Hierarchy of probabilistic systems

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CWI and TU/e

Outline

- Introduction
- Probabilistic system types
- Comparison of system types
  * expressiveness criterion
  * translation of coalgebras
  * preservation and reflection of bisimulation
- Building the hierarchy
- Conclusions

Transition systems

TS is a pair $\langle S, \alpha : S \to \mathcal{P}S \rangle$

Hence a coalgebra of the powerset functor $\mathcal{P}$

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LTS is a pair $\langle S, \alpha : S \to \mathcal{P}S^A \rangle$

A - a fixed set of actions (labels)

Coalgebra of the functor $\mathcal{P}^A$

Note: $\mathcal{P}^A \cong \mathcal{P}(A \times \mathcal{I})$

Introduction of probabilities

There are many ways to do it ...

Examples:

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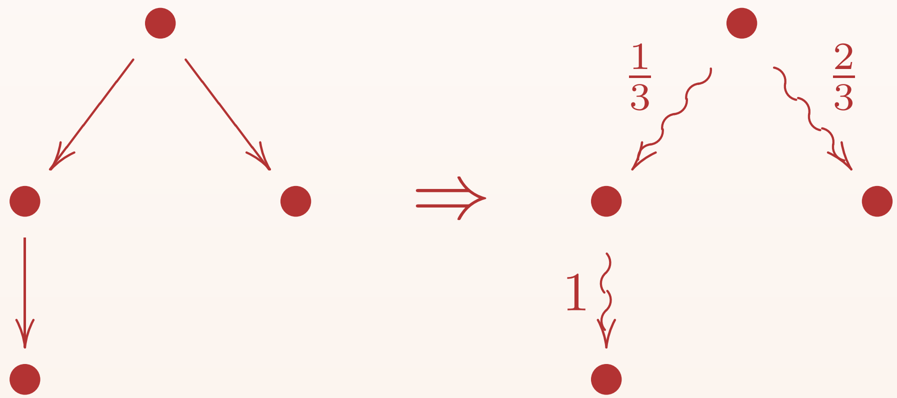

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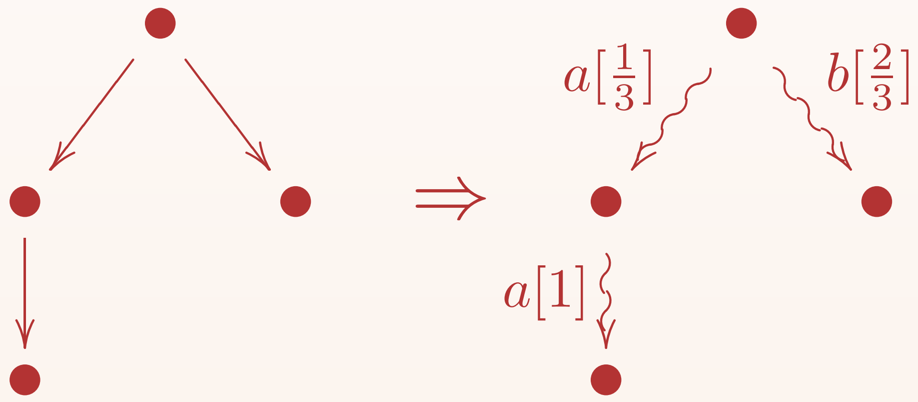

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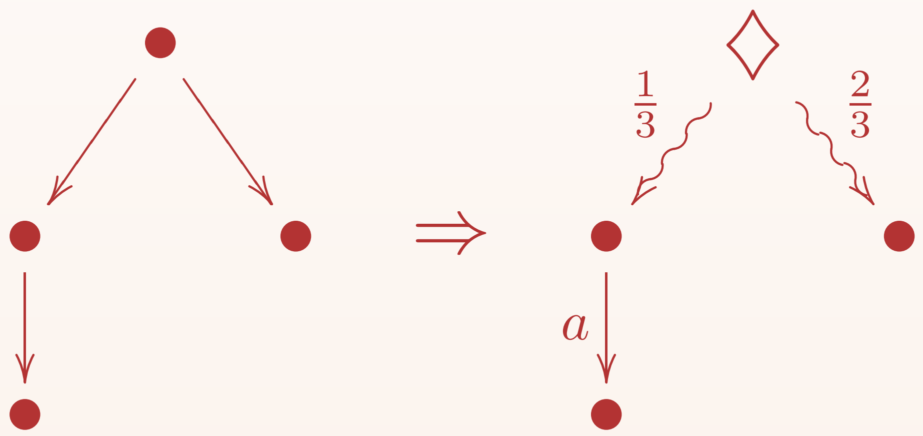

