# Probabilistic automata: types and semantics

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# Outline

- introduction
- probabilistic systems and coalgebras
- bisimilarity - the strong end of the spectrum
- expressiveness hierarchy
- other semantics - at the weak end of the spectrum

# Formal methods

are mathematically based techniques for

- specification
- development
- verification

of software and hardware systems

# Formal methods

In general:

- models - transition systems, automata, terms,...
  with a clear semantics
- analysis - model checking
  process algebra
  theorem proving...

# Formal methods

Here:

- models - probabilistic transition systems
- semantics - behavior equivalences

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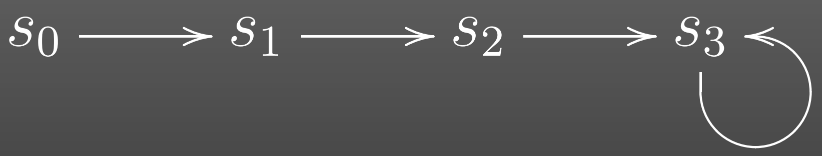
- semantics - behavior equivalences

Aim: one framework for many probabilistic models and semantics - compare expressiveness

# Example models

deterministic systems



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states $+$ transitions $\alpha : S \to S$

$$\alpha(s_0) = s_1,\ \alpha(s_1) = s_2,\ \ldots$$

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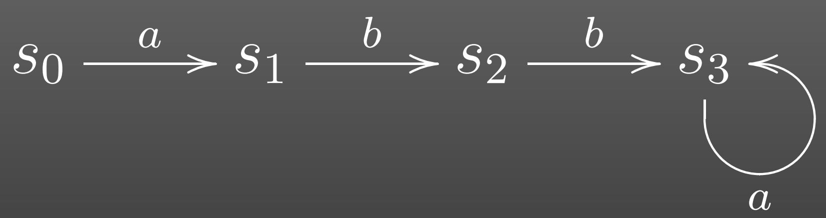
labelled deterministic systems      $A$ - labels



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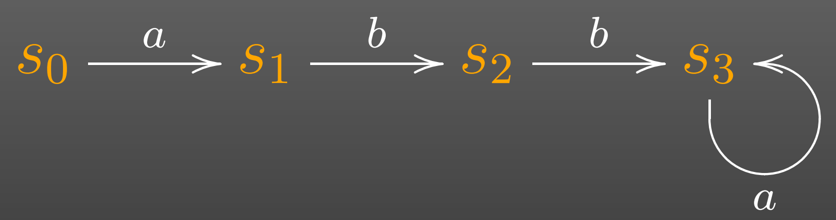
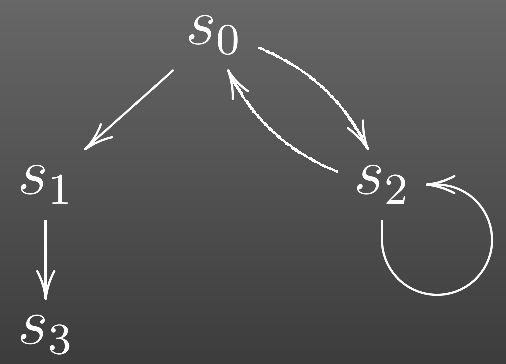



states + transitions $\alpha : S \to A \times S$

$$\alpha(s_0) = \langle a, s_1 \rangle,\ \alpha(s_1) = \langle b, s_2 \rangle,\ \ldots$$

# Example models

transition systems



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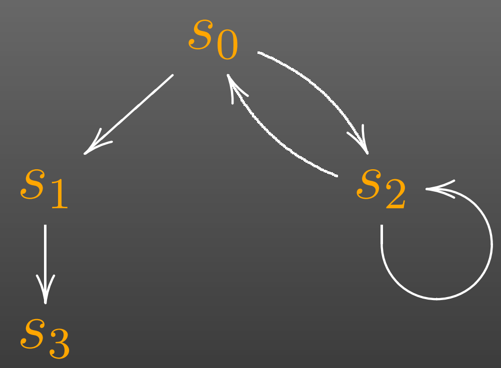


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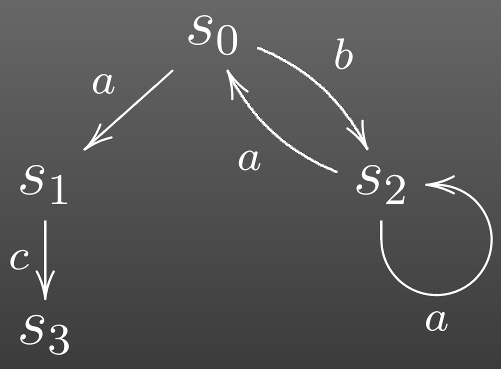

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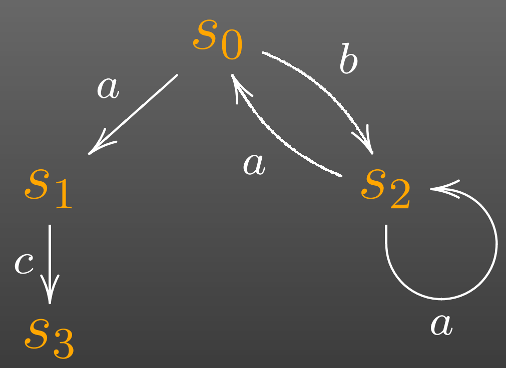


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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

- rich mathematical structure
- a uniform way for treating transition systems
- general notions and results, generic notion of bisimulation

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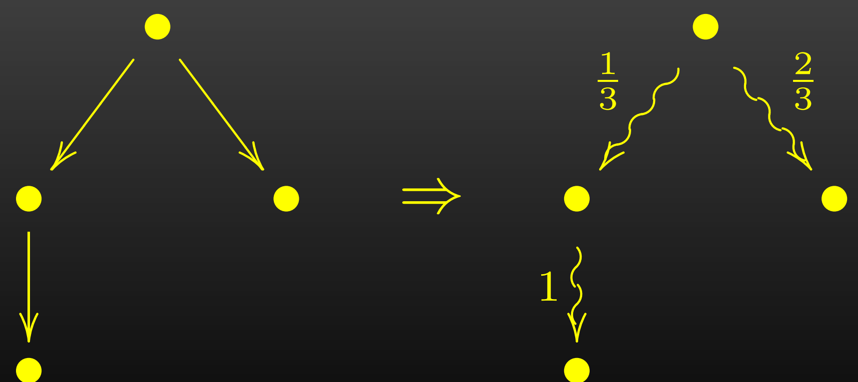

