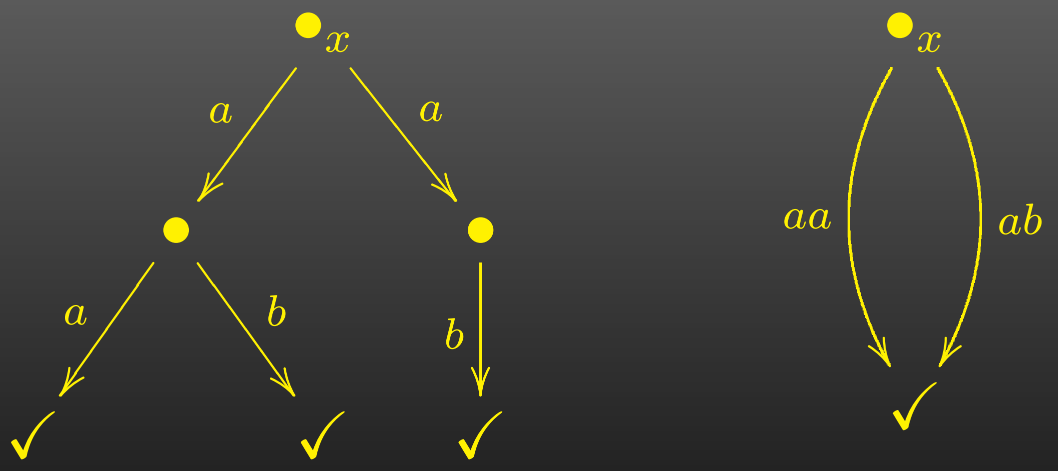

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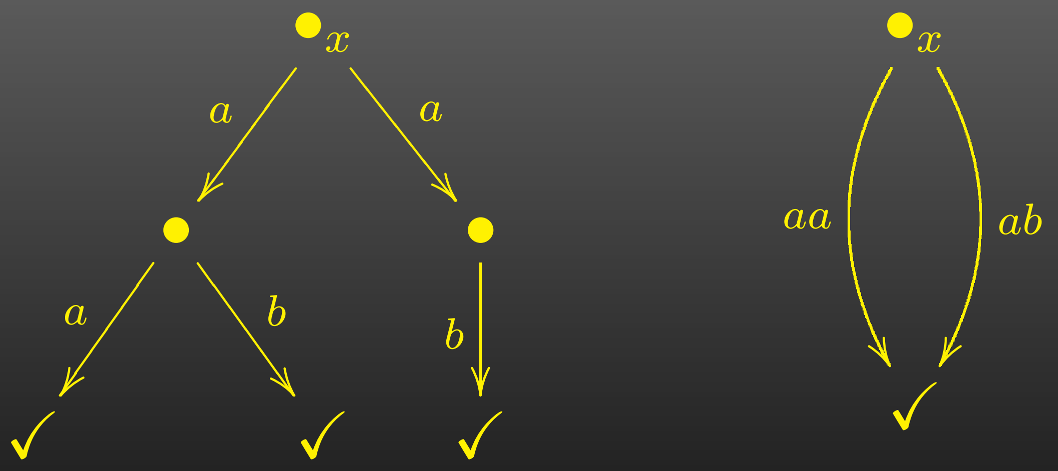

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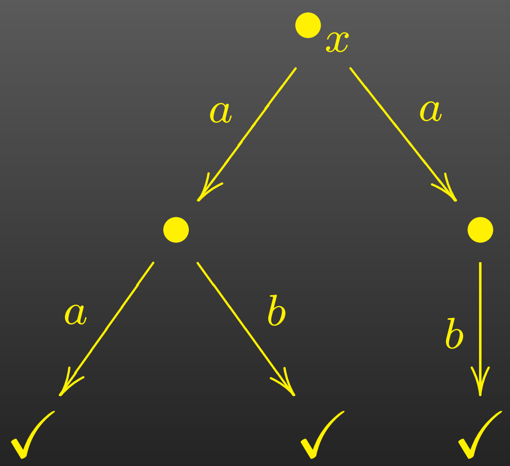

