Hierarchy of probabilistic systems

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TU/e and CWI

Outline

- Basic coalgebraic notions

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  - ∗ labelled transition systems

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  - bisimulation

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  - $\ast$ bisimulation
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  - $\ast$ expressiveness criterion
  - $\ast$ translation of coalgebras

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  - preservation and reflection of bisimulation
- Building the hierarchy

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- Comparison of system types
  - ∗ expressiveness criterion
  - ∗ translation of coalgebras
  - ∗ preservation and reflection of bisimulation
- Building the hierarchy
- Conclusions

Labelled transition systems

LTS is a pair $\langle S, \alpha : S \to \mathcal{P}(A \times S) \rangle$

A - a fixed set of actions

Labelled transition systems

LTS is a pair $\langle S, \alpha : S \rightarrow \mathcal{P}(A \times S)\rangle$

A - a fixed set of actions

Hence a coalgebra

$$\langle S, \alpha \rangle,\ \alpha : S \rightarrow \mathcal{F}S$$

of the functor $\mathcal{F} = \mathcal{P}(A \times \mathcal{I})$

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A - a fixed set of actions

Example:

$$\begin{tikzcd}
 & s_2 \arrow[r, "a"] \arrow[dr, "a"'] & s_4 & & \\
s_1 \arrow[ur, "b"] \arrow[dr, "a"'] & & s_5 \arrow[r, "b"] & s_6 & \cdots \\
 & s_3 & & &
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Note: $\mathcal{P}(A \times \mathcal{I}) \cong \mathcal{P}^A$

Concrete bisimulation for LTS

$\langle S, \alpha \rangle$, $\langle T, \beta \rangle$ - LTS

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$$s \xrightarrow{a} s' \Rightarrow (\exists t')t \xrightarrow{a} t', \; \langle s', t' \rangle \in R \text{ and}$$

$$t \xrightarrow{a} t' \Rightarrow (\exists s')s \xrightarrow{a} s', \; \langle s', t' \rangle \in R$$

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$s \approx t$ - there exists a concrete bisimulation

R with $\langle s, t \rangle \in R$

Coalgebraic bisimulation

A *bisimulation* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a relation $R \subseteq S \times T$ such that there exists a $\mathcal{F}$-coalgebra structure γ on R making

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$$\begin{tikzcd} S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\ \mathcal{F}S & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"] \arrow[r, "\mathcal{F}\pi_2"'] & \mathcal{F}T \end{tikzcd}$$

Concrete vs coalgebraic (LTS)

$\langle S, \alpha \rangle$, $\langle T, \beta \rangle$ - LTS

(coalgebras of type $\mathcal{F} = \mathcal{P}(A \times \mathcal{I})$)

$s \sim t$ - there exists a
bisimulation R with $\langle s, t \rangle \in R$.

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$s \sim t$ - there exists a
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known:
$s \approx t$ if and only if $s \sim t$

Introduction of probabilities

There are many ways to do it ...

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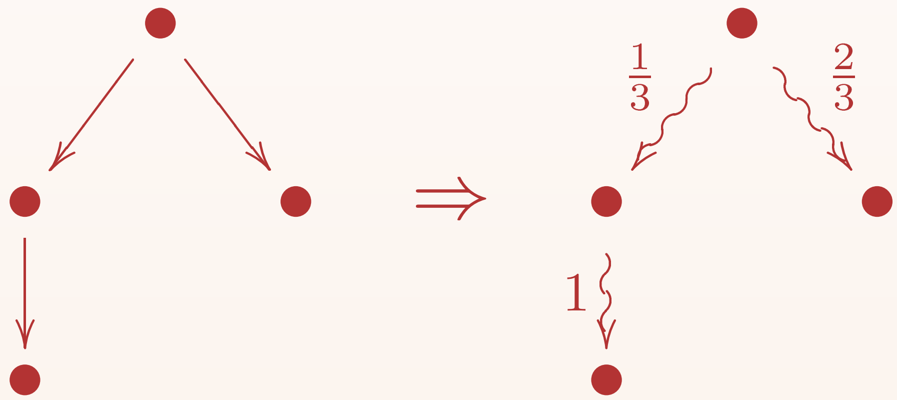

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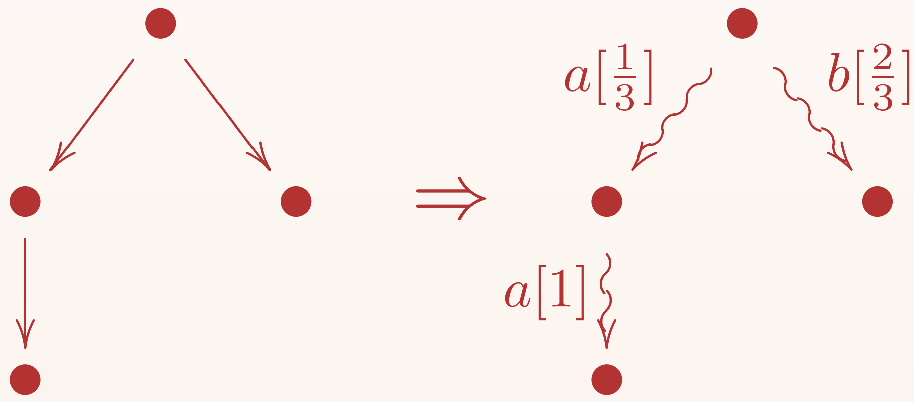

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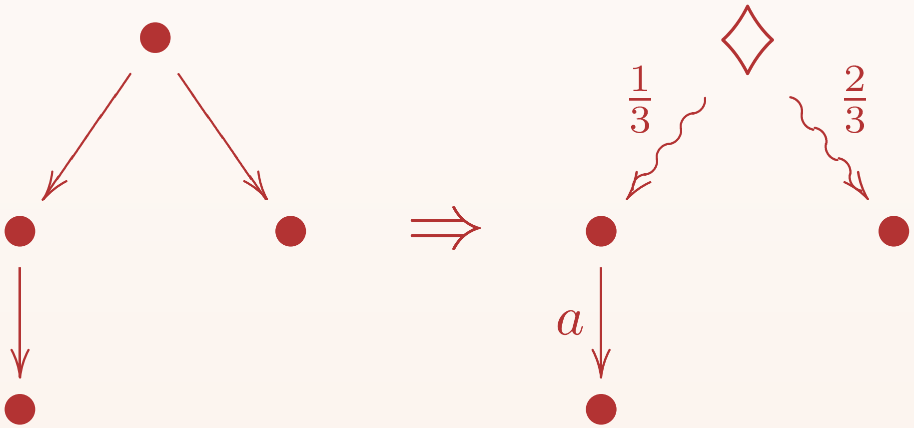