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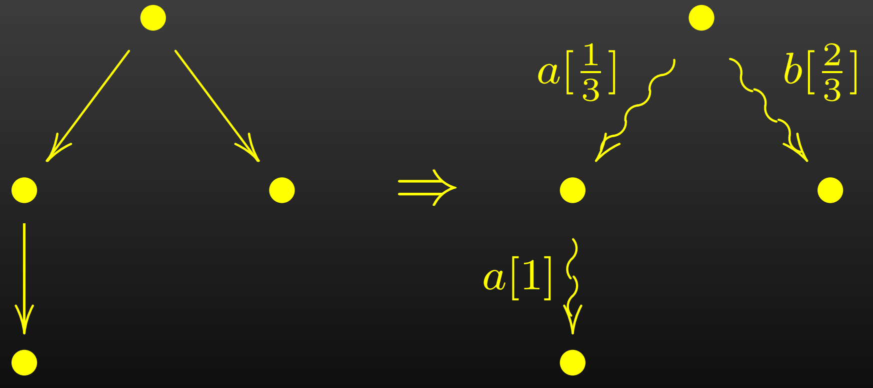

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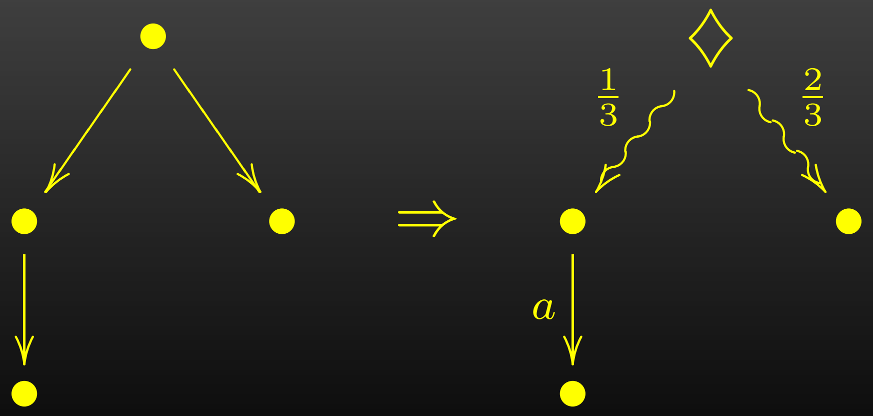

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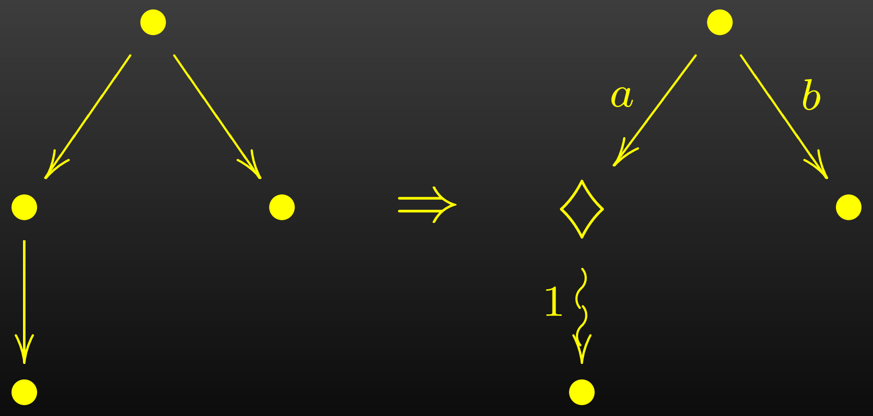

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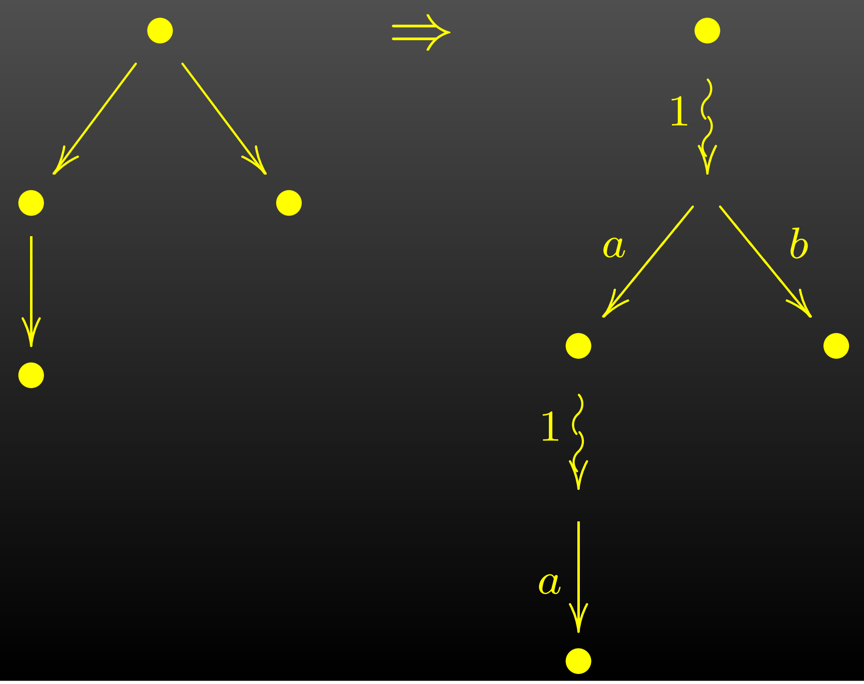

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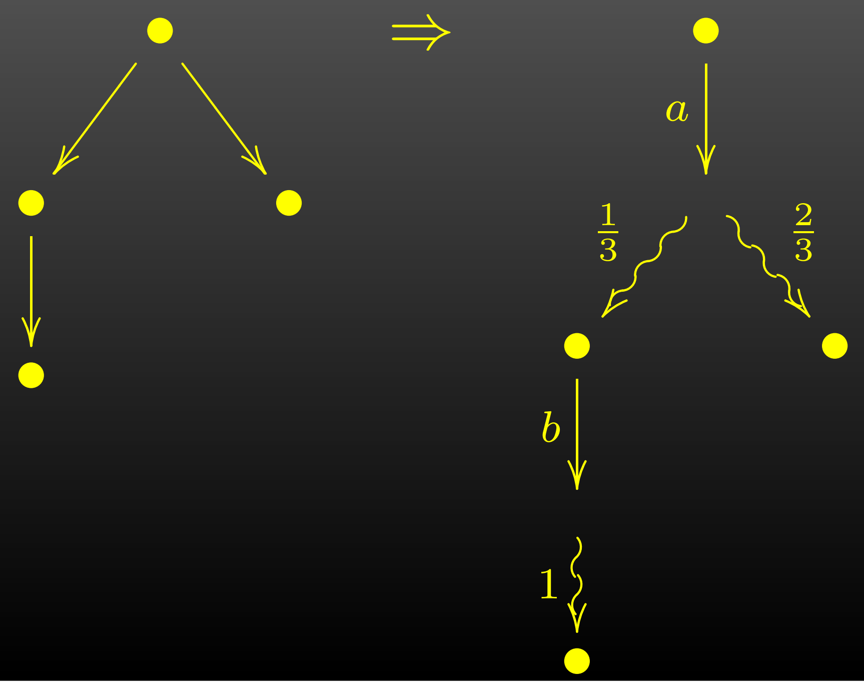