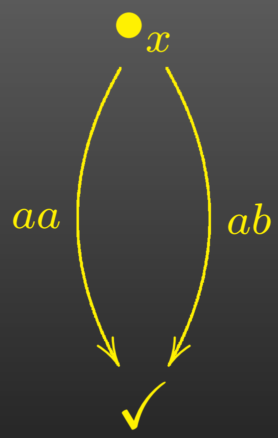

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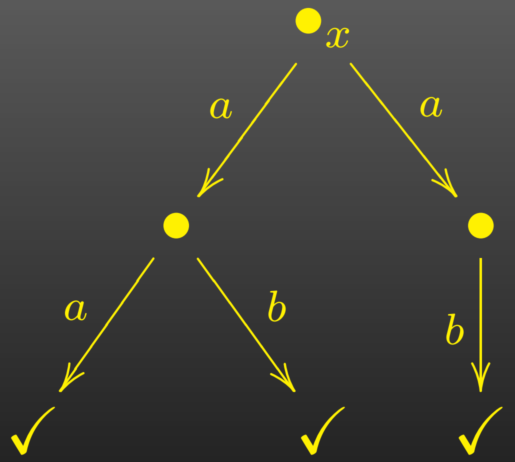

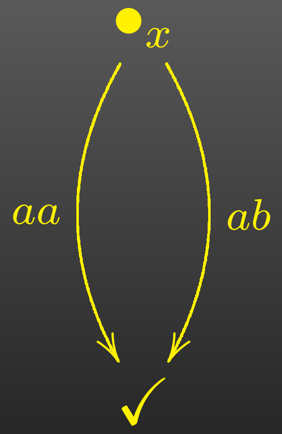

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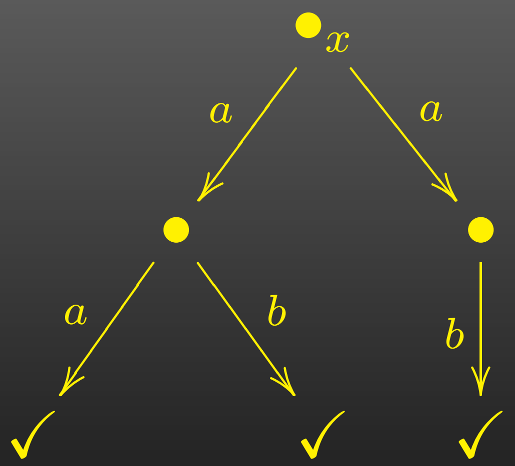

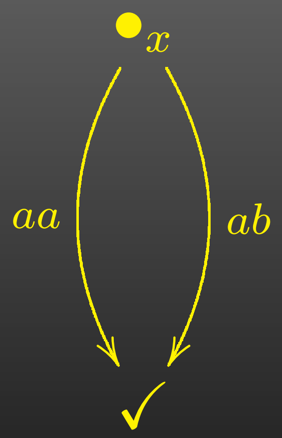

$$X \xrightarrow{c} \mathcal{PF}X \xrightarrow{\mathcal{PF}c} \mathcal{PFPF}X$$

Distributive law

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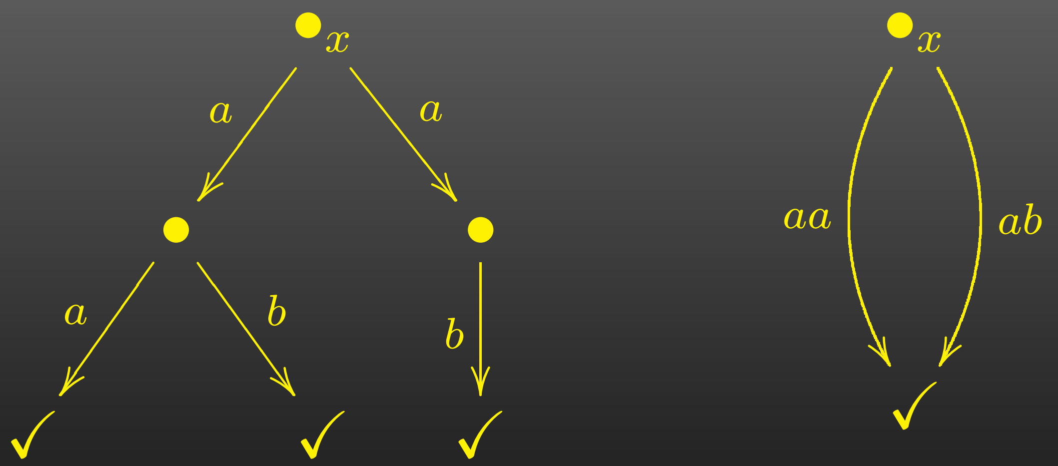

