Hierarchy of probabilistic systems

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CWI and TU/e

Outline

- Introduction

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  * labelled transition systems

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  * bisimulation

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  * adding probabilities

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  - * preservation and reflection of bisimulation
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  - * translation of coalgebras
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- Conclusions

Labelled transition systems

LTS is a pair $\langle S, \alpha : S \to \mathcal{P}S^A \rangle$

A - a fixed set of actions

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Hence a coalgebra

$$\langle S, \alpha \rangle,\ \alpha : S \to \mathcal{F}S$$

of the functor $\mathcal{F} = \mathcal{P}^A$

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Example:

$$\begin{tikzcd}
& s_2 \arrow[r, "a"] \arrow[dr, "a"'] & s_4 & & \\
s_1 \arrow[ur, "b"] \arrow[dr, "a"'] & & s_5 \arrow[r, "b"] & s_6 & \cdots \\
& s_3 & & &
\end{tikzcd}$$

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Note: $\mathcal{P}^A \cong \mathcal{P}(A \times \mathcal{I})$

Concrete bisimulation for LTS

$\langle S, \alpha \rangle$, $\langle T, \beta \rangle$ - LTS

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$$s \xrightarrow{a} s' \Rightarrow (\exists t')t \xrightarrow{a} t',\ \langle s', t' \rangle \in R \text{ and}$$

$$t \xrightarrow{a} t' \Rightarrow (\exists s')s \xrightarrow{a} s',\ \langle s', t' \rangle \in R$$

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$s \approx t$ - there exists a concrete bisimulation

R with $\langle s, t \rangle \in R$

Coalgebraic bisimulation

A *bisimulation* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a relation

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$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
\mathcal{F}S & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"'] \arrow[r, "\mathcal{F}\pi_2"] & \mathcal{F}T
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Concrete vs coalgebraic (LTS)

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(coalgebras of type $\mathcal{F} = \mathcal{P}^A$)

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known: $s \approx t$ if and only if $s \sim t$

Introduction of probabilities

There are many ways to do it ...

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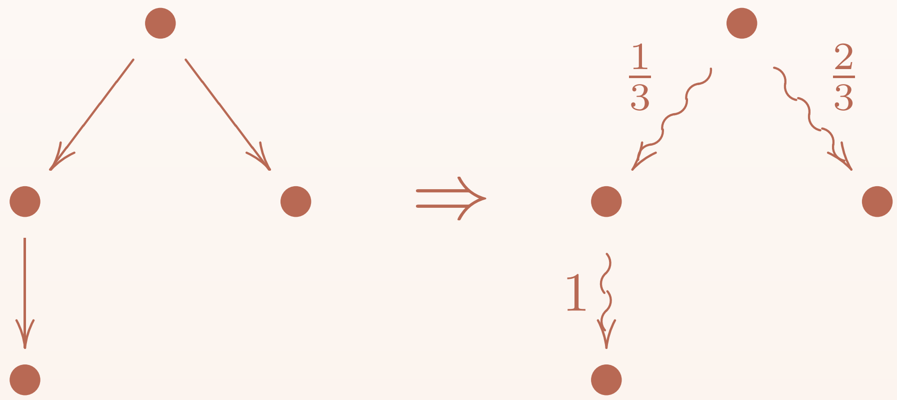

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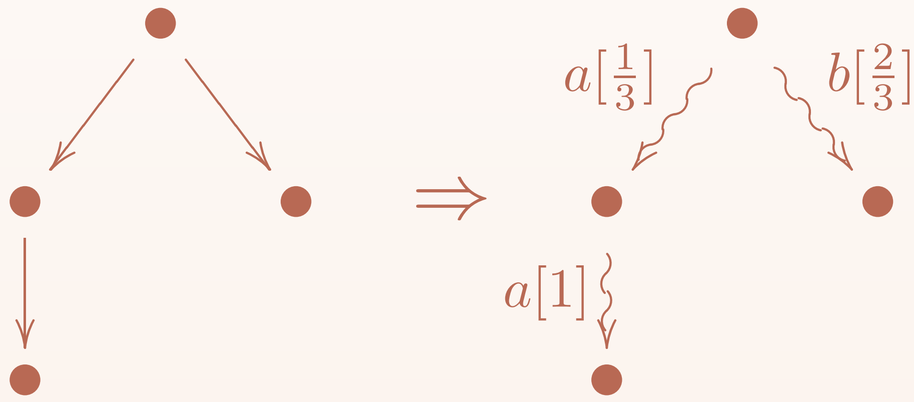

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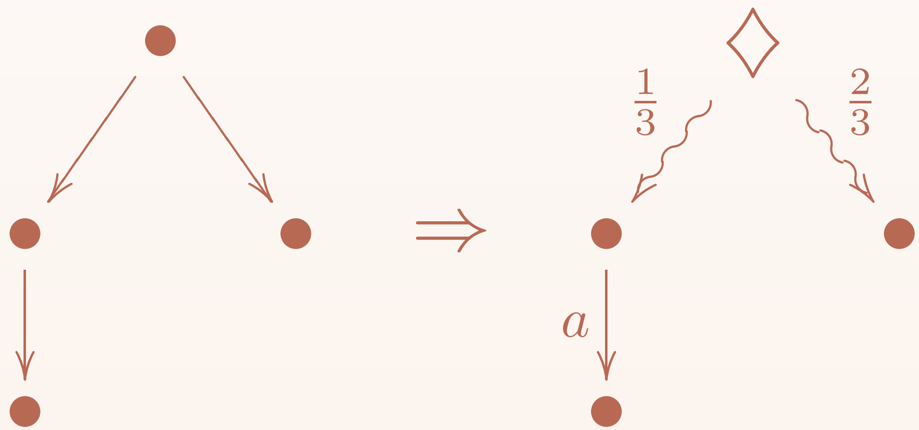