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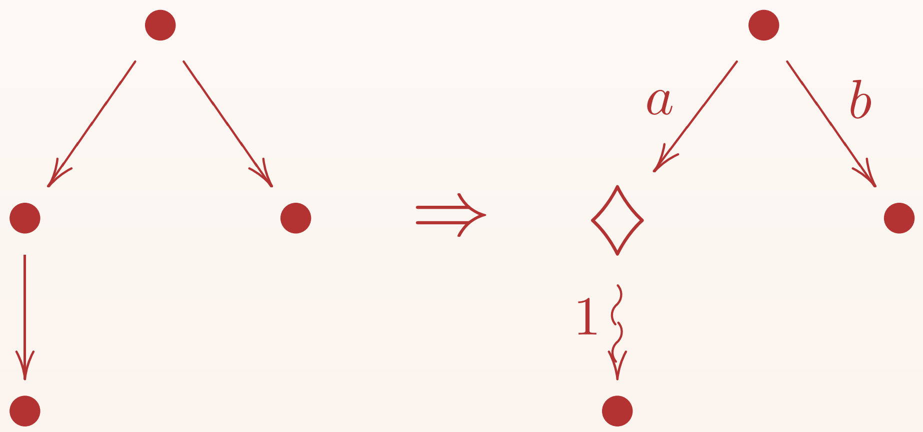

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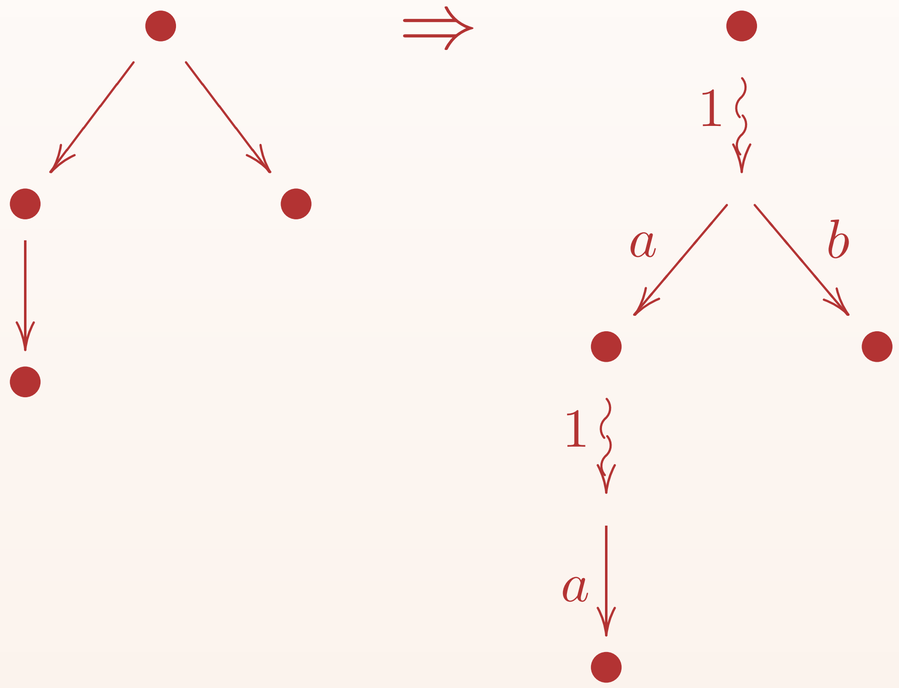

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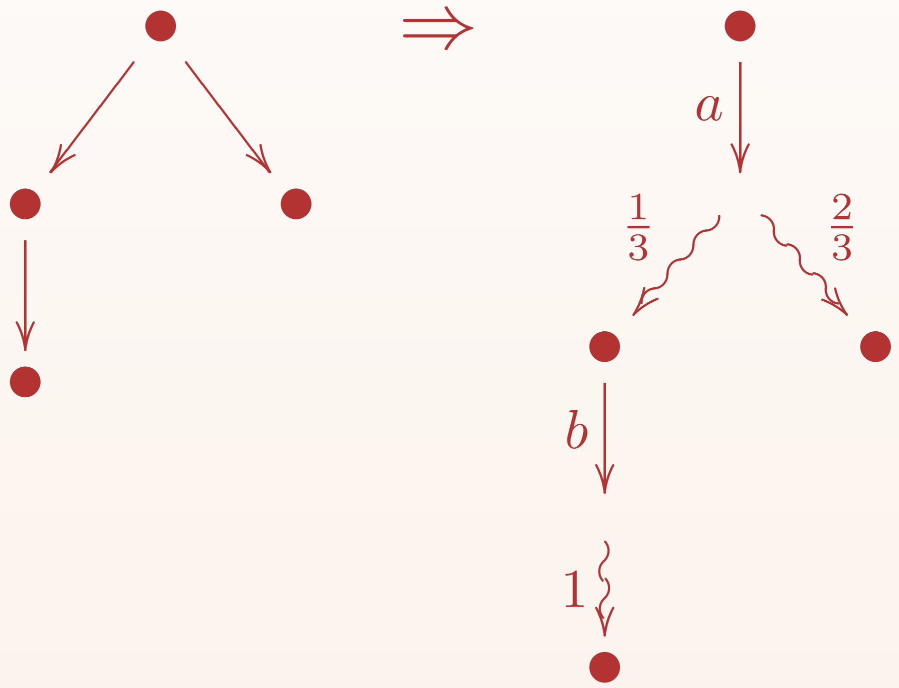