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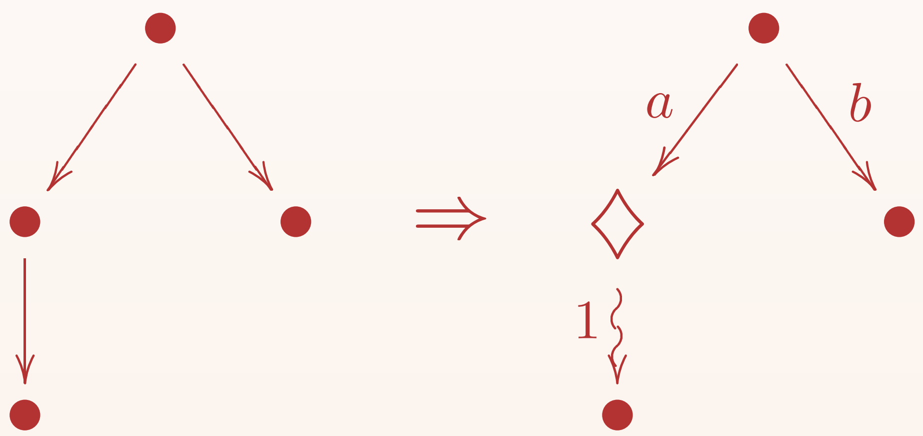

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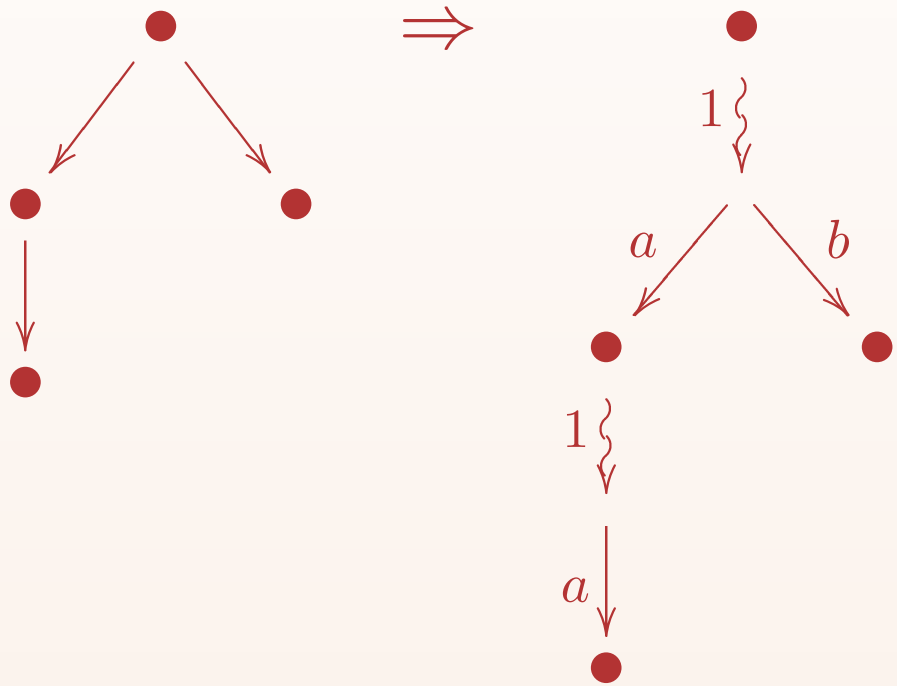

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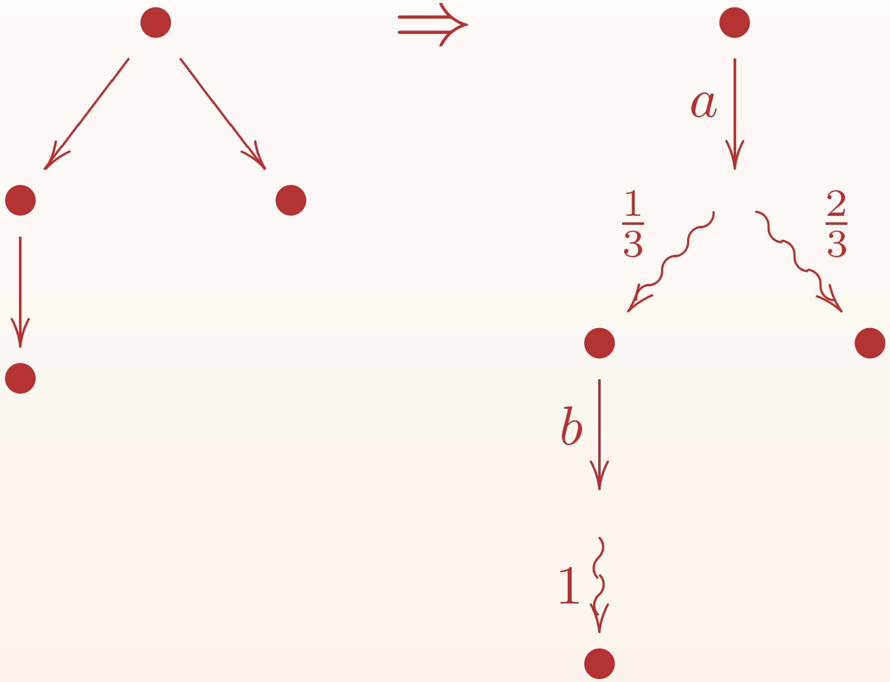

Introduction of probabilities

There are many ways to do it ...