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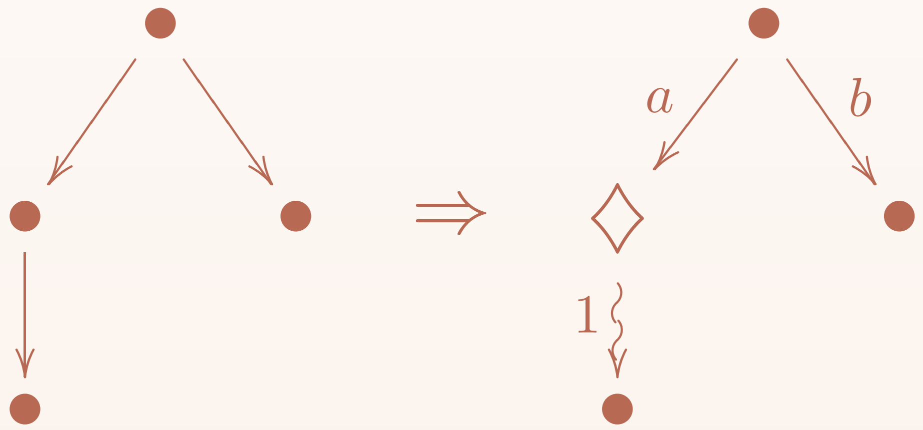

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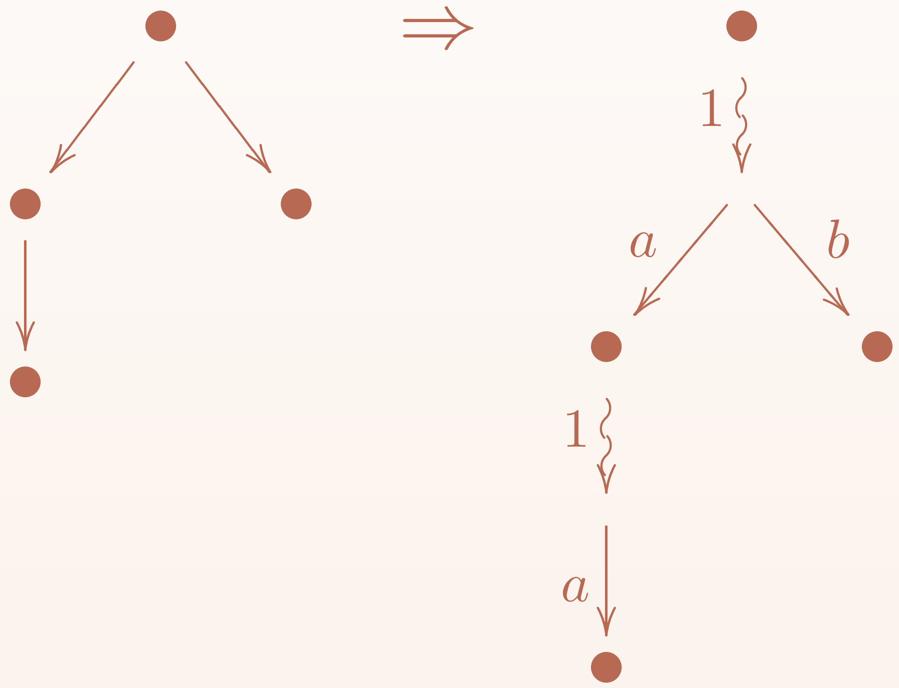

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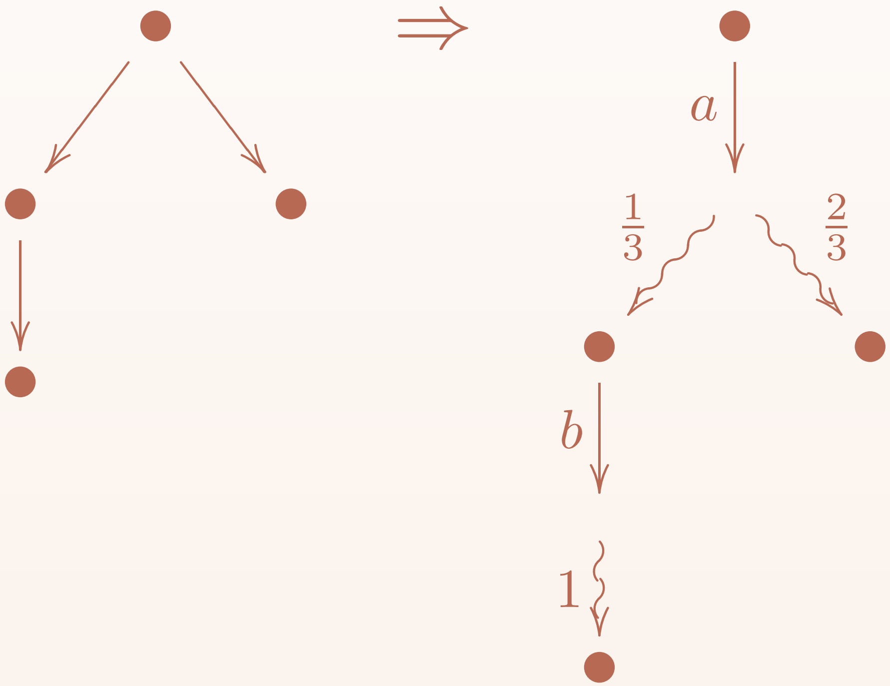

Introduction of probabilities

There are many ways to do it ...

13 types of systems - from the literature

with (or without):

- action labels
- nondeterminism
- probabilities

Existing system types

MG

PZ

Alt **Seg** **Bun**

SSeg **Var**

Str **React** **NA** **Gen**

MC **DA**

System types

The (probabilistic) models of systems we consider are coalgebras

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for a functor $\mathcal{F}$ built by the following syntax

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for a functor $\mathcal{F}$ built by the following syntax

$$\mathcal{F} ::= A \mid \mathcal{I} \mid \mathcal{P} \mid \mathcal{D}_\omega \mid \mathcal{F} + \mathcal{F} \mid \mathcal{F} \times \mathcal{F} \mid \mathcal{F}^A \mid \mathcal{F}\mathcal{F}$$

Reactive and generative systems

evolve from LTS - functor $\mathcal{P}^{A} \cong \mathcal{P}(A \times \mathcal{I})$

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functor $(\mathcal{D}_\omega + 1)(A \times \mathcal{I}) = \mathcal{D}_\omega(A \times \mathcal{I}) + 1$