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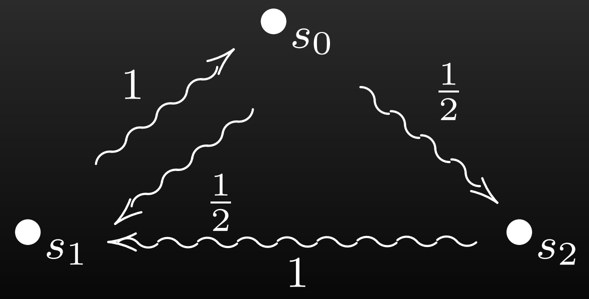
Example: $\alpha : S \to \mathcal{D}S$



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... of functors built by the following syntax

$$\mathcal{F} ::= \_ \mid A \mid \mathcal{P} \mid \mathcal{D} \mid \mathcal{G} + \mathcal{H} \mid \mathcal{G} \times \mathcal{H} \mid \mathcal{G}^A \mid \mathcal{G} \circ \mathcal{H}$$

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note: in the probabilistic case

$$(\mathcal{D} + 1)^A \not\cong \mathcal{D}(A \times \_) + 1$$

# Probabilistic system types

| | |
|---|---|
| **MC** | $\mathcal{D}$ |
| **DLTS** | $(\_ + 1)^A$ |
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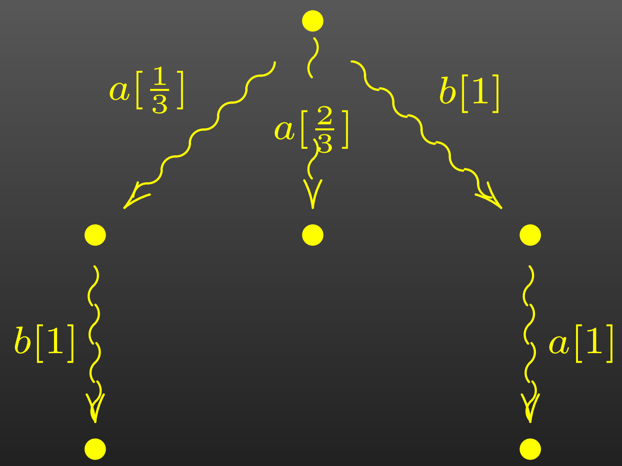

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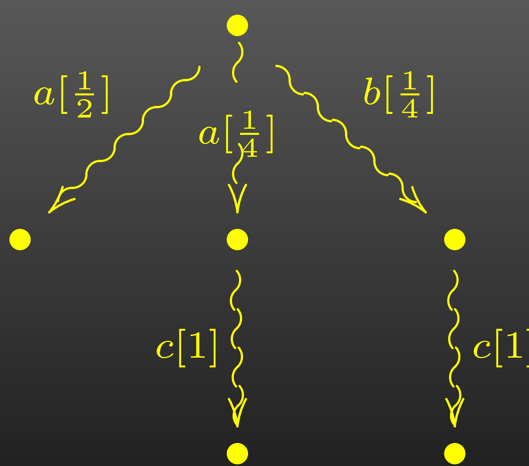

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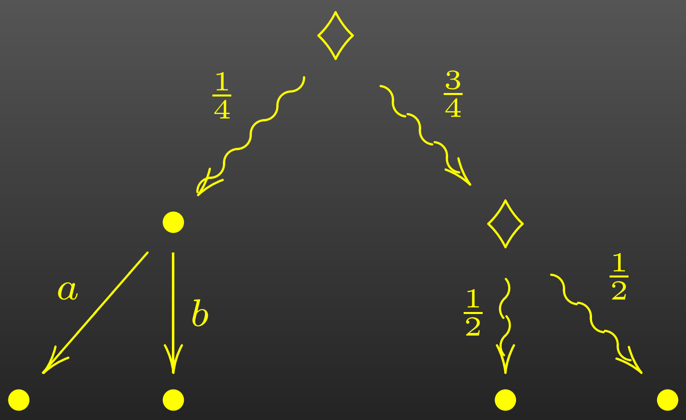

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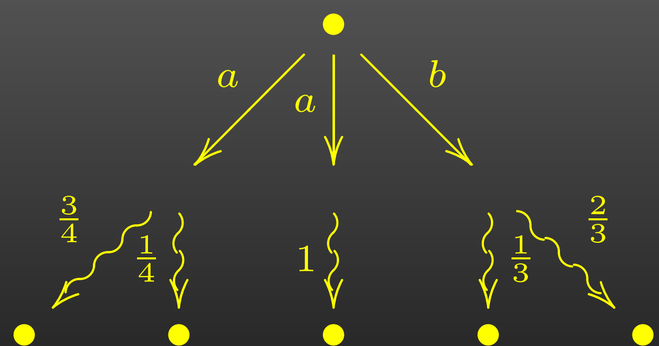

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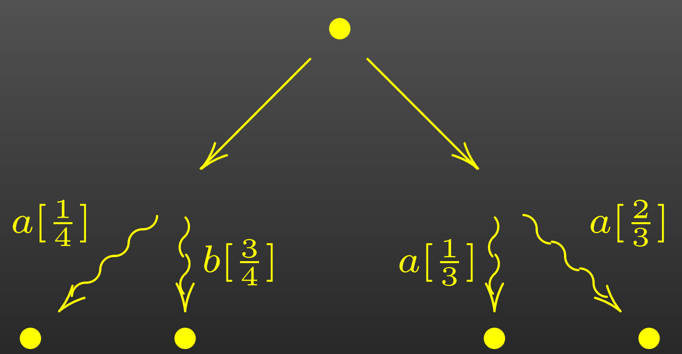

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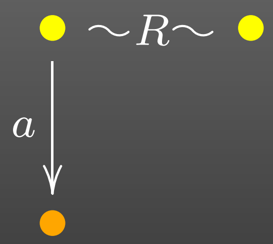