Introduction of probabilities

There are many ways to do it ...

13 types of systems - from the literature

with (or without):

- action labels
- nondeterminism
- probabilities

Existing system types

MG

PZ

Alt **Seg** **Bun**

SSeg **Var**

Str **React** **NA** **Gen**

MC **DA**

System types

The (probabilistic) models of systems we consider are coalgebras

$$\langle S, \alpha \rangle, \ \alpha : S \to \mathcal{F}S$$

for a functor $\mathcal{F}$ built by the following syntax

$$\mathcal{F} ::= \mathcal{C} \mid \mathcal{I} \mid \mathcal{P} \mid \ \mathcal{D}_\omega \ \mid \mathcal{F} + \mathcal{F} \mid \mathcal{F} \times \mathcal{F} \mid \mathcal{F}^{\mathcal{C}} \mid \mathcal{F}\mathcal{F}$$

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$\mathcal{D}_\omega S = \{\mu : S \to [0,1], \mu[S] = 1\}$, $\quad \mu[X] = \sum_{s \in X} \mu(x)$

$\mathcal{D}_\omega f : \mathcal{D}_\omega S \to \mathcal{D}_\omega T,\ \mathcal{D}_\omega f(\mu)(t) = \mu[f^{-1}(\{t\})]$

Probabilistic system types

MC	$\mathcal{D}_\omega$
DA	$(\mathcal{I}+1)^A$
NA	$\mathcal{P}(A\times\mathcal{I})\cong\mathcal{P}^A$
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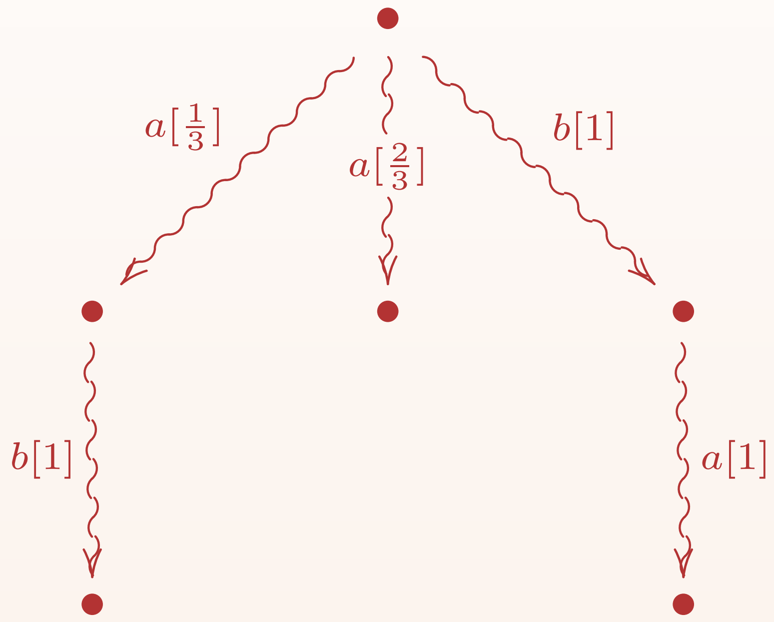