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Reactive systems - input type

functor $(\mathcal{D}_\omega + 1)^A$

Generative systems - output type

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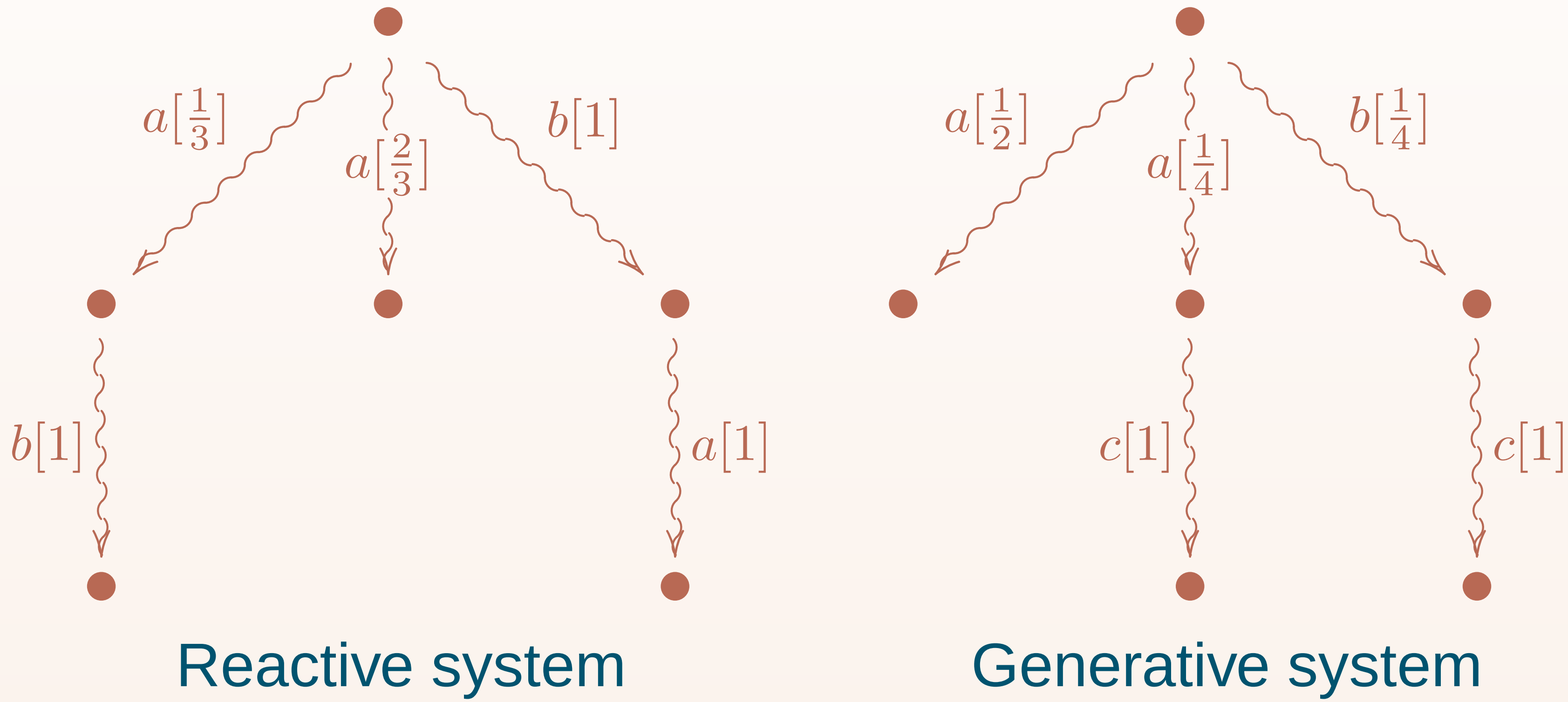

Bisimulation - generative systems

$\langle S, \alpha \rangle, \ \langle T, \beta \rangle$ - generative systems

$R \subseteq S \times T$ is a concrete bisimulation if for all $a \in A$, for all $\langle s, t \rangle \in R$ with $\mu = \alpha(s)$, $\nu = \beta(t)$ and every component C of R:

$$\mu(a, C) = \nu(a, C)$$

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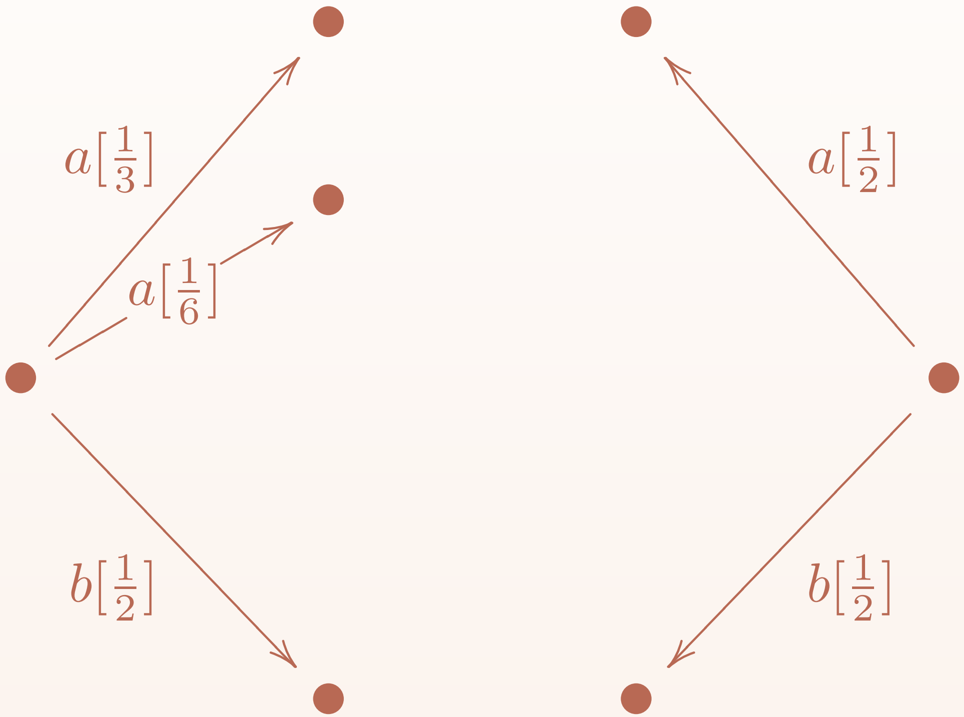