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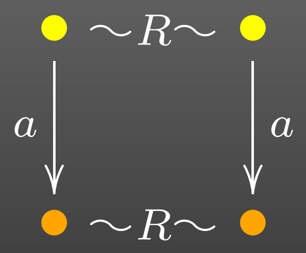

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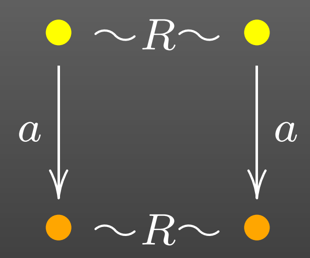


... two states are bisimilar if they are related by some bisimulation

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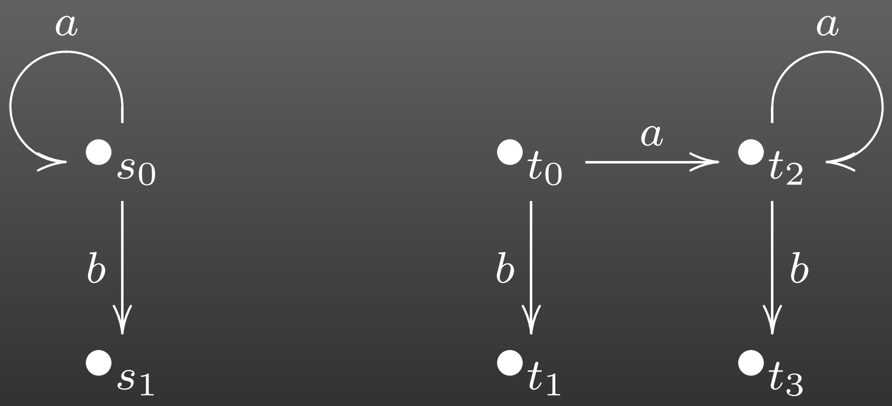
Example: Consider the LTS



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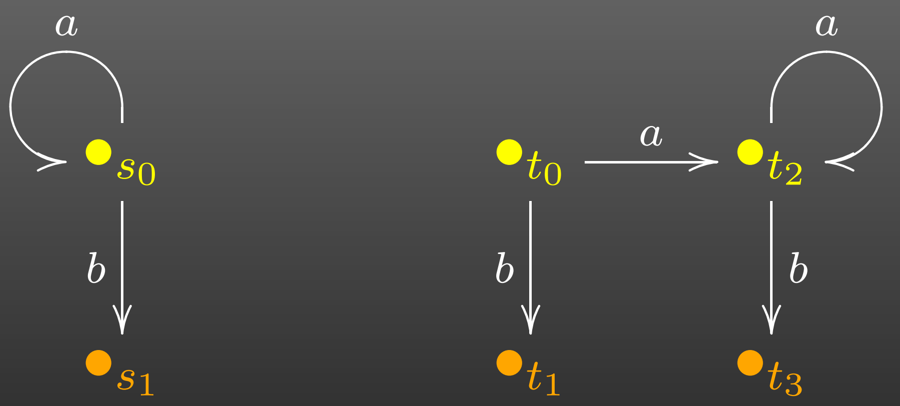


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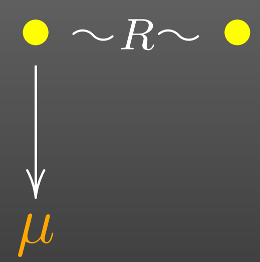

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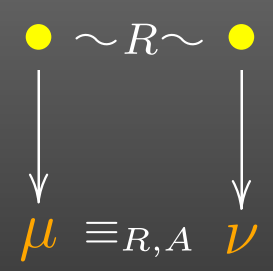


$\equiv_{R,A}$ relates distributions that assign the same probability to each label and each $R$-class

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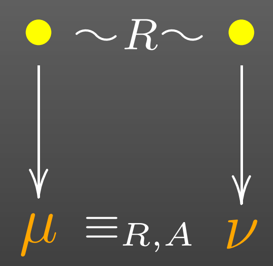


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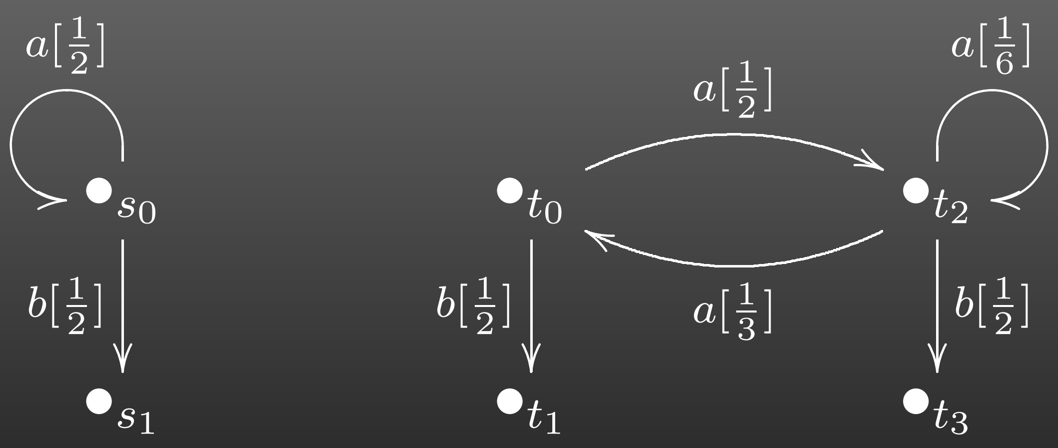

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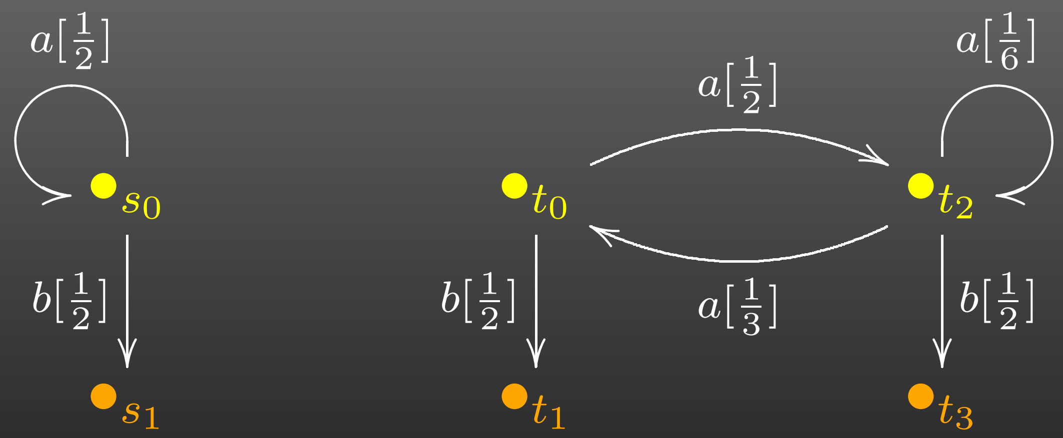