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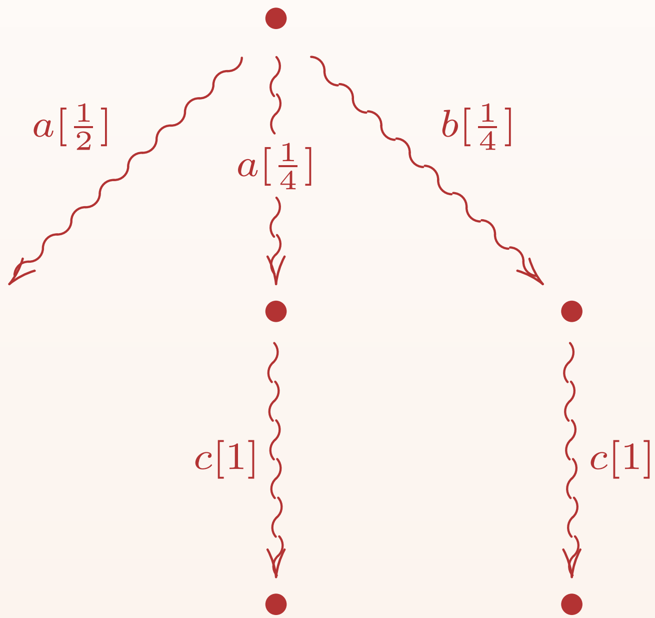

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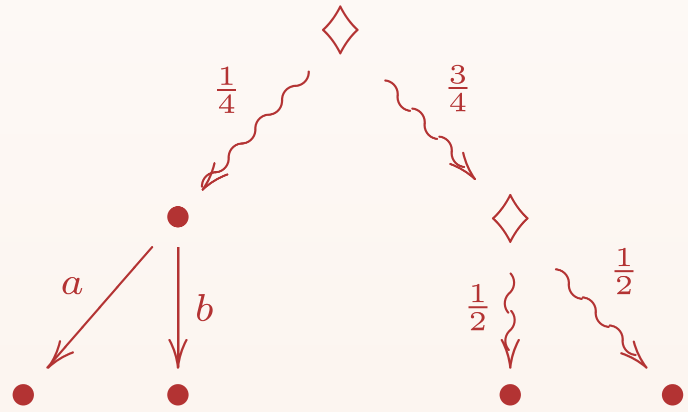

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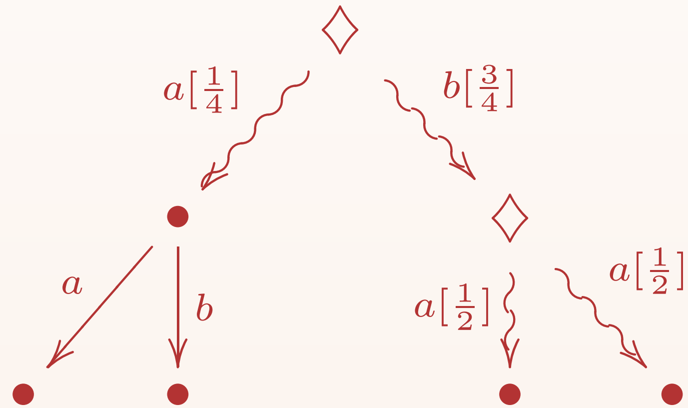

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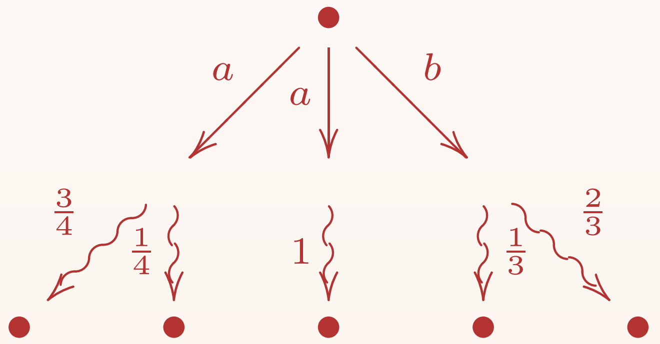

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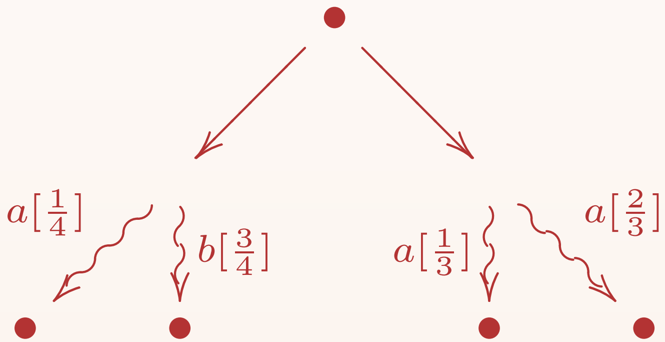

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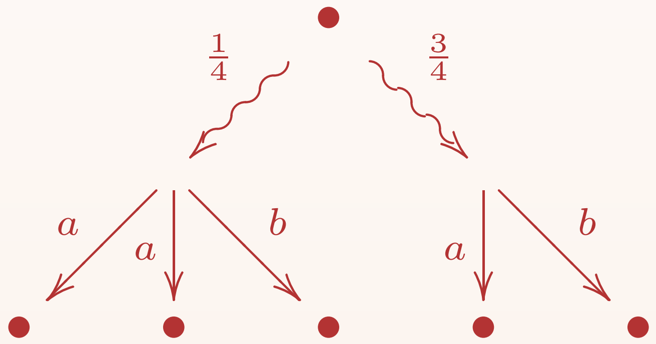

Coalgebraic bisimulation

A *bisimulation* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *span*

$$\langle R, r_1 : R \to S, r_2 : R \to T \rangle$$

such that there exists a $\mathcal{F}$-coalgebra structure γ on R making ...