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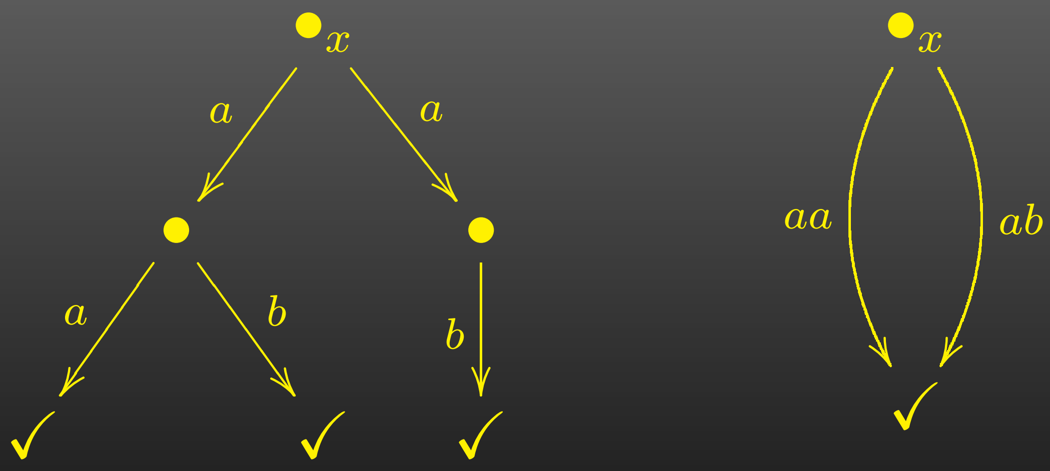

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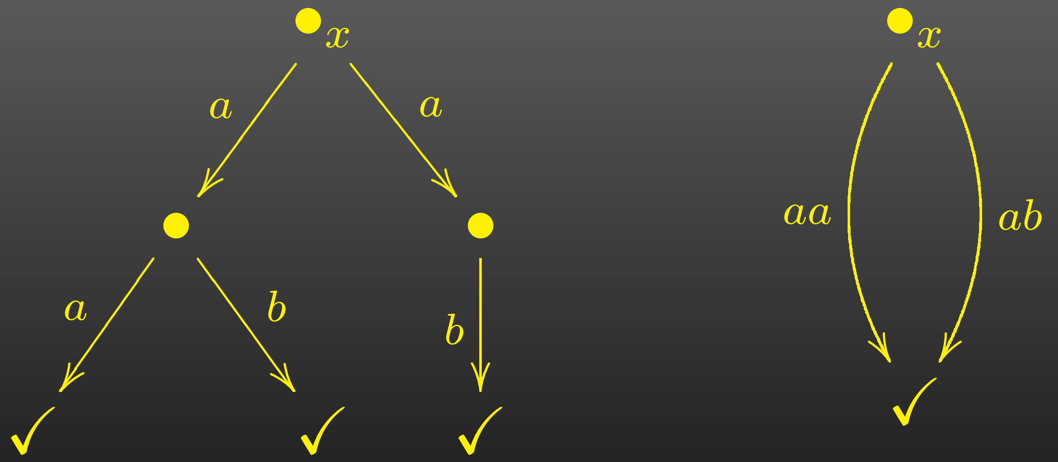

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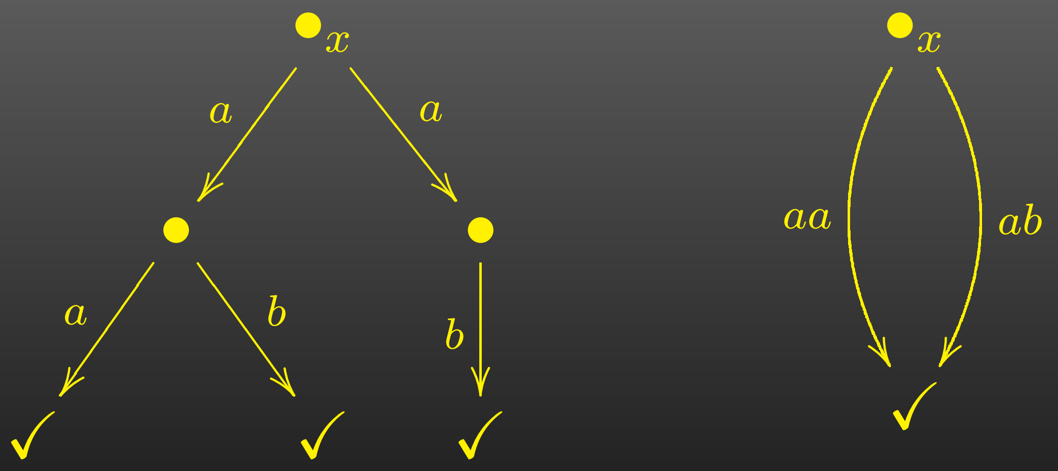