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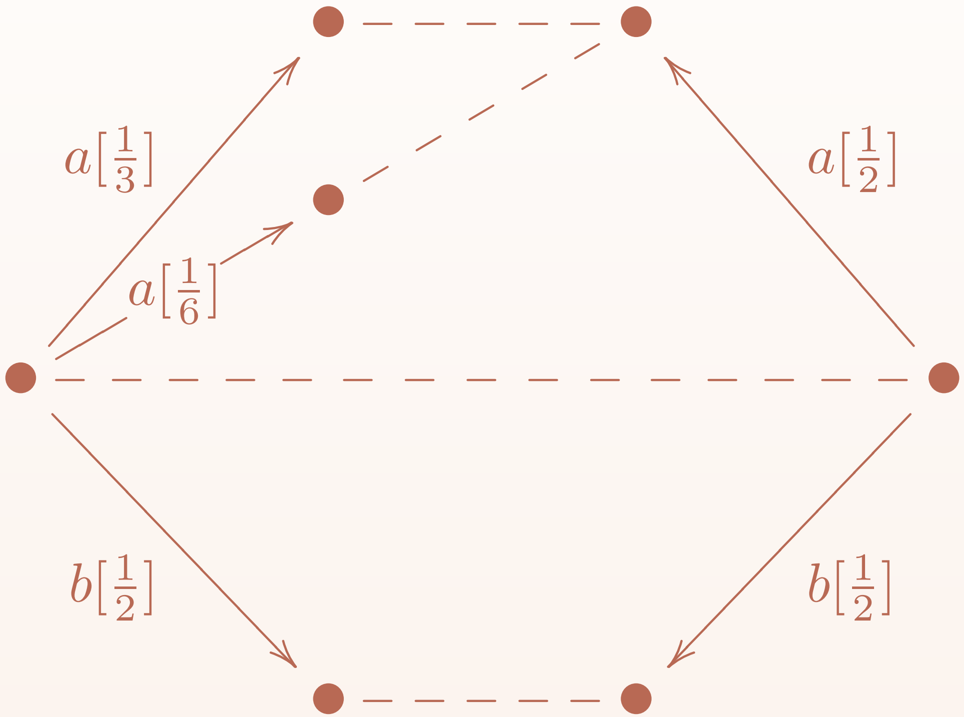

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Property:

$s \approx t$ if and only if $s \sim t$

Systems with distinction of states

Alternating systems - functor $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$

Vardi systems - functor $\mathcal{D}_\omega(A \times \mathcal{I}) + \mathcal{P}(A \times \mathcal{I})$

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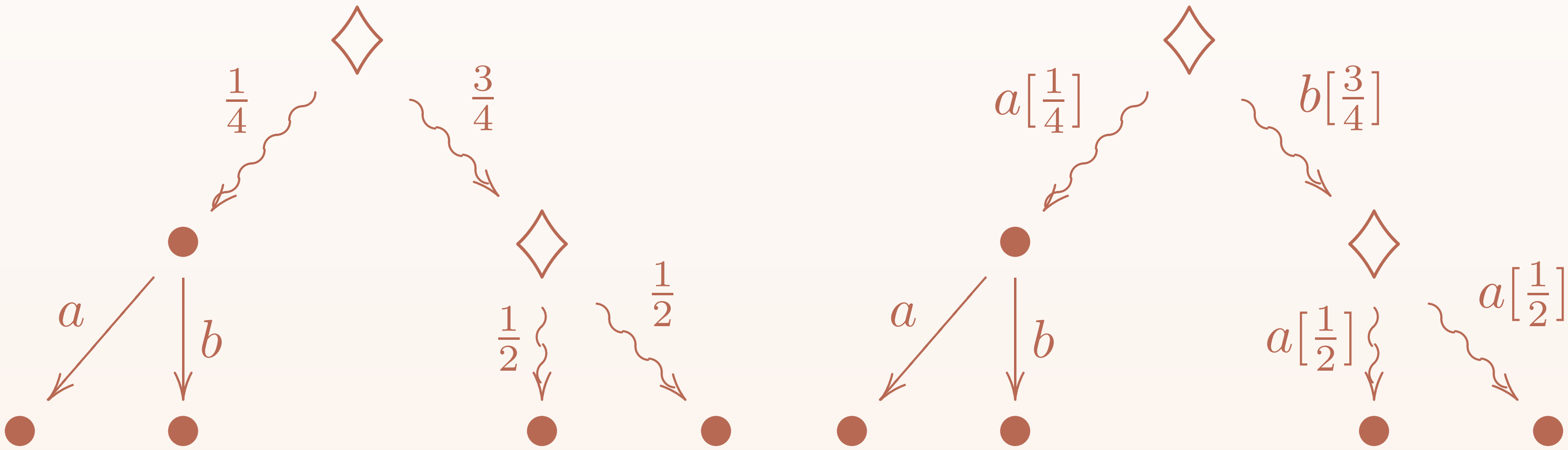


Alternating system Vardi system

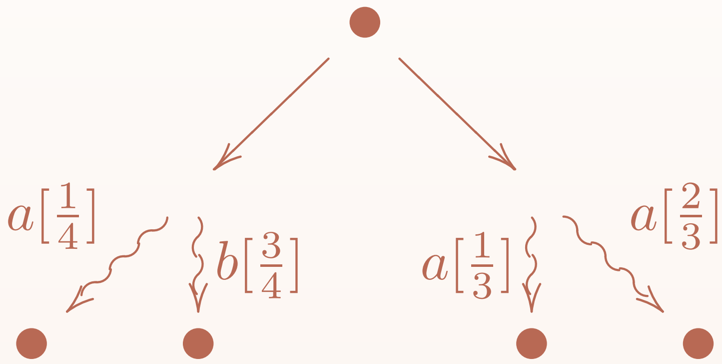
Structured transition function

Segala systems - functor $\mathcal{P}\mathcal{D}_\omega(A \times \mathcal{I})$

Structured transition function

Segala systems - functor $\mathcal{PD}_\omega(A \times \mathcal{I})$

Example:



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simple Segala systems - functor $\mathcal{P}(A \times \mathcal{D}_\omega)$

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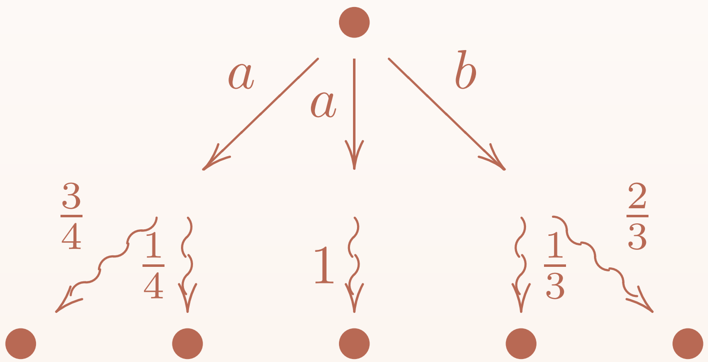