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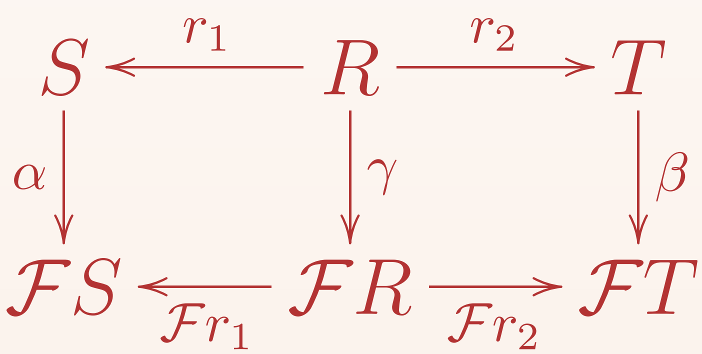


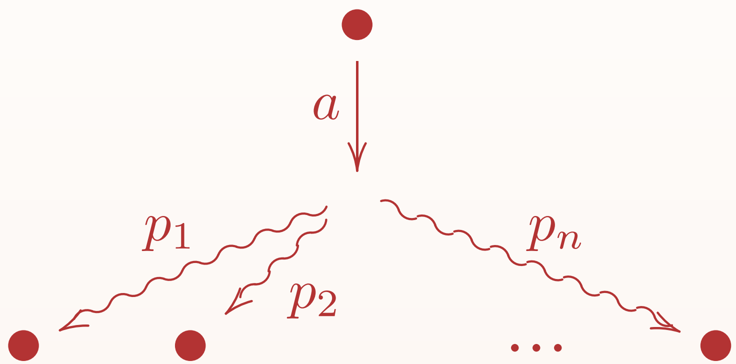
Coalgebraic bisimulation

$$
\begin{array}{ccccc}
S & \xleftarrow{r_1} & R & \xrightarrow{r_2} & T \\
\downarrow \alpha & & \downarrow \gamma & & \downarrow \beta \\
\mathcal{F}S & \xleftarrow[\mathcal{F}r_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}r_2]{} & \mathcal{F}T
\end{array}
$$

Note:
Concrete strong bisimulation is coalgebraic bisimulation !

An intuitive translation

simple Segala system $\longrightarrow$ Segala system



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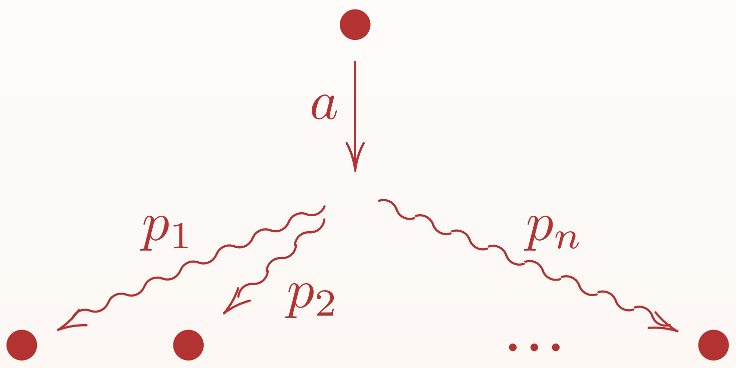