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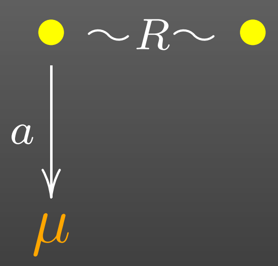

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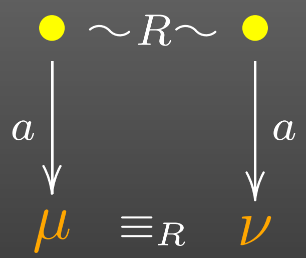


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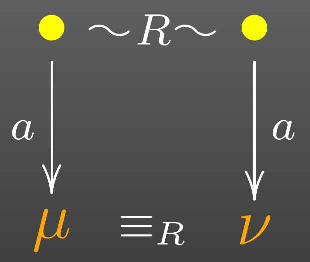


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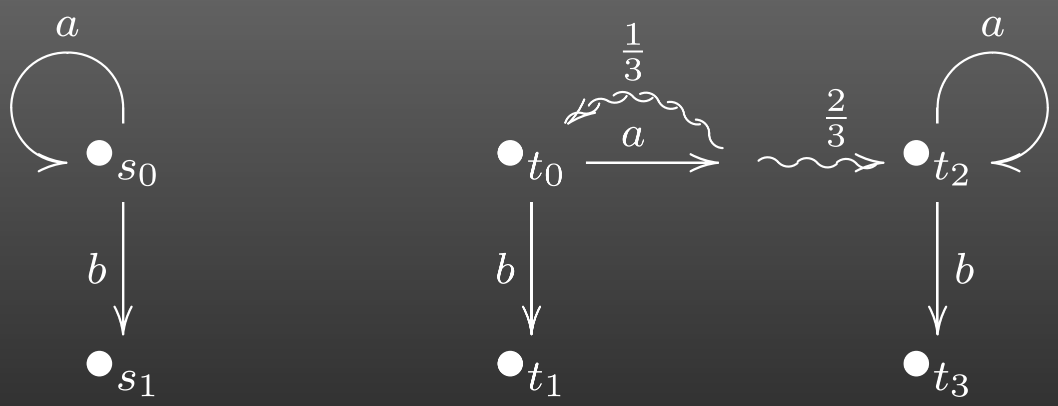

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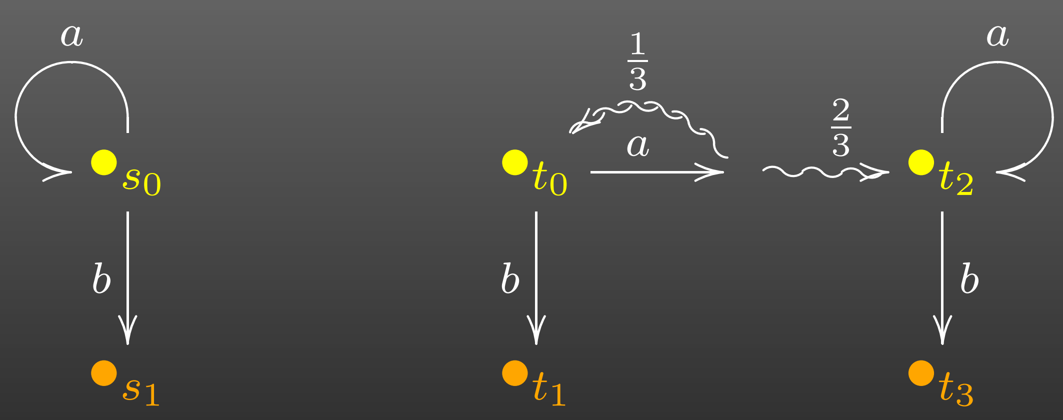


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$$\begin{array}{ccccc}
S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & S \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \alpha} \\
\mathcal{F}S & \xleftarrow[\mathcal{F}\pi_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}\pi_2]{} & \mathcal{F}S
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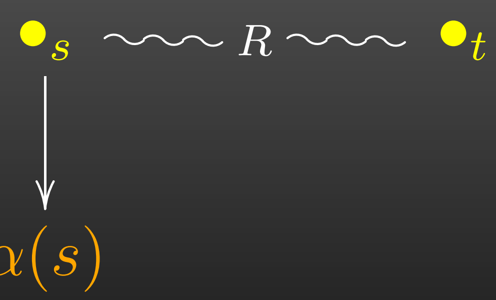

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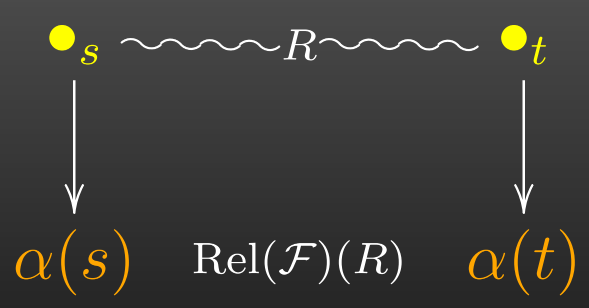

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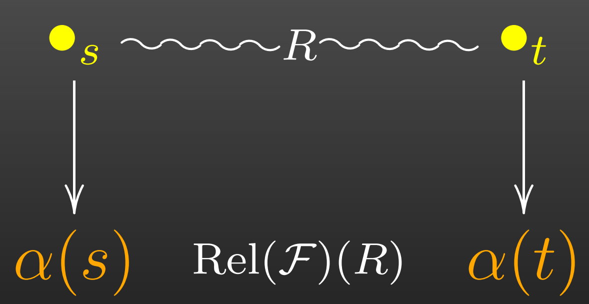