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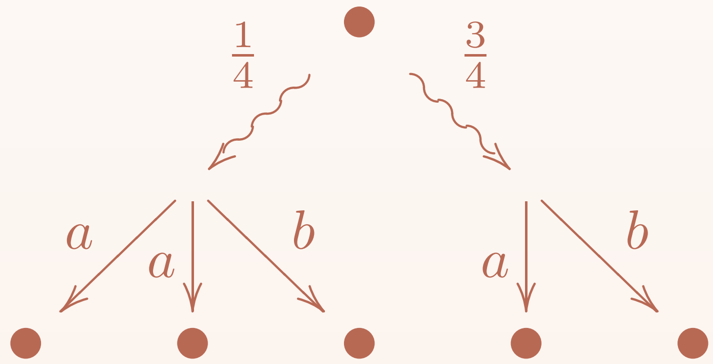

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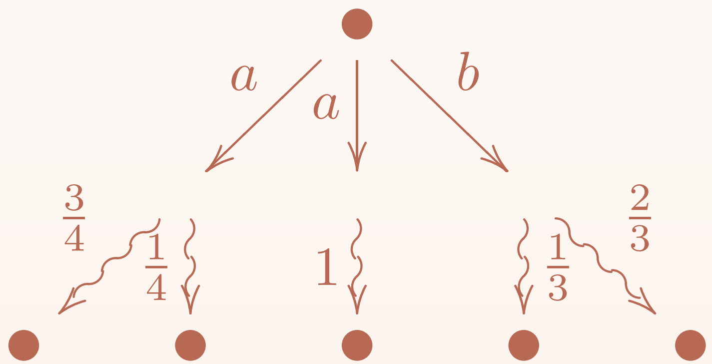
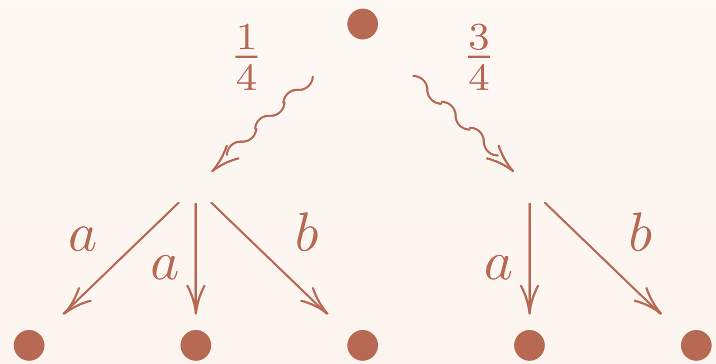
Simple Segala system $\quad\perp\quad$ Bundle system





Complicated system types

Pnueli-Zuck systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I})$

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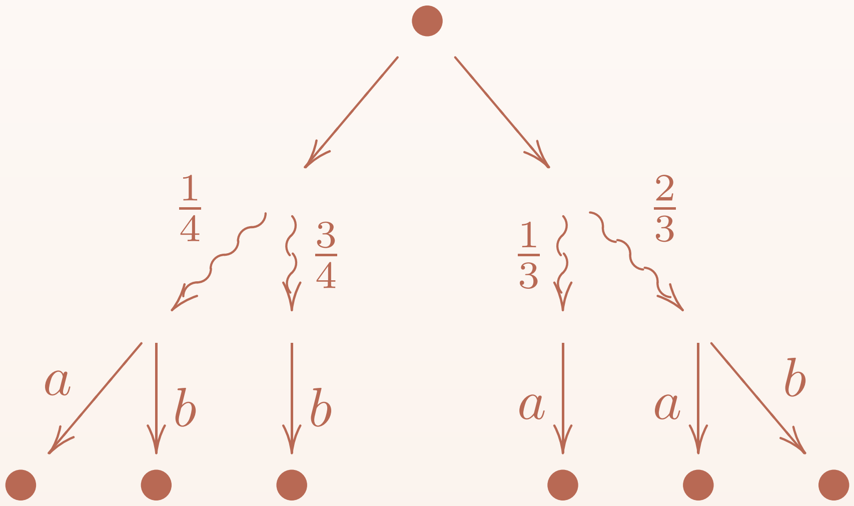
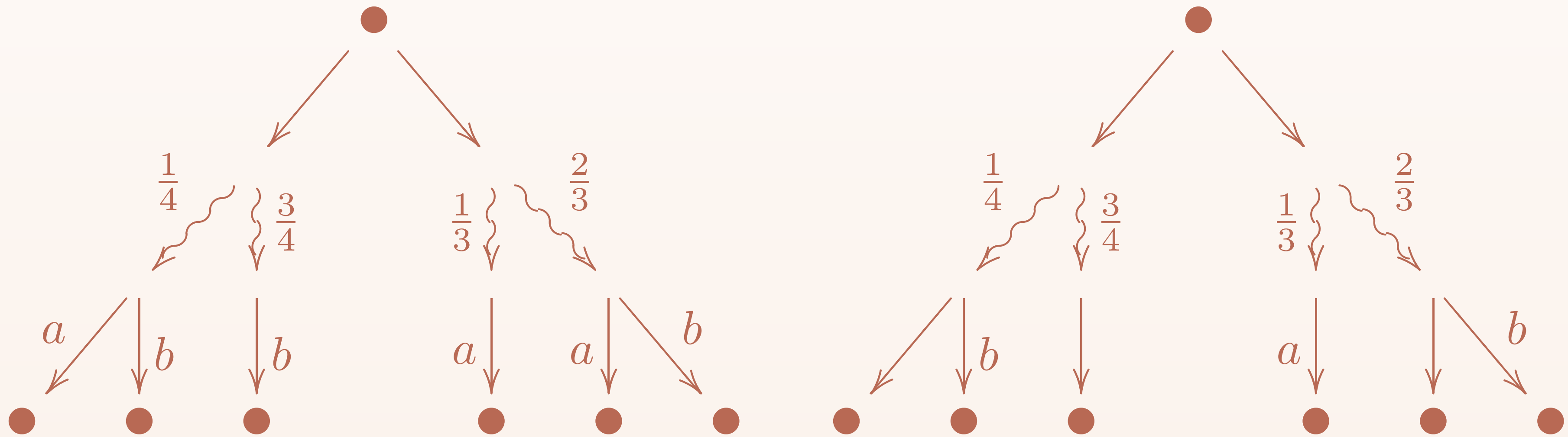