13 types of systems - from the literature with (or without):

- action labels
- nondeterminism
- probabilities

Existing system types

MG

PZ

Alt **Seg** **Bun**

SSeg **Var**

Str **React** **NA** **Gen**

MC **DA**

System types

The (probabilistic) models of systems we consider are coalgebras

$$\langle S, \alpha \rangle, \ \alpha : S \to \mathcal{F}S$$

for a functor $\mathcal{F}$ built by the following syntax

$$\mathcal{F} ::= \mathcal{C} \mid \mathcal{I} \mid \mathcal{P} \mid \mathcal{D}_\omega \mid \mathcal{F}_1 + \mathcal{F}_2 \mid \mathcal{F}_1 \times \mathcal{F}_2 \mid \mathcal{F}^{\mathcal{C}} \mid \mathcal{F}_2\mathcal{F}_1$$

where

$$\mathcal{D}_\omega X := \{\mu : X \to [0,1] \mid \text{supp}(\mu) \text{ finite}, \ \mu[X] = 1\}$$

Reactive and generative systems

evolve from LTS - functor $\mathcal{P}(A \times \mathcal{I}) \cong \mathcal{P}^A$

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functor $(\mathcal{D}_\omega + 1)(A \times \mathcal{I}) = \mathcal{D}_\omega(A \times \mathcal{I}) + 1$

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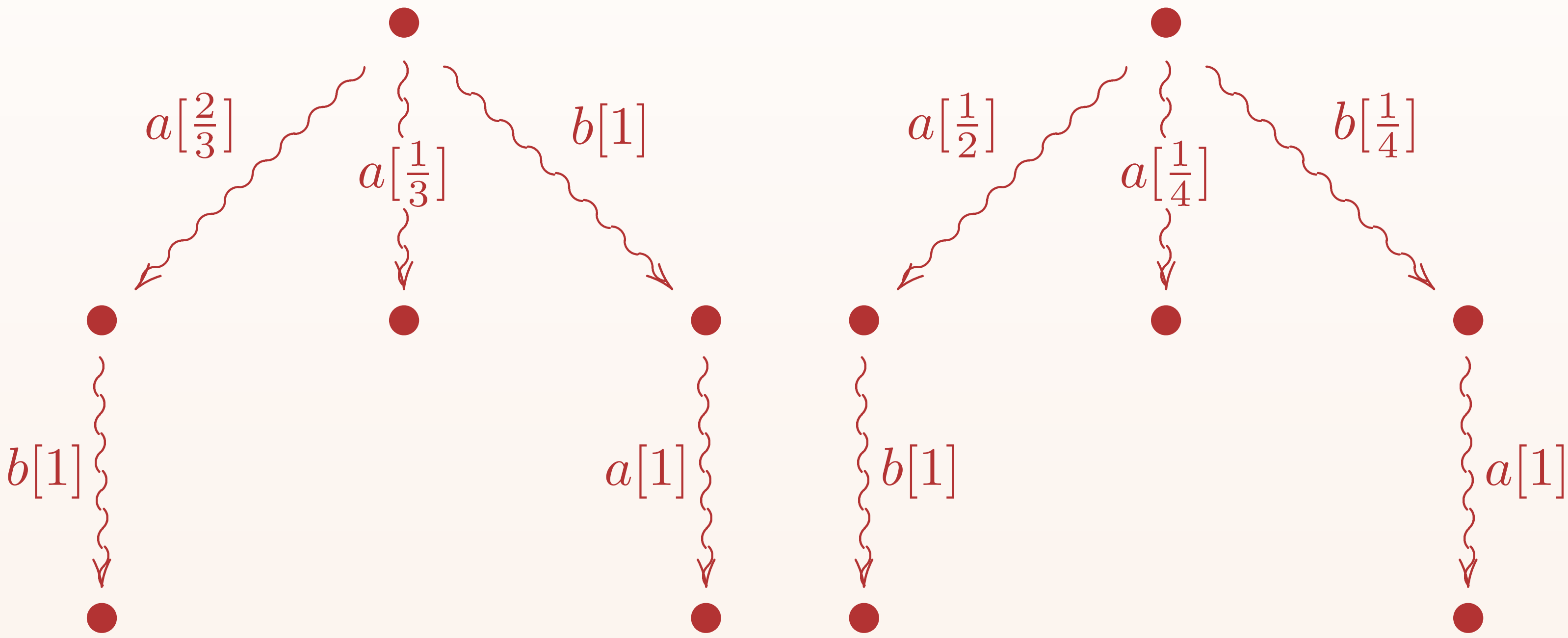
note:

In the probabilistic case

$(\mathcal{D}_\omega + 1)^A \ncong \mathcal{D}_\omega(A \times \mathcal{I}) + 1$

Reactive and generative systems

Example:



Reactive system

Generative system

Bisimulation - generative systems

$\langle S, \alpha \rangle,\ \langle T, \beta \rangle$ - generative systems

$R \subseteq S \times T$ is a concrete bisimulation if for all $a \in A$, for all $\langle s, t \rangle \in R$ with $\mu = \alpha(s)$, $\nu = \beta(t)$ and every component C of R:

$$\mu(a, C) = \nu(a, C)$$

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$$\sum_{s' \in \pi_1(C)} \mu(a, s') = \sum_{t' \in \pi_2(C)} \nu(a, t')$$

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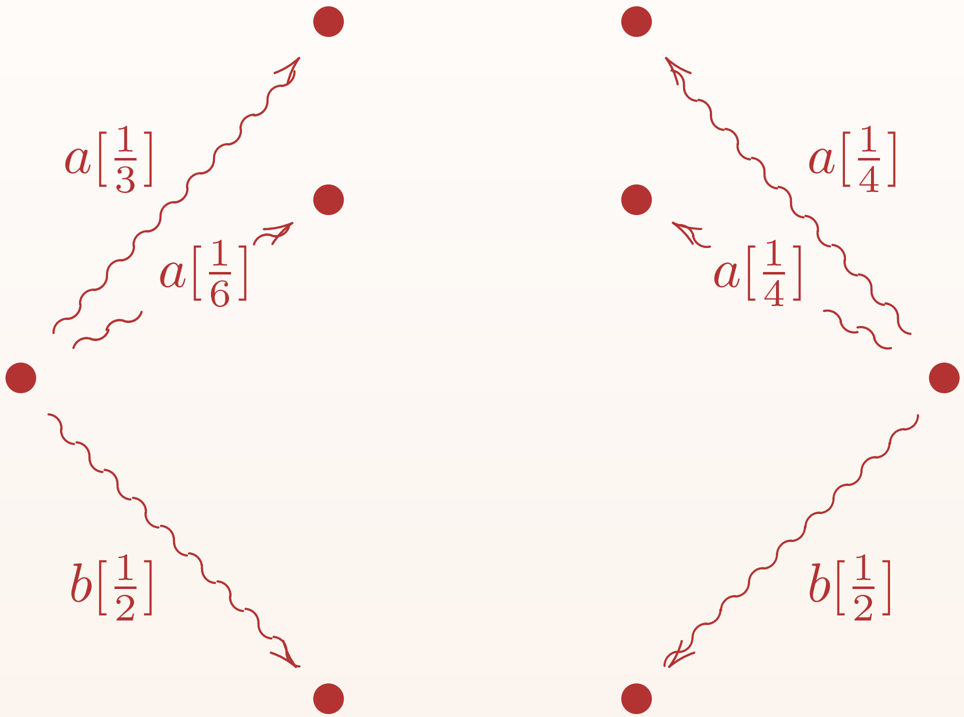