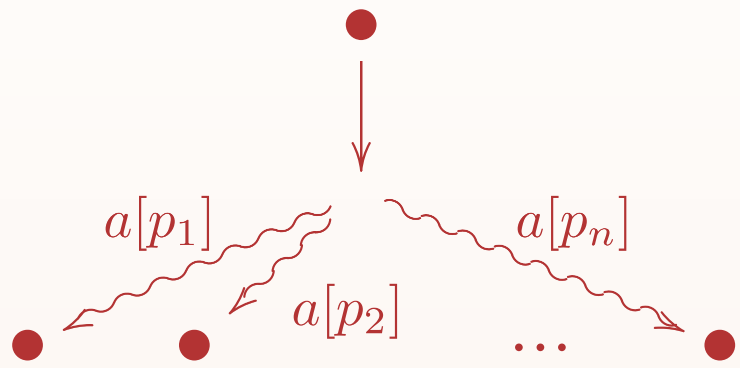

$$\langle S, \alpha \rangle$$

$$\alpha : S \to \mathcal{P}(A \times \mathcal{D}_\omega S)$$

$$\alpha(s) = \{(a_i, \mu_i) \mid i \in I\}$$

$$\langle S, \alpha' \rangle$$

$$\alpha' : S \to \mathcal{P}\mathcal{D}_\omega(A \times S)$$

$$\alpha'(s) = \{\delta_{a_i} \cdot \mu_i \mid i \in I\}$$

where $(\mu \cdot \mu')(x, x') = \mu(x) \cdot \mu'(x')$

and $\delta_a(b) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise.} \end{cases}$

An intuitive translation

simple Segala system $\longrightarrow$ Segala system

When do we consider one type of systems more expressive than another?

Expressiveness

Our approach:

Systems of type $\mathcal{F}$ are at most as expressive as systems of type $\mathcal{G}$, if there is a mapping

$$\mathcal{T} : \mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$$

with

$$\langle S, \alpha \rangle \stackrel{\mathcal{T}}{\longmapsto} \langle S, \tilde{\alpha} \rangle$$

that *preserves* and *reflects* bisimilarity:

$$s_{\langle S,\alpha \rangle} \sim t_{\langle T,\beta \rangle} \iff s_{\mathcal{T}\langle S,\alpha \rangle} \sim t_{\mathcal{T}\langle T,\beta \rangle}$$

Translation of coalgebras

In the simple vs. ordinary Segala System example with $\vartheta_X(a,\mu) := \delta_a \cdot \mu$ we find

$$\mathcal{P}\vartheta : \mathcal{P}(A \times \mathcal{D}_\omega) \Rightarrow \mathcal{P}\mathcal{D}_\omega(A \times \mathcal{I}).$$

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Generally, $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ induces $\mathcal{T}_\tau : \mathsf{Coalg}_\mathcal{F} \to \mathsf{Coalg}_\mathcal{G}$:

$$\begin{array}{ccc} S & & S \\ & & \downarrow \alpha \\ \downarrow \alpha & \overset{\mathcal{T}_\tau}{\Longrightarrow} & \mathcal{F}S \\ & & \downarrow \tau_S \\ \mathcal{F}S & & \mathcal{G}S \end{array}$$

Reflection of bisimilarity

$\mathcal{T}_\tau$ always *preserves* but need not *reflect* bisimilarity.

Counter-example:

$$\tau := [\mathsf{supp}, \emptyset] : \mathcal{D}_\omega + 1 \Rightarrow \mathcal{P}$$

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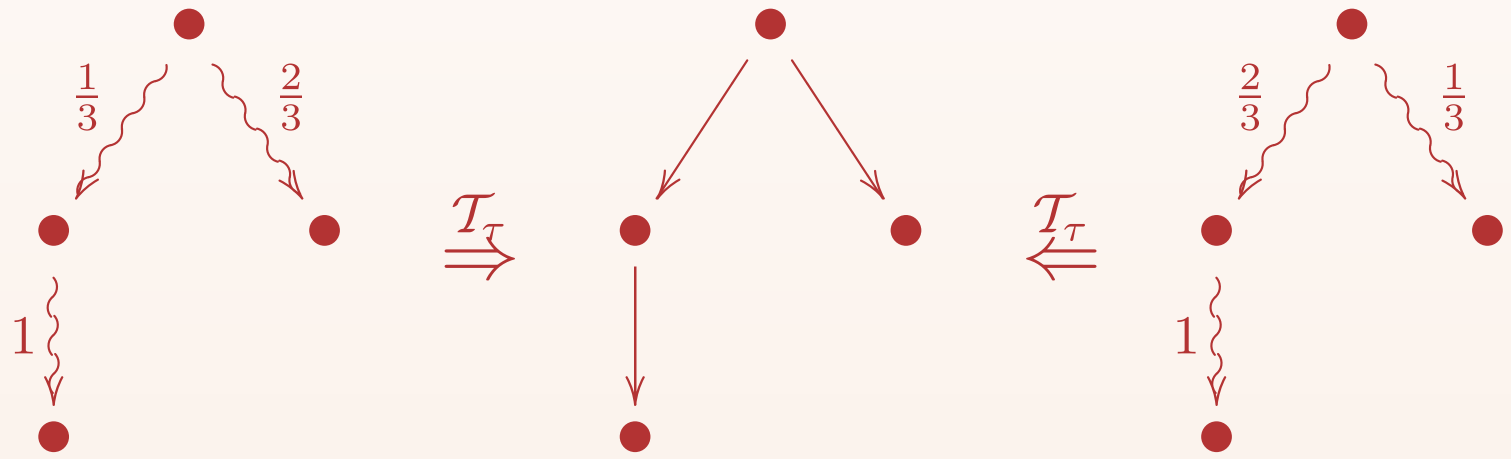