Complicated system types

Pnueli-Zuck systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I})$

most general systems - functor $\mathcal{P}\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I}+\mathcal{I})$

Example:

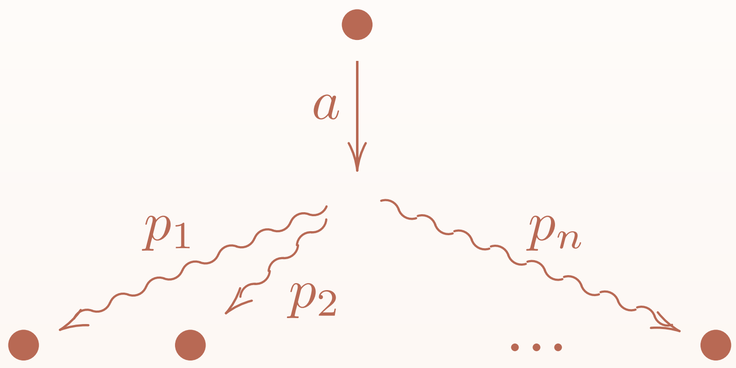


Pnueli-Zuck system

most general system

An intuitive translation

simple Segala system $\longrightarrow$ Segala system



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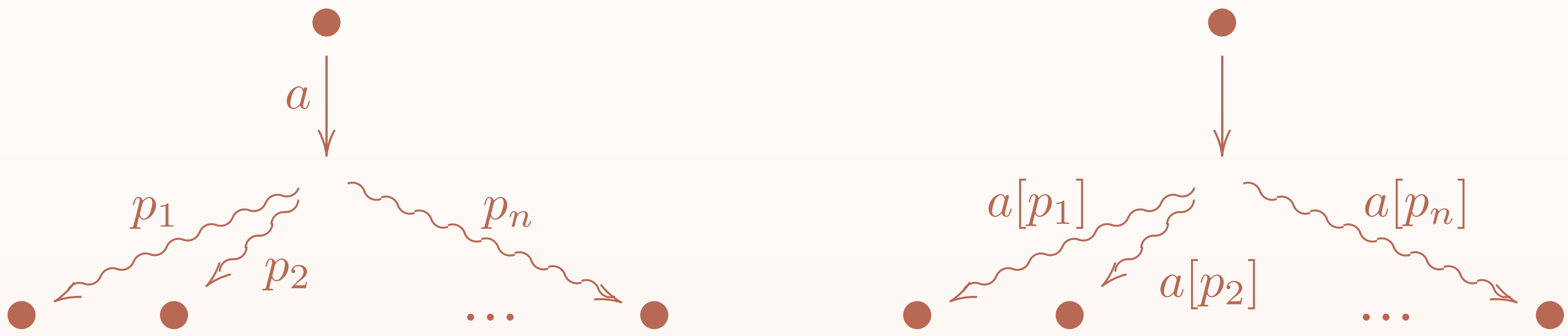

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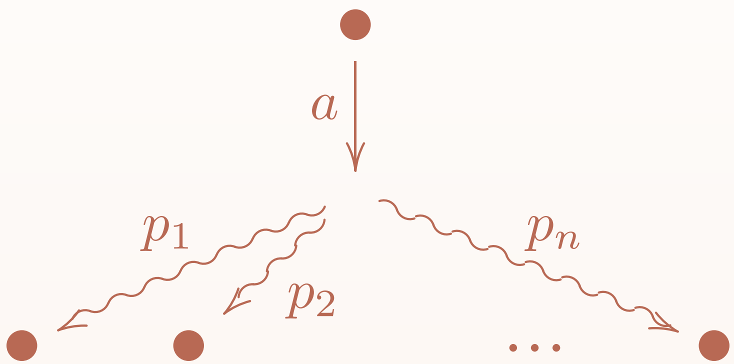

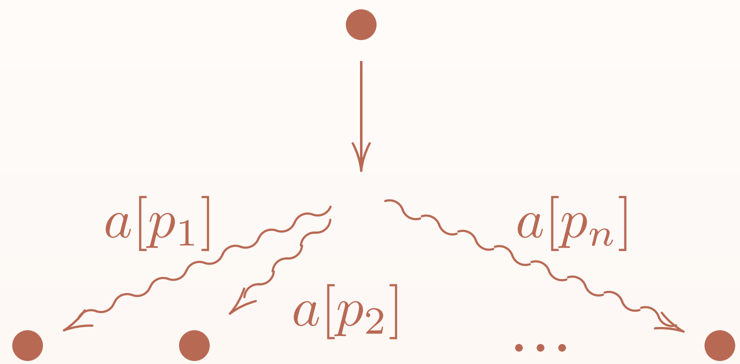

$$\langle S, \alpha \rangle \qquad \langle S, \alpha' \rangle$$

$$\alpha : S \to \mathcal{P}(A \times \mathcal{D}_\omega S) \qquad \alpha' : S \to \mathcal{P}\mathcal{D}_\omega(A \times S)$$

$$\alpha(s) = \{(a_i, \mu_i) \mid i \in I\} \qquad \alpha'(s) = \{\delta_{a_i} \cdot \mu_i \mid i \in I\}$$

where $(\mu \cdot \mu')(x, x') = \mu(x) \cdot \mu'(x')$

and $\delta_a(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$

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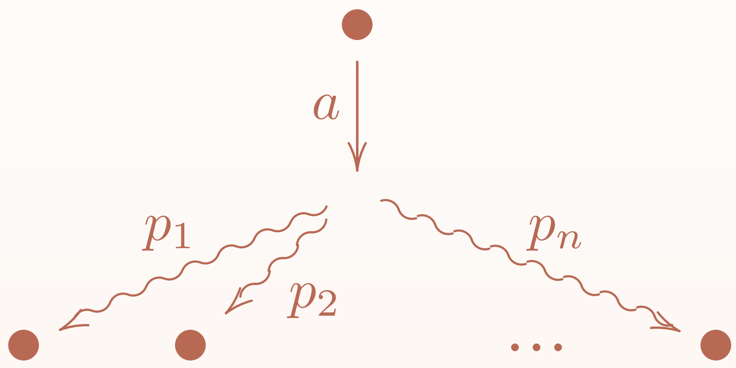