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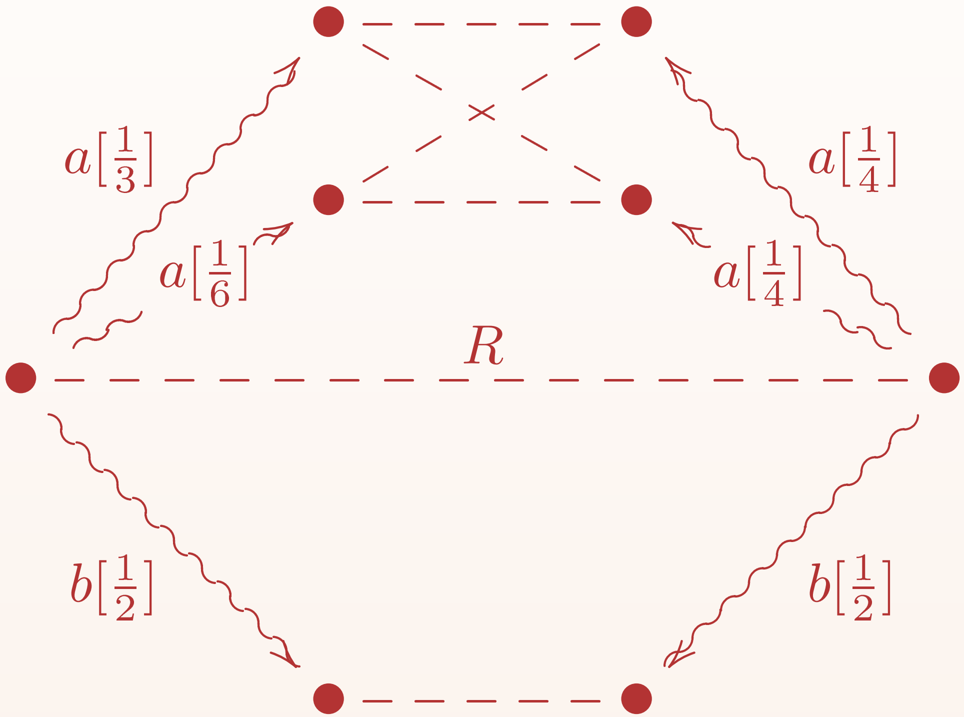

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$s \approx t$ - there exists a concrete bisimulation R with $\langle s, t \rangle \in R$

Property:

$s \approx t$ if and only if $s \sim t$

Systems with distinction of states

Alternating systems - functor $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$

Vardi systems - functor $\mathcal{D}_\omega(A \times \mathcal{I}) + \mathcal{P}(A \times \mathcal{I})$

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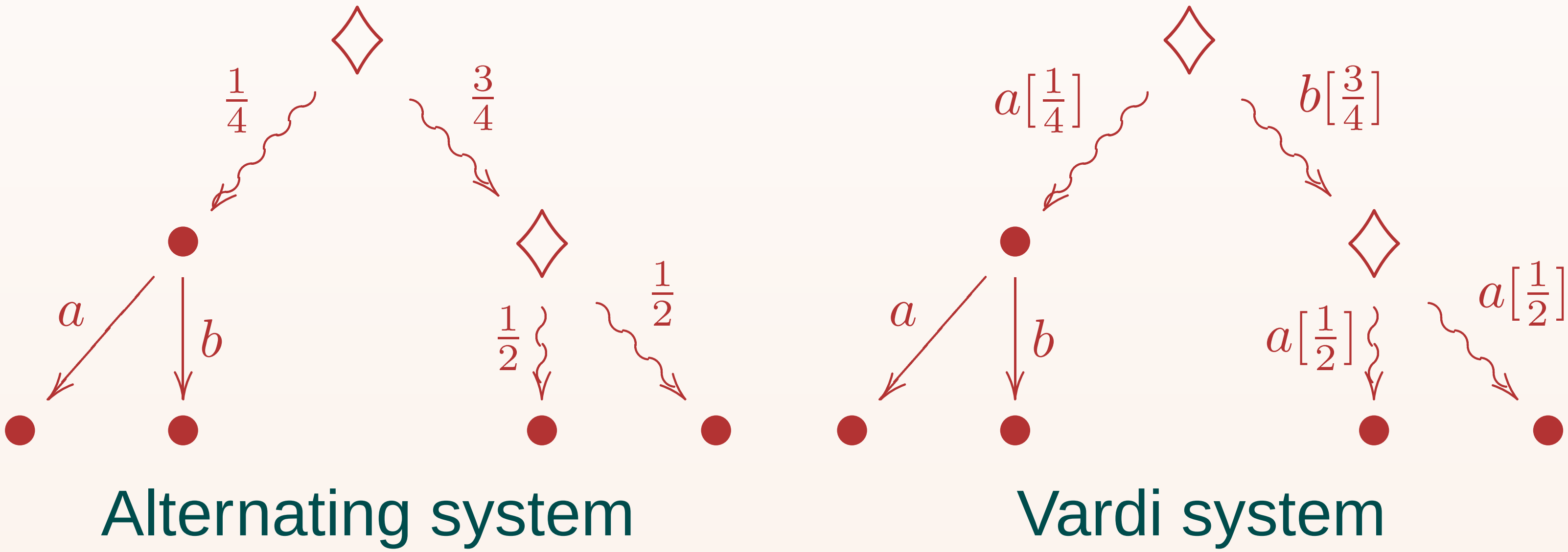