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- objects - sets
- arrows - $X \xrightarrow{f} Y$ are functions $f : X \to \mathcal{T}Y$

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Main Theorem

If $\clubsuit$, then

$$\begin{array}{c} \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \\ {\scriptstyle \eta_A \circ \alpha}\big\downarrow{\scriptstyle \cong} \\ A \end{array} \qquad \qquad \begin{array}{c} \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \\ {\scriptstyle \eta_{\mathcal{F}A} \circ \alpha^{-1}}\big\uparrow{\scriptstyle \cong} \\ A \end{array}$$

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proof: via limit-colimit coincidence Smyth&Plotkin ’82

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- $\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}$ should be locally monotone

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$$\begin{array}{ccc} \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}X & \xrightarrow{\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}(\mathrm{tr}_c)} & \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}A \\ {\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \cong} \\ X & \xrightarrow[\mathrm{tr}_c]{} & A \end{array}$$

It works for

- branching types:
  - lift monad $1 + \_$

    systems with non-termination, exception
  - powerset monad $\mathcal{P}$

    non-deterministic systems
  - subdistribution monad $\mathcal{D}$

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$\mathcal{D}X = \{\mu : X \to [0, 1] \mid \sum_{x \in X} \mu(x) \leq 1\}$

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all with pointwise order !

together with

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$$\mathcal{F} = \textit{id} \mid \Sigma \mid F \times F \mid \coprod_i F_i$$

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- linear I/O types: shapely functors

$$\mathcal{F} = \mathit{id} \mid \Sigma \mid F \times F \mid \coprod_i F_i$$

  * modular distributive law between commutative monads and shapely functors

  * our monads are commutative

Hence, it works

- for LTS with explicit termination

$$\mathcal{P}(1 + \Sigma \times \_)$$

Hence, it works

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$$\mathcal{P}(1 + \Sigma \times \_\,)$$

- for generative systems with explicit termination

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Note: Initial $1 + \Sigma \times \_$ - algebra is

$$\Sigma^* \xrightarrow[\cong]{[\mathrm{nil},\mathrm{cons}]} 1 + \Sigma \times \Sigma^*$$

Finite traces - LTS with ✓

the finality diagram in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{array}{ccc}
\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}X & \xrightarrow{\;\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}(\mathrm{tr}_c)\;} & \mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}\Sigma^* \\
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Finite traces - LTS with $\checkmark$

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amounts to

- $\langle\rangle \in \mathrm{tr}_c(x) \iff \checkmark \in c(x)$
- $a \cdot w \in \mathrm{tr}_c(x) \iff (\exists x')\langle a, x'\rangle \in c(x),\ w \in \mathrm{tr}_c(x')$

Finite traces - generative ✓

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\end{array}$$

amounts to $\mathrm{tr}_c(x)$:

- $\langle\rangle \mapsto c(x)(\checkmark)$
- $a \cdot w \mapsto \sum_{y\in X} c(x)(a,y)\cdot c(y)(w)$

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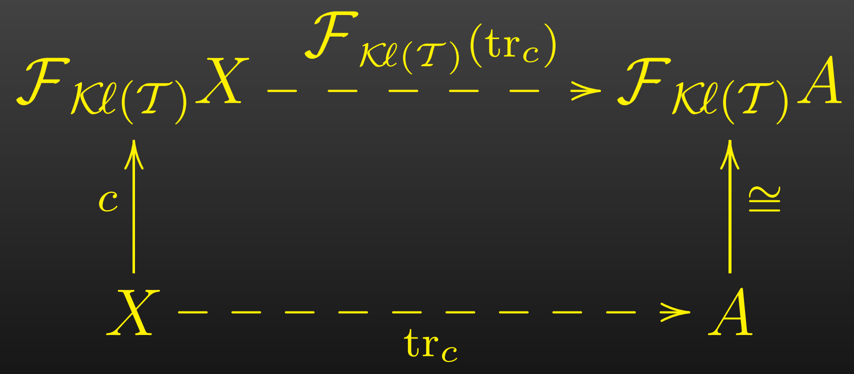

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- Main technical result: initial algebra = final coalgebra