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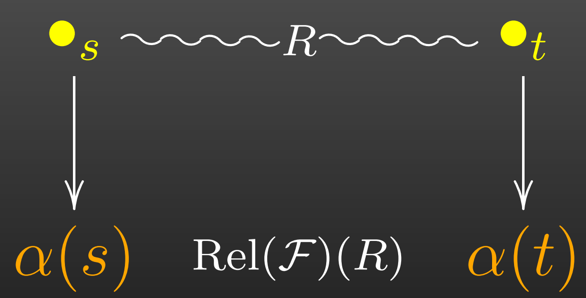
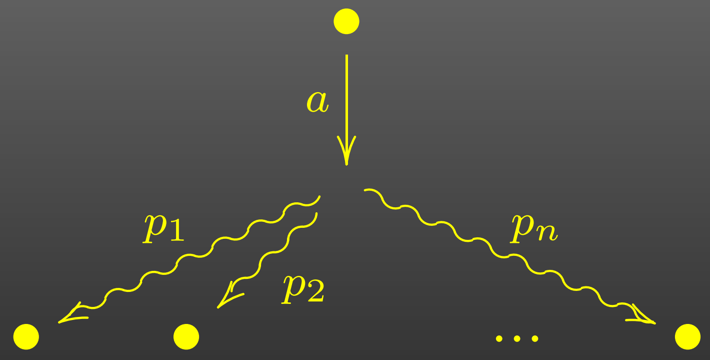



Theorem: Coalgebraic and concrete bisimilarity coincide !

# Expressiveness

simple Segala system → Segala system



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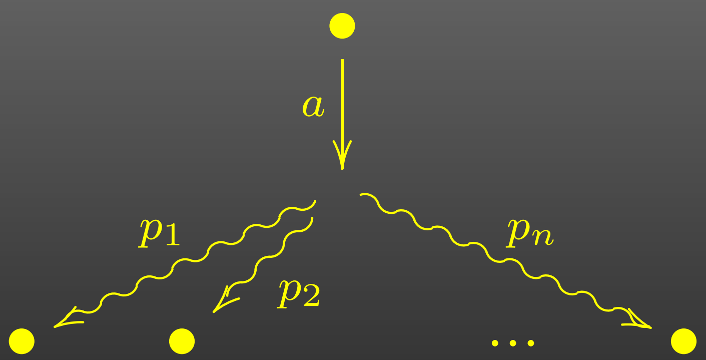


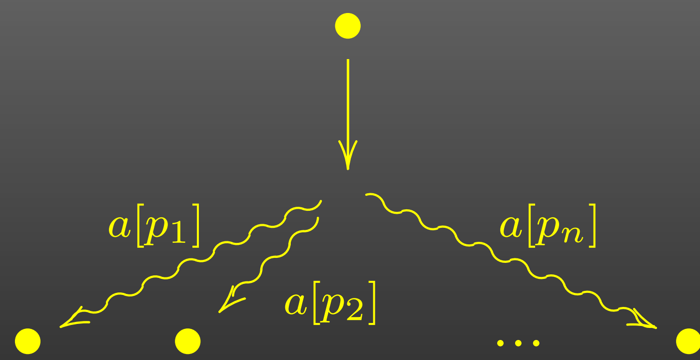

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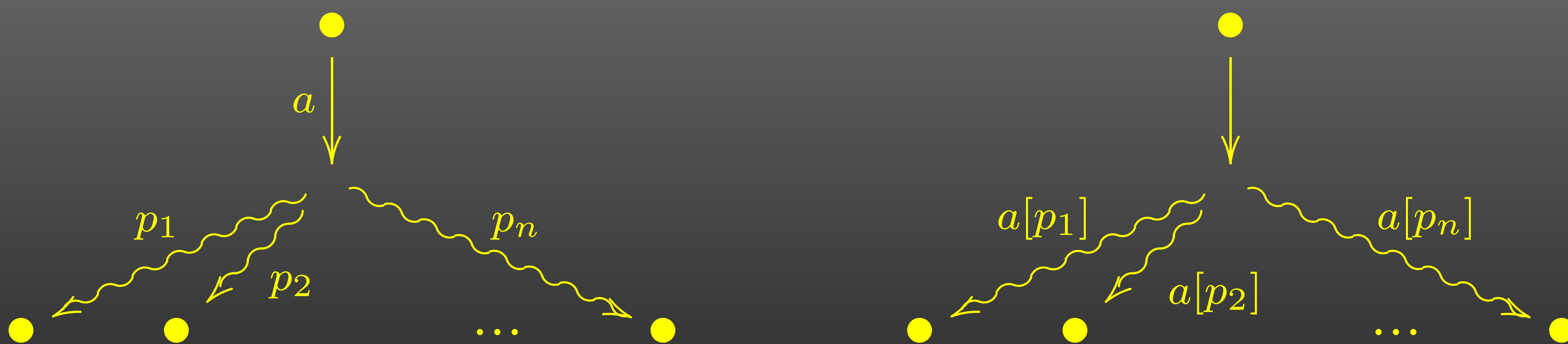


When do we consider one type of systems more expressive than another?

# Comparison criterion

$$\mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$$

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states are bisimilar in the original system iff they are bisimilar in the translation

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proof via cocongruences - behavioral equivalence

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Indeed $\mathbf{SSeg} \rightarrow \mathbf{Seg}$ since

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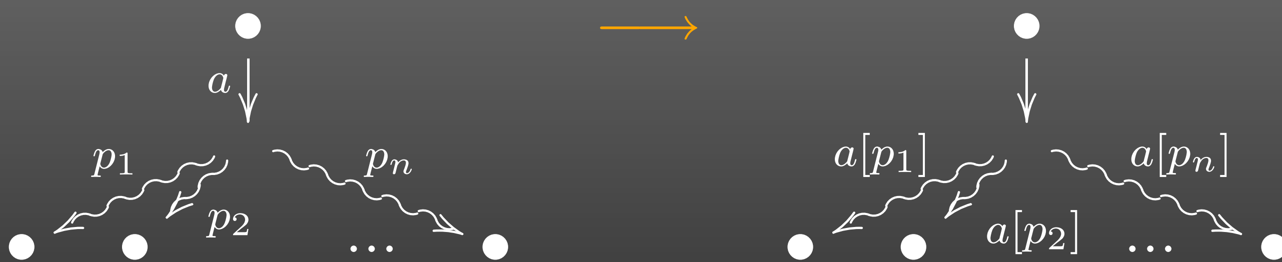