Structured transition function

Segala systems - functor $\mathcal{P}\mathcal{D}_\omega(A \times \mathcal{I})$

Structured transition function

Segala systems - functor $\mathcal{PD}_\omega(A \times \mathcal{I})$

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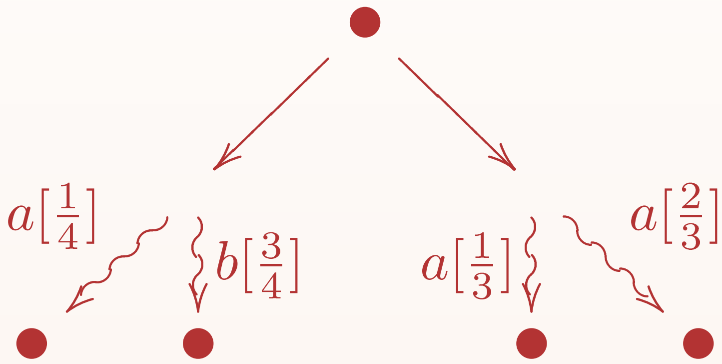

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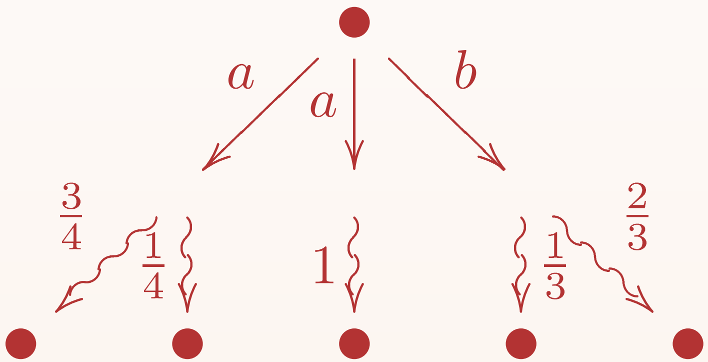

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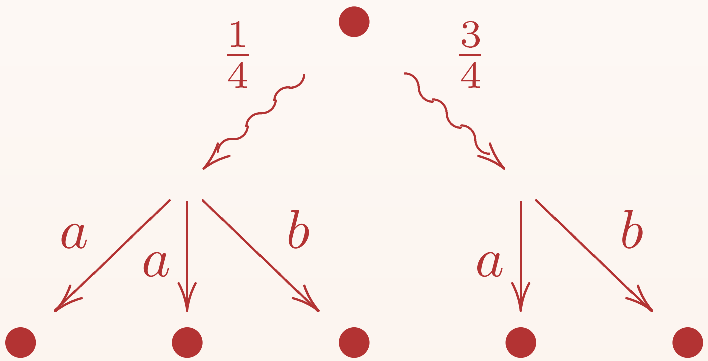

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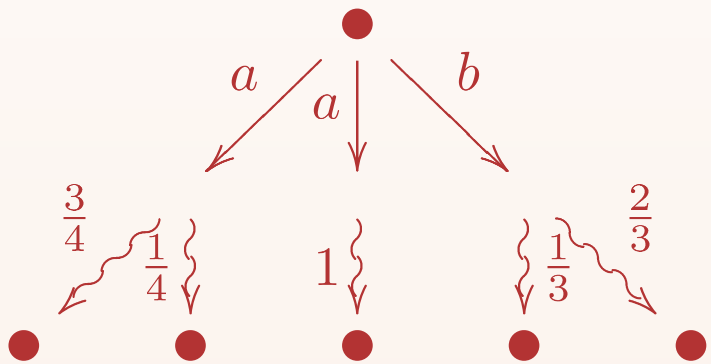
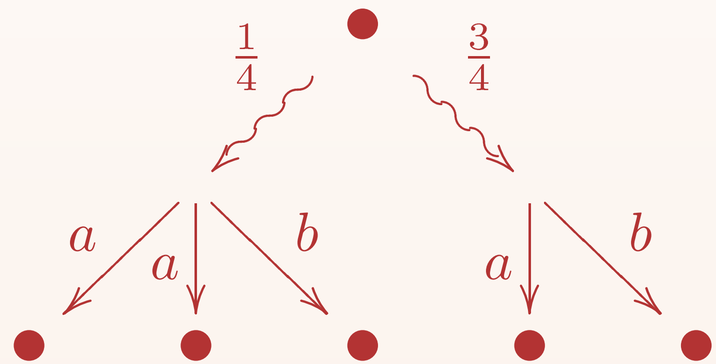
Simple Segala system $\perp$ Bundle system





Complicated system types

Pnueli-Zuck systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I})$

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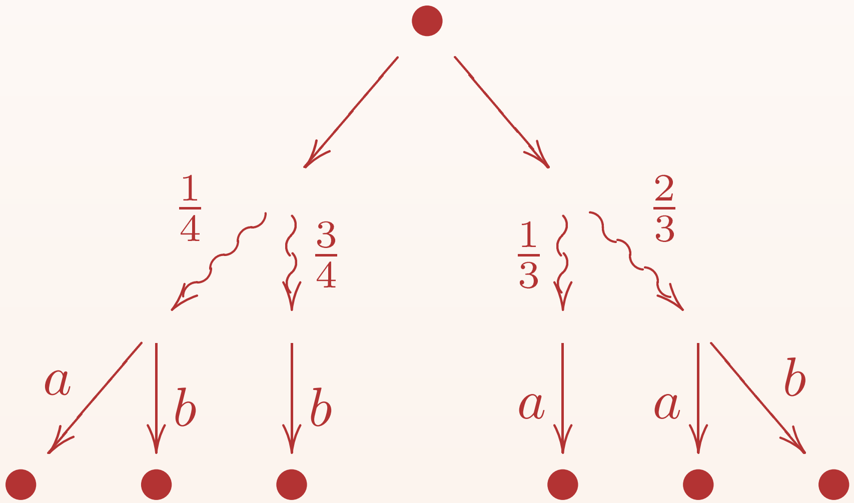

Complicated system types

Pnueli-Zuck systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I})$

most general systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I} + \mathcal{I})$