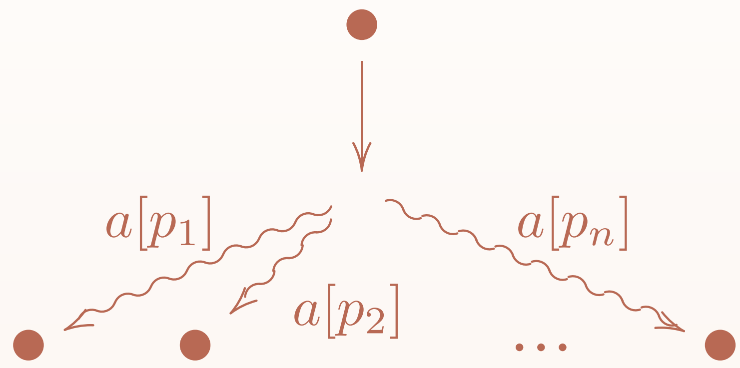

When do we consider one type of systems more expressive than another?

Expressiveness

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Example:

LTS (functor: $\mathcal{P}(A \times \mathcal{I})$)
are clearly less expressive than
Alternating Systems (functor: $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$):

Any LTS can be viewed as an Alternating System that never uses the option to do a probabilistic step.

Expressiveness (2)

Our approach:

Systems of type $\mathcal{F}$ are at most as expressive as systems of type $\mathcal{G}$, if there is a mapping

$$\mathcal{T} : \mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$$

with

$$\langle S, \alpha \rangle \overset{\mathcal{T}}{\longmapsto} \langle S, \tilde{\alpha} \rangle$$

that *preserves* and *reflects* bisimilarity:

$$s_{\langle S,\alpha\rangle} \sim t_{\langle T,\beta\rangle} \iff s_{\mathcal{T}\langle S,\alpha\rangle} \sim t_{\mathcal{T}\langle T,\beta\rangle}$$

Translation of coalgebras

Note that in the LTS vs. Alternating Systems example there exists a natural transformation

$$\iota_r : \mathcal{P}(A \times \mathcal{I}) \Rightarrow \mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I}).$$

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Generally, a natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ gives rise to a translation of coalgebras $\mathcal{T}_\tau : \mathsf{Coalg}_{\mathcal{F}} \rightarrow \mathsf{Coalg}_{\mathcal{G}}$ as follows:

$$\begin{tikzcd} S \arrow[dd, "\alpha"'] & & S \arrow[d, "\alpha"] \\ & \arrow[r, phantom, "\overset{\mathcal{T}_\tau}{\Longrightarrow}"] & \mathcal{F}S \arrow[d, "\tau_S"] \\ \mathcal{F}S & & \mathcal{G}S \end{tikzcd}$$

Preservation of bisimulations

The translation $\mathcal{T}_\tau$ preserves bisimulations:

A bisimulation $R \subseteq S \times T$ between $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$

$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
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\mathcal{F}S \arrow[d, "\tau_S"'] \arrow[dr, phantom, "\text{nat. } \tau"] & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"'] \arrow[r, "\mathcal{F}\pi_2"] \arrow[d, "\tau_R"] \arrow[dr, phantom, "\text{nat. } \tau"] & \mathcal{F}T \arrow[d, "\tau_T"] \\
\mathcal{G}S & \mathcal{G}R \arrow[l, "\mathcal{G}\pi_1"] \arrow[r, "\mathcal{G}\pi_2"'] & \mathcal{G}T
\end{tikzcd}
$$

is a bisimulation between $\mathcal{T}_\tau \langle S, \alpha \rangle$ and $\mathcal{T}_\tau \langle T, \beta \rangle$ as well.

Reflection of bisimilarity

But: $\mathcal{T}_\tau$ need not reflect bisimilarity.

Example:

Let τ be the natural transformation

$$\widetilde{\mathsf{supp}} : \mathcal{D}_\omega + 1 \Rightarrow \mathcal{P}$$

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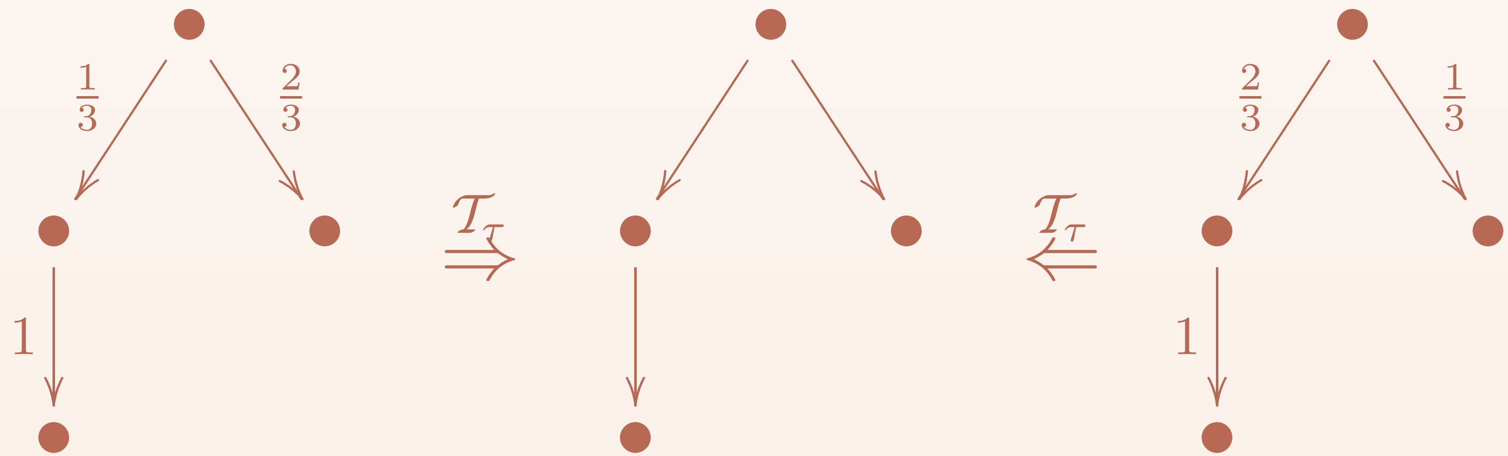

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... although intuitive, sufficiency proof is not immediate...