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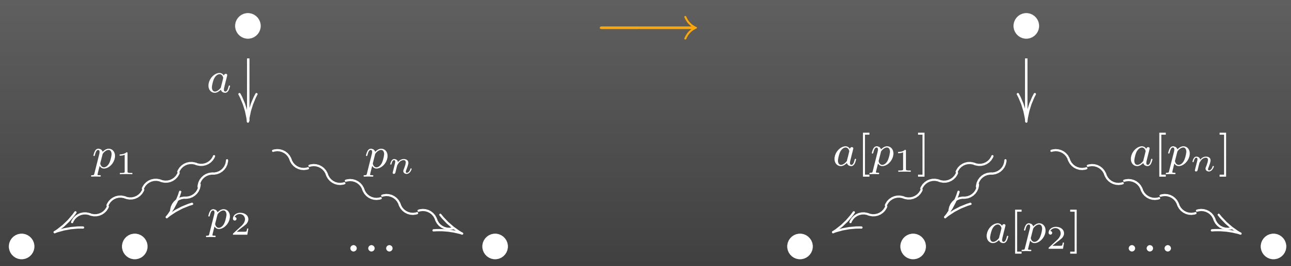
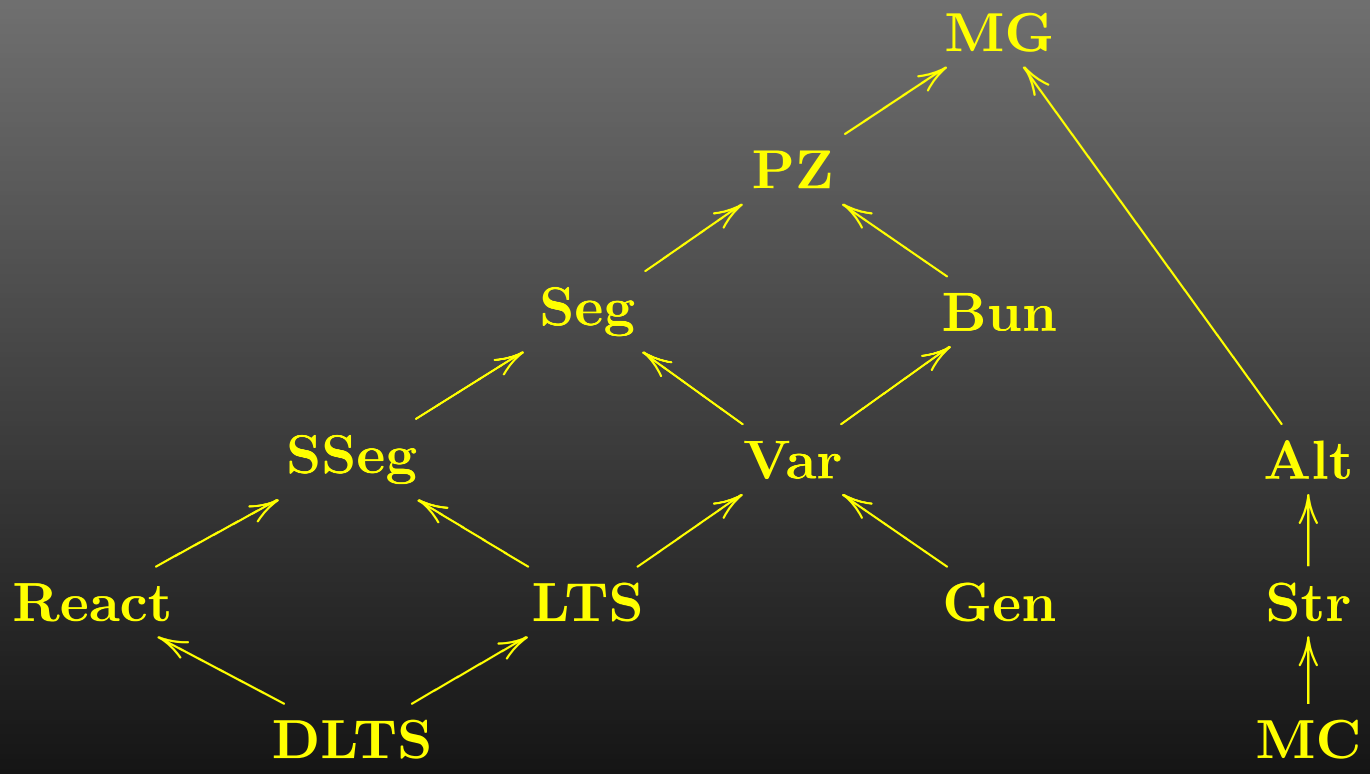



gives us an injective natural transformation

$$\mathcal{P}(A \times \mathcal{D}) \Longrightarrow \mathcal{P}\mathcal{D}(A \times \_\,)$$

# The hierarchy...



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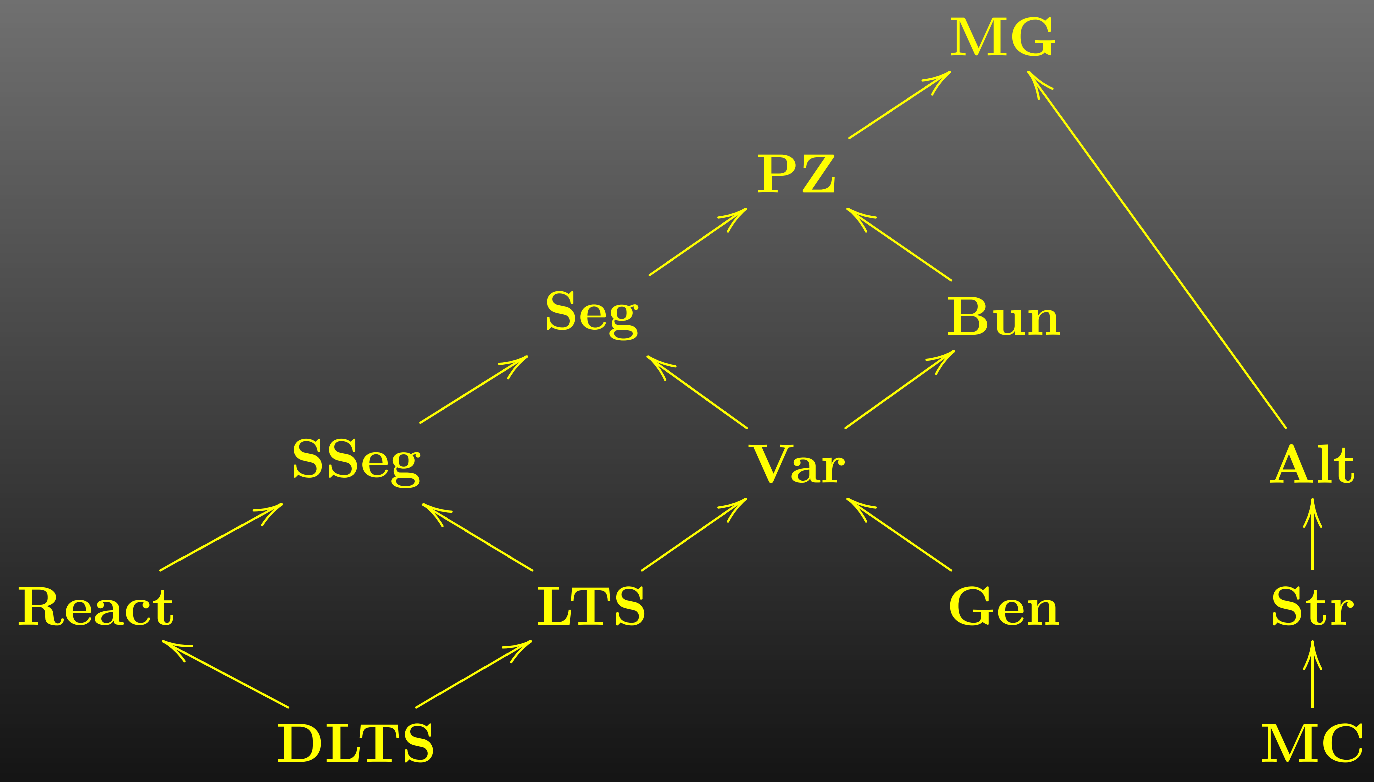