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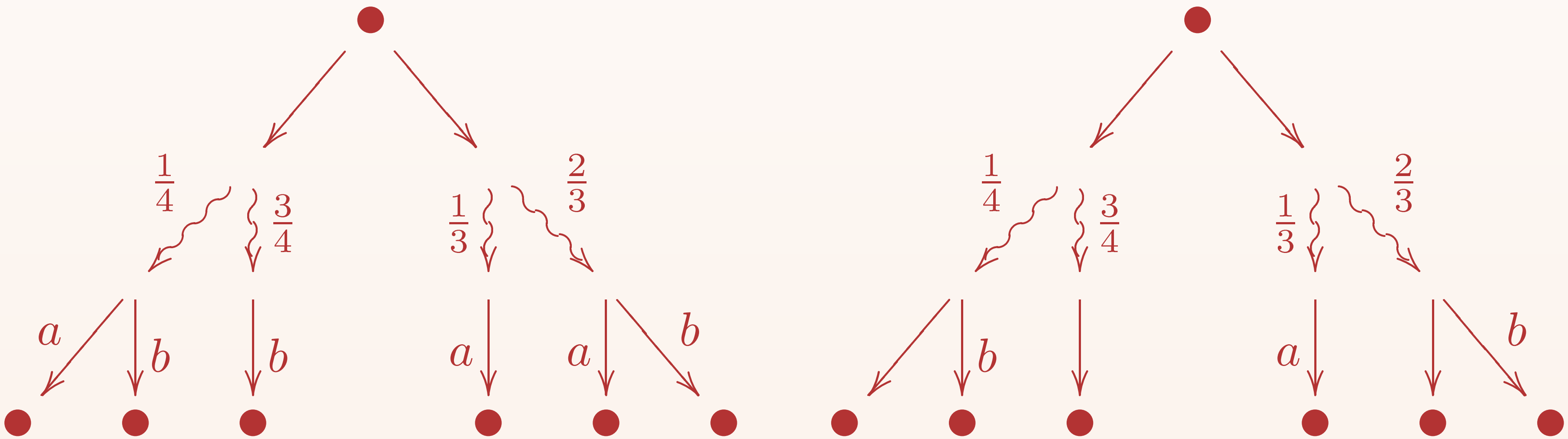
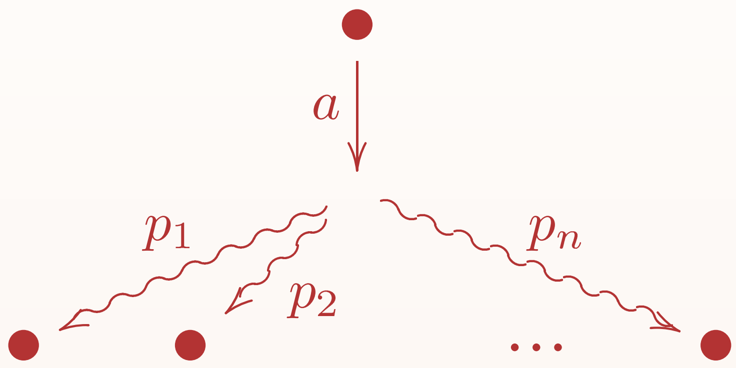


Pnueli-Zuck system

most general system

An intuitive translation

simple Segala system $\longrightarrow$ Segala system



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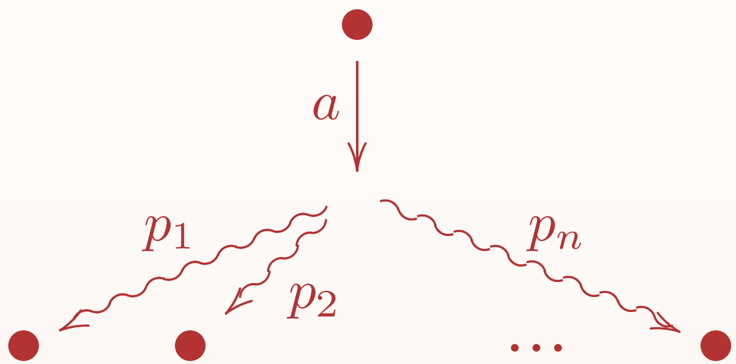

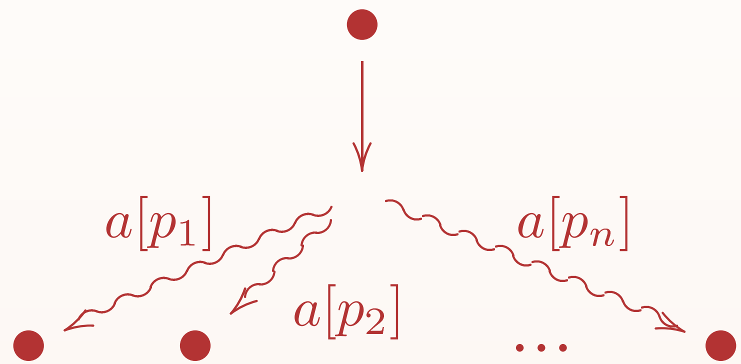

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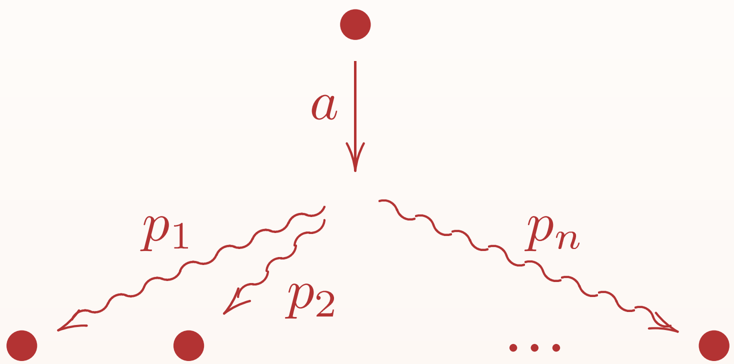

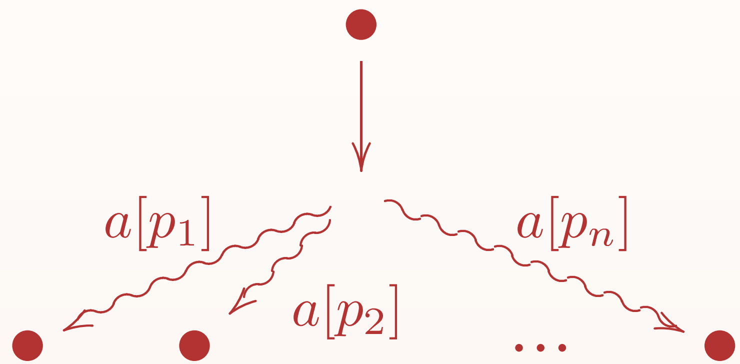

$$\langle S, \alpha \rangle$$
$$\alpha : S \to \mathcal{P}(A \times \mathcal{D}_\omega S)$$
$$\alpha(s) = \{(a_i, \mu_i) \mid i \in I\}$$

$$\langle S, \alpha' \rangle$$
$$\alpha' : S \to \mathcal{P}\mathcal{D}_\omega(A \times S)$$
$$\alpha'(s) = \{\delta_{a_i} \cdot \mu_i \mid i \in I\}$$

where $(\mu \cdot \mu')(x, x') = \mu(x) \cdot \mu'(x')$

and $\delta_a(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$

An intuitive translation

simple Segala system $\longrightarrow$ Segala system

When do we consider one type of systems more expressive than another?