Cocongruences

A *cocongruence* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *cospan*

$$\langle Q, q_1 : S \to Q, q_2 : T \to Q \rangle$$

such that there exists a $\mathcal{F}$-coalgebra structure γ on Q making the diagram below commute.

$$\begin{tikzcd} S \arrow[r, "q_1"] \arrow[d, "\alpha"'] & Q \arrow[d, "\gamma"] & T \arrow[l, "q_2"'] \arrow[d, "\beta"] \\ \mathcal{F}S \arrow[r, "\mathcal{F}q_1"'] & \mathcal{F}Q & \mathcal{F}T \arrow[l, "\mathcal{F}q_2"] \end{tikzcd}$$

Behavioural equivalence

Two states s and t in two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ respectively are called *behavioural equivalent* if they are identified by some cocongruence.

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Result:
If all components of the natural transformation

$$\tau : \mathcal{F} \Rightarrow \mathcal{G}$$

are injective, then $\mathcal{T}_\tau$ reflects behavioural equivalence.

Bisimularity vs. beh. equivalence

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Corollary:
If $\mathcal{F}$ preserves weak pullbacks and all components of $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ are injective, then $\mathcal{T}_\tau$ reflects bisimilarity.

Need of w.p. preservation

The corollary is not valid without assuming that $\mathcal{F}$ preserves weak pullbacks.

Counter-example:
Consider the functors

$$\mathcal{F}X := \left\{ \langle x, y, z \rangle \in X^3 \;\middle|\; |\{x, y, z\}| \leq 2 \right\}$$

and

$$\mathcal{G}X := X^3$$

and let $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ be the set inclusion.

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$\Rightarrow$ All functors used to define the different probabilistic system types preserve weak pullbacks.

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- $\ldots$

One expressiveness statement

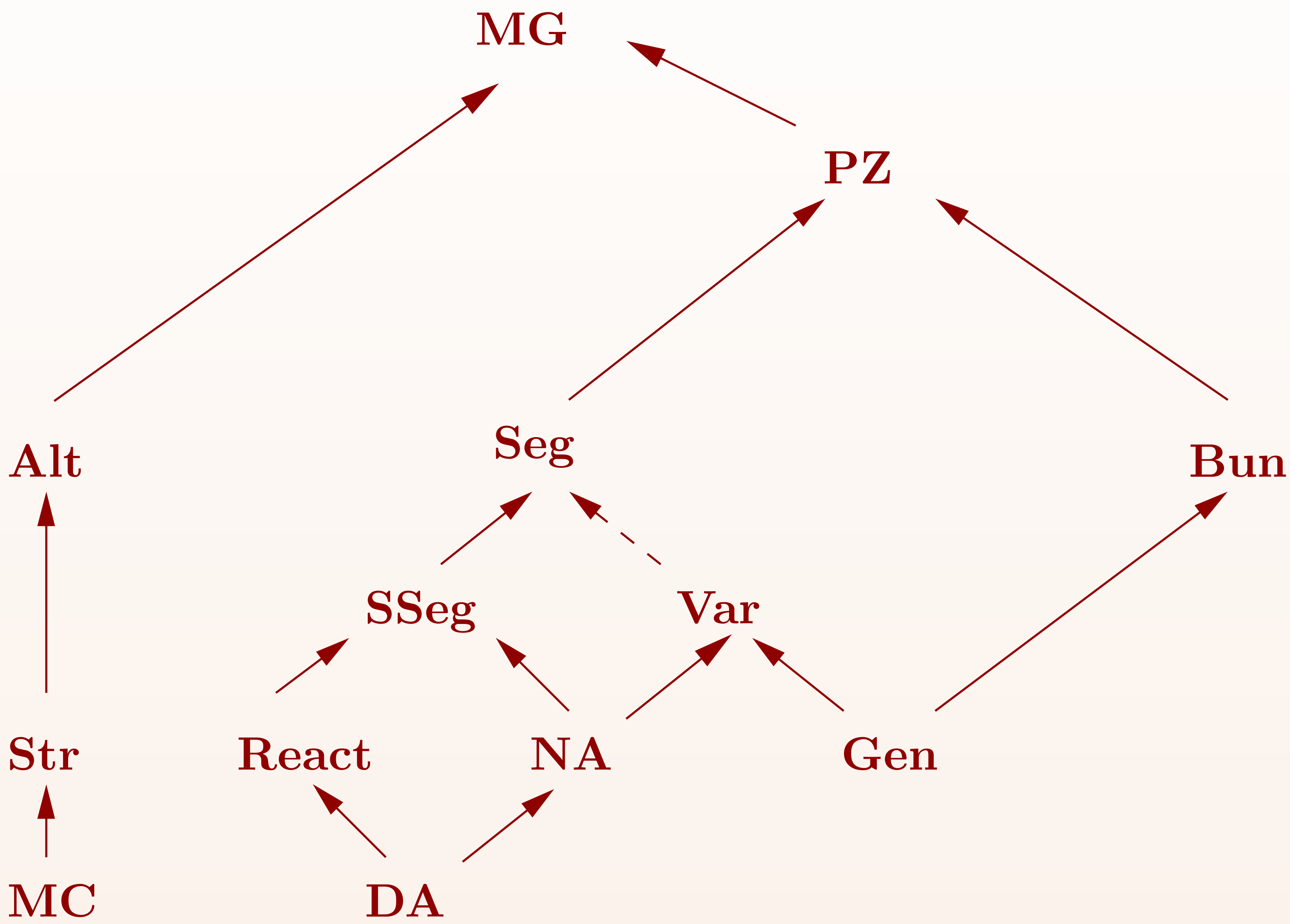
To show for instance that
Generative systems
(functor: $\mathcal{F} := \mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + 1$)

are at most as expressive as
Vardi systems
(functor: $\mathcal{G} := \mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + \mathcal{P}(\mathrm{A} \times \mathcal{I})$)

we employ the natural transformation

$$\mathcal{D}_\omega(\mathrm{A} \times \mathcal{I}) + \eta(\mathrm{A} \times \mathcal{I}) : \mathcal{F} \Rightarrow \mathcal{G}.$$

The hierarchy of system types



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