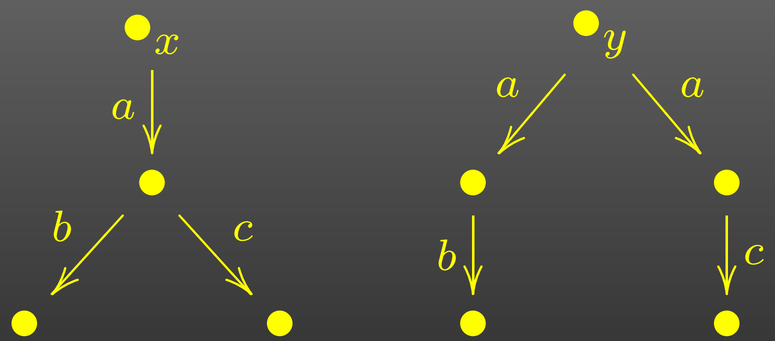

* Falk Bartels, AS, Erik de Vink

# LT/BT spectrum

Bisimilarity is not the only semantics...

# LT/BT spectrum

Are these non-deterministic systems equal ?



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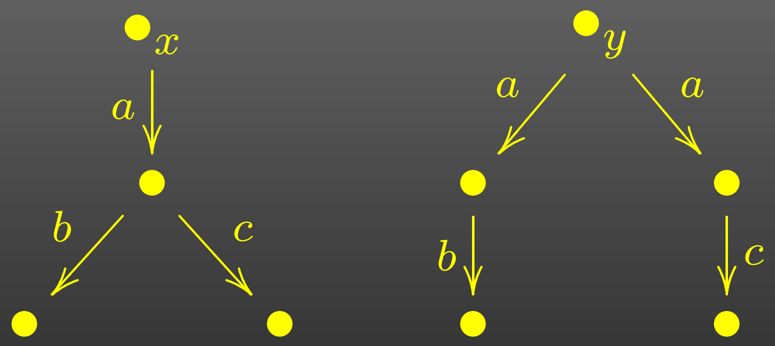


$x$ and $y$ are:

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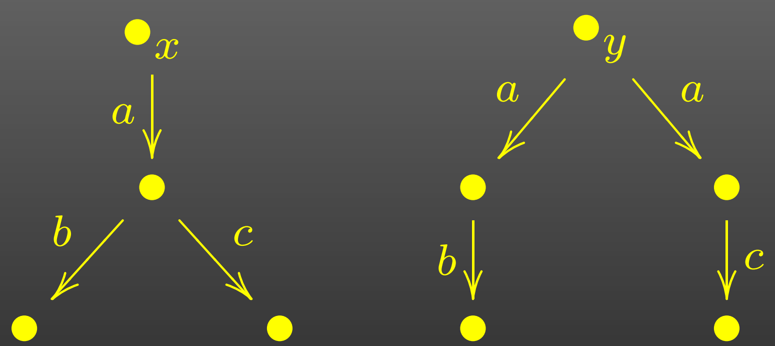


$x$ and $y$ are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

# Traces - LTS

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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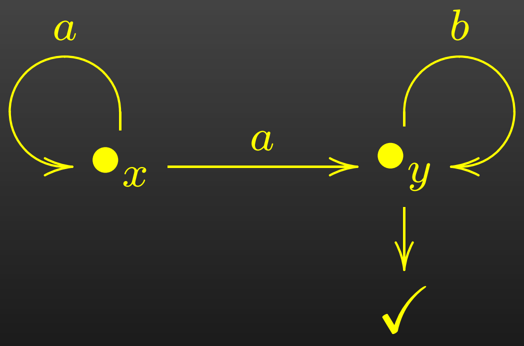


$$\mathrm{tr}(y) = b^*, \qquad \mathrm{tr}(x) = a^+ \cdot \mathrm{tr}(y) = a^+ \cdot b^*$$

# Traces - generative

For generative probabilistic systems with ex. termination

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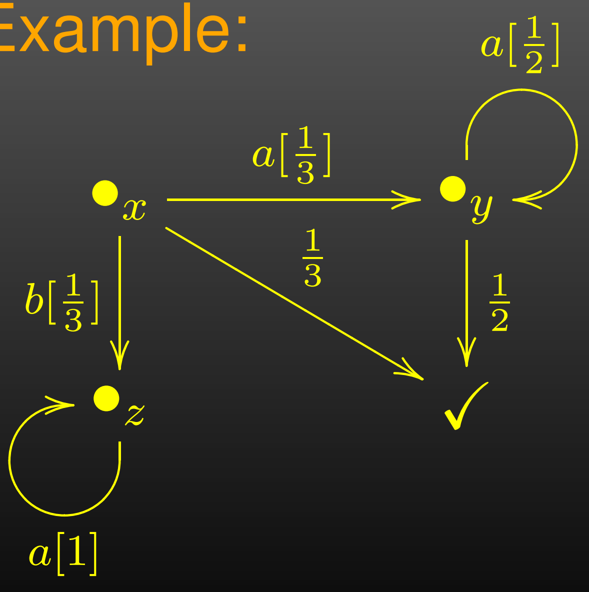


$$\mathrm{tr}(x): \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3} \cdot \frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\ldots$$

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main idea: coinduction in a Kleisli category

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- coalgebras allow for a unified treatment of transition systems and bisimulation
- comparison of systems is then easy
- we have built an expressiveness hierarchy w.r.t bisimulation semantics
- trace semantics can also be captured coalgebraically