Expressiveness

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Example:

LTS ($\mathcal{P}(A \times \mathcal{I})$)

less expressive than

Alternating Systems ($\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$)

Expressiveness (2)

Our approach:

Systems of type $\mathcal{F}$ are at most as expressive as systems of type $\mathcal{G}$, if there is a mapping

$$\mathcal{T} : \mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$$

with

$$\langle S, \alpha \rangle \stackrel{\mathcal{T}}{\mapsto} \langle S, \tilde{\alpha} \rangle$$

that *preserves* and *reflects* bisimilarity:

$$s_{\langle S,\alpha \rangle} \sim t_{\langle T,\beta \rangle} \iff s_{\mathcal{T}\langle S,\alpha \rangle} \sim t_{\mathcal{T}\langle T,\beta \rangle}$$

Translation of coalgebras

For LTS vs. Alternating Systems there exists a natural transformation

$$\iota_r : \mathcal{P}(A \times \mathcal{I}) \Rightarrow \mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I}).$$

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Generally, a natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ gives rise to a translation of coalgebras $\mathcal{T}_\tau : \mathsf{Coalg}_{\mathcal{F}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}$ as follows:

$$\begin{tikzcd}
S \arrow[dd, "\alpha"'] & & S \arrow[d, "\alpha"] \\
 & {} \arrow[r, phantom, "\overset{\mathcal{T}_\tau}{\Longrightarrow}"] & \mathcal{F}S \arrow[d, "\tau_S"] \\
\mathcal{F}S & & \mathcal{G}S
\end{tikzcd}$$

Preservation of bisimulations

The translation $\mathcal{T}_\tau$ preserves bisimulations:

A bisimulation $R \subseteq S \times T$ between $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$

$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
\mathcal{F}S & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"'] \arrow[r, "\mathcal{F}\pi_2"] & \mathcal{F}T
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\mathcal{F}S \arrow[d, "\tau_S"'] \arrow[r, phantom, shift right=6ex, "\text{nat. } \tau"] & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"'] \arrow[r, "\mathcal{F}\pi_2"] \arrow[d, "\tau_R"] \arrow[r, phantom, shift right=6ex, "\text{nat. } \tau"'] & \mathcal{F}T \arrow[d, "\tau_T"] \\
\mathcal{G}S & \mathcal{G}R \arrow[l, "\mathcal{G}\pi_1"] \arrow[r, "\mathcal{G}\pi_2"'] & \mathcal{G}T
\end{tikzcd}$$

is a bisimulation between $\mathcal{T}_\tau\langle S, \alpha\rangle$ and $\mathcal{T}_\tau\langle T, \beta\rangle$ as well.

Reflection of bisimilarity

But: $\mathcal{T}_\tau$ need not reflect bisimilarity.

Example:

Let τ be the natural transformation

$$\widetilde{\mathsf{supp}} : \mathcal{D}_\omega + 1 \Rightarrow \mathcal{P}$$

that *forgets* the probabilities.

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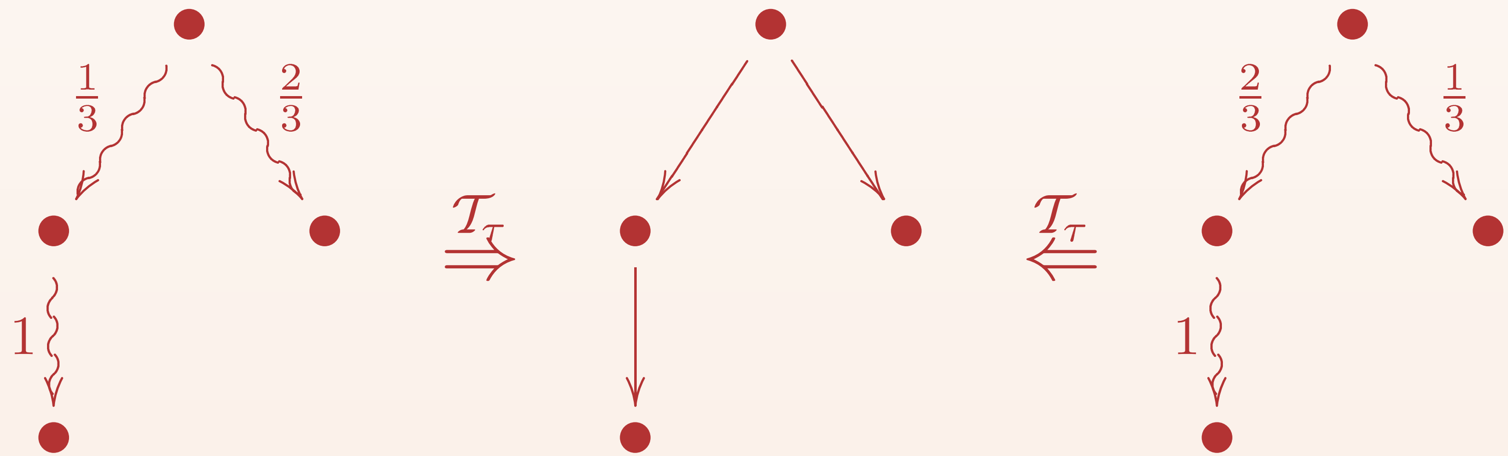

Assumption: Injectivity

Observation:
the components of $\widetilde{\mathrm{supp}}$ are not injective.

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Injectivity is not necessary
Example:
$\mathrm{supp} : \mathcal{D}_\omega \Rightarrow \mathcal{P}$

Assumption: Injectivity

Observation:
the components of $\widetilde{\mathsf{supp}}$ are not injective.

$\Rightarrow$ to obtain reflection of bisimilarity we assume that τ is componentwise injective.

Injectivity is not necessary
Example:
$\mathsf{supp} : \mathcal{D}_\omega \Rightarrow \mathcal{P}$

... for the proof another notion of behaviour equivalence is needed...