Bisimulation reflection result

Lemma

If all components of the natural transformation

$$\tau : \mathcal{F} \Rightarrow \mathcal{G}$$

are injective and $\mathcal{F}$ preserves weak pullbacks, then $\mathcal{T}_\tau$ reflects bisimularity.

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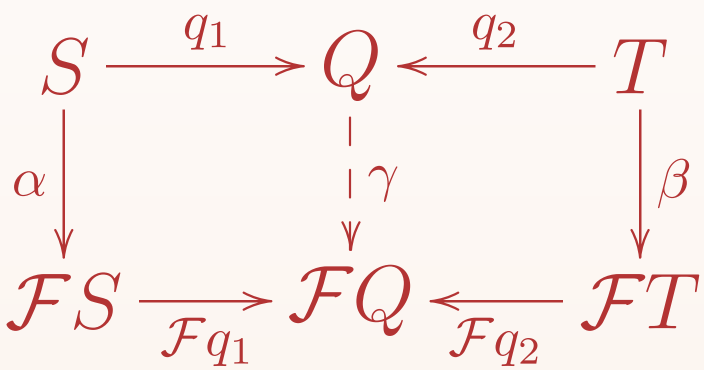
Used in the proof:

Behavioural equivalence defined in terms of *cospans*.

Cocongruences

A *cocongruence* between two $\mathcal{F}$-coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *cospan* $\langle Q, q_1, q_2 \rangle$ such that there exists $\gamma : Q \to \mathcal{F}Q$ with



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$$\begin{array}{ccccc} S & \xrightarrow{q_1} & Q & \xleftarrow{q_2} & T \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\ \mathcal{F}S & \xrightarrow[\mathcal{F}q_1]{} & \mathcal{F}Q & \xleftarrow[\mathcal{F}q_2]{} & \mathcal{F}T \end{array}$$

States identified by some cocongruence are called *behaviourally equivalent*.

Weak pullback preservation

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Note:
All functors for the probabilistic system types preserve weak pullbacks.

Assumptions of the Lemma

- The assumption on the functor cannot be omitted.
  Counter-example:
  Built on

$$\mathcal{F}X := \left\{ \langle x, y, z \rangle \in X^3 \;\middle|\; |\{x, y, z\}| \leq 2 \right\}.$$

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- Componentwise injectivity of τ is not necessary.

  Example: $\mathsf{supp} : \mathcal{D}_\omega \Rightarrow \mathcal{P}$

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Injective natural transformations:

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- $\kappa : \mathrm{A} \times \mathcal{P} \Rightarrow \mathcal{P}(\mathrm{A} \times \mathcal{I})$ with
$$\kappa_X(a, M) := \{\langle a, x \rangle \mid x \in M\},$$

Some basic transformations

Injective natural transformations:

- $\eta : 1 \Rightarrow \mathcal{P}$ with $\eta_X(*) := \emptyset$,
- $\sigma : \mathcal{I} \Rightarrow \mathcal{P}$ with $\sigma_X(x) := \{x\}$,
- $\delta : \mathcal{I} \Rightarrow \mathcal{D}_\omega$ with $\delta_X(x) := \delta_x$ (*Dirac*),
- $\iota_l : \mathcal{F} \Rightarrow \mathcal{F} + \mathcal{G}$ and $\iota_r : \mathcal{G} \Rightarrow \mathcal{F} + \mathcal{G}$,
- $\phi + \psi : \mathcal{F} + \mathcal{G} \Rightarrow \mathcal{F}' + \mathcal{G}'$ for $\phi : \mathcal{F} \Rightarrow \mathcal{F}'$ and $\psi : \mathcal{G} \Rightarrow \mathcal{G}'$ (both with i.c.),
- $\kappa : \mathrm{A} \times \mathcal{P} \Rightarrow \mathcal{P}(\mathrm{A} \times \mathcal{I})$ with
  $$\kappa_X(a, M) := \{\langle a, x\rangle \mid x \in M\},$$
- $\vartheta : \mathrm{A} \times \mathcal{D}_\omega \Rightarrow \mathcal{D}_\omega(\mathrm{A} \times \mathcal{I})$ with
  $$\vartheta_X(a, \mu) := \delta_a \cdot \mu.$$

Expressiveness in the example

Simple Segala Systems
(functor: $\mathcal{F} := \mathcal{P}(A \times \mathcal{D}_\omega)$)

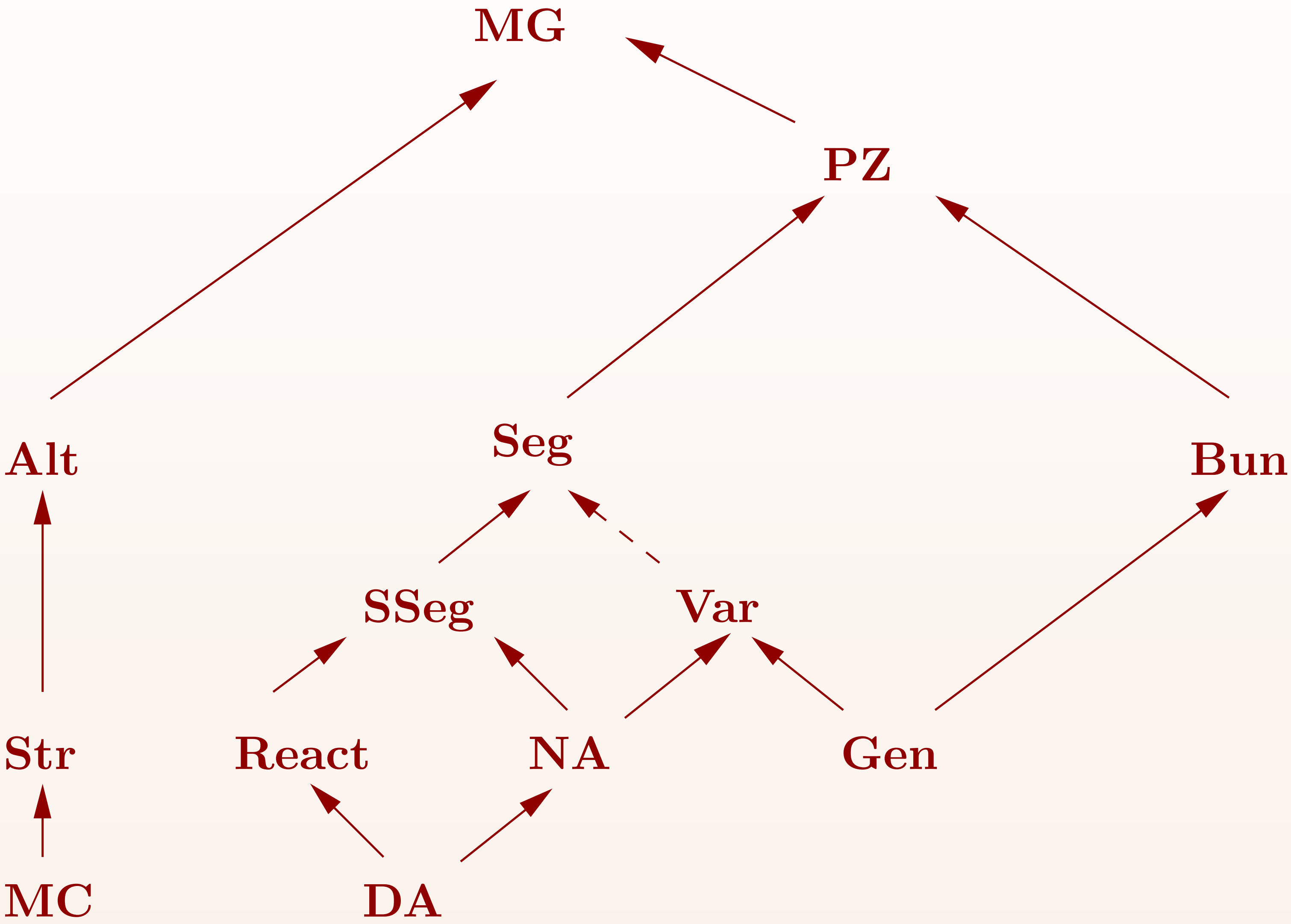
are at most as expressive as
(ordinary) Segala Systems
(functor: $\mathcal{G} := \mathcal{P}\mathcal{D}_\omega(A \times \mathcal{I})$).

Proof:

$$\mathcal{P}\vartheta : \mathcal{F} \Rightarrow \mathcal{G}$$

has injective components.

The hierarchy of system types



Conclusion

- Various probabilistic system types were compared
- The coalgebraic approach proved useful for:
  - providing a uniform framework
  - a general notion of bisimulation
  - proving a comparison result