Cocongruences

A *cocongruence* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *cospan*

$$\langle Q, q_1 : S \to Q, q_2 : T \to Q \rangle$$

such that there exists a $\mathcal{F}$-coalgebra structure γ on Q making the diagram below commute.

$$\begin{tikzcd}
S \arrow[r, "q_1"] \arrow[d, "\alpha"'] & Q \arrow[d, "\gamma"] & T \arrow[l, "q_2"'] \arrow[d, "\beta"] \\
\mathcal{F}S \arrow[r, "\mathcal{F}q_1"'] & \mathcal{F}Q & \mathcal{F}T \arrow[l, "\mathcal{F}q_2"]
\end{tikzcd}$$

Behavioural equivalence

states s and t in two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ are *behavioural equivalent* if they are identified by some cocongruence.

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Result:
If all components of the natural transformation

$$\tau : \mathcal{F} \Rightarrow \mathcal{G}$$

are injective, then $\mathcal{T}_\tau$ reflects behavioural equivalence.

Bisimularity vs. beh. equivalence

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Corollary:
If $\mathcal{F}$ preserves weak pullbacks and all components of $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ are injective, then $\mathcal{T}_\tau$ reflects bisimilarity.

Need of w.p. preservation

The assumption
that $\mathcal{F}$ preserves weak pullbacks is necessary.

Example:
Consider the functors

$$\mathcal{F}X := \{\langle x, y, z\rangle \in X^3 \mid |\{x, y, z\}| \leq 2\}$$

and

$$\mathcal{G}X := X^3$$

and let $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ be the set inclusion.

Weak pullback preservation here

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$\Rightarrow$ All functors used to define the different probabilistic system types preserve weak pullbacks.

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Examples of natural transformations with injective components:

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- $\ldots$

One expressiveness statement

Generative systems

$$(\mathcal{F} := \mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + 1)$$

are at most as expressive as
Vardi systems

$$(\mathcal{G} := \mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + \mathcal{P}(\mathrm{A} \times \mathcal{I}))$$

natural transformation

$$\mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + \eta(\mathrm{A} \times \mathcal{I}) : \mathcal{F} \Rightarrow \mathcal{G}.$$

Another expressiveness statement

simple Segala systems

$$(\mathcal{F} := \mathcal{P}(\mathrm{A} \times \mathcal{D}_\omega))$$

are at most as expressive as
Segala systems

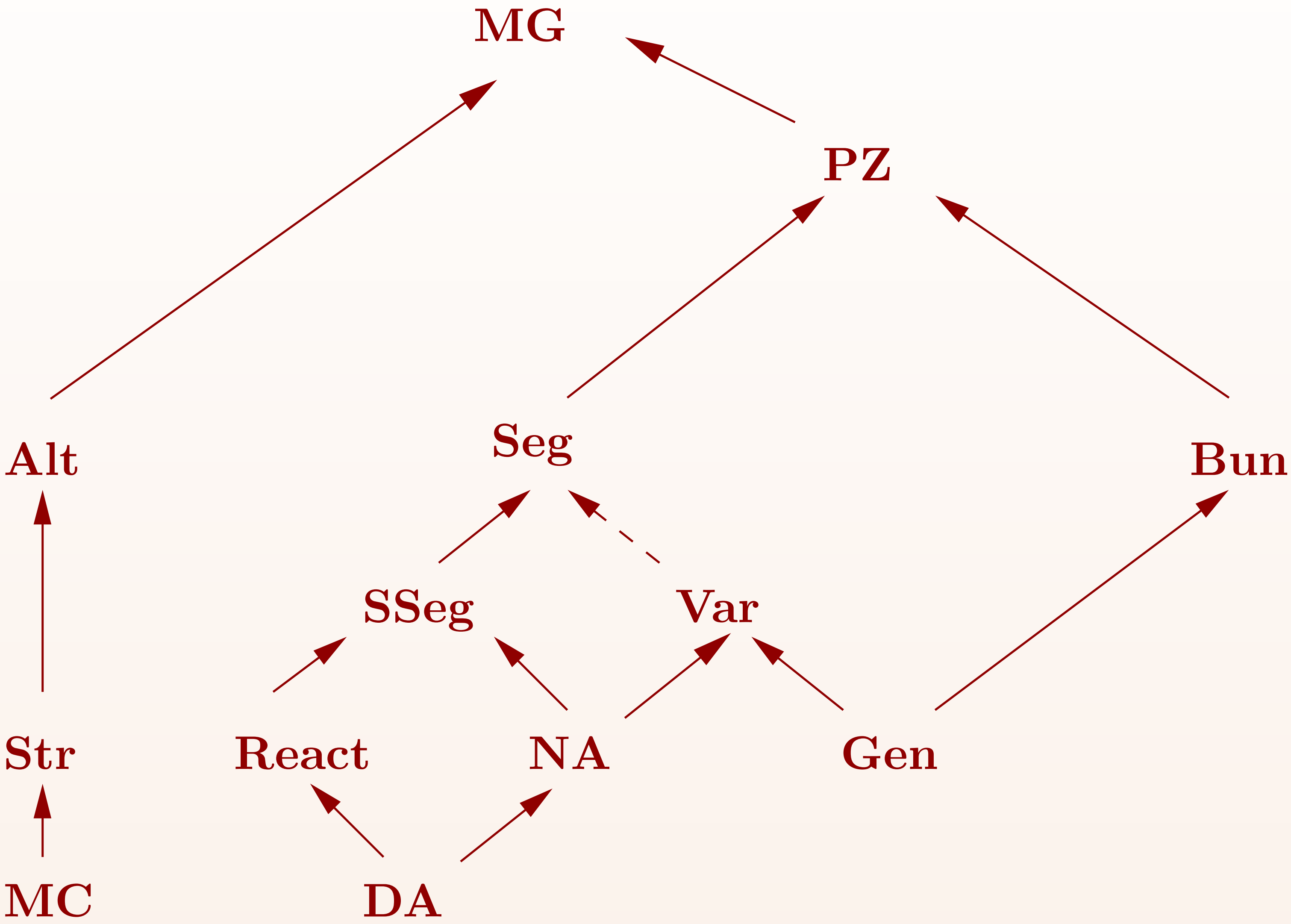
$$(\mathcal{G} := \mathcal{P}\mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}))$$

natural transformation

$$\mathcal{P}\tau : \mathcal{F} \Rightarrow \mathcal{G}$$

for $\tau_X(\langle a, \mu \rangle) = \delta_a \cdot \mu$
where $(\mu \cdot \mu')(x, x') = \mu(x) \cdot \mu'(x')$
and ...

The hierarchy of system types



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  - $*$ providing a uniform framework
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  - $*$ proving a comparison result