

Hierarchy of probabilistic systems

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TU/e and CWI

Outline

- Introduction

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 - * labelled transition systems

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 - * bisimulation

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 - * preservation and reflection of bisimulation

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 - * translation of coalgebras
 - * preservation and reflection of bisimulation
- Building the hierarchy

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- Comparison of system types
 - * expressiveness criterion
 - * translation of coalgebras
 - * preservation and reflection of bisimulation
- Building the hierarchy
- Conclusions

Labelled transition systems

LTS is a pair $\langle S, \alpha : S \rightarrow \mathcal{P}S^A \rangle$

A - a fixed set of actions

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Hence a coalgebra

$$\langle S, \alpha \rangle, \alpha : S \rightarrow \mathcal{F}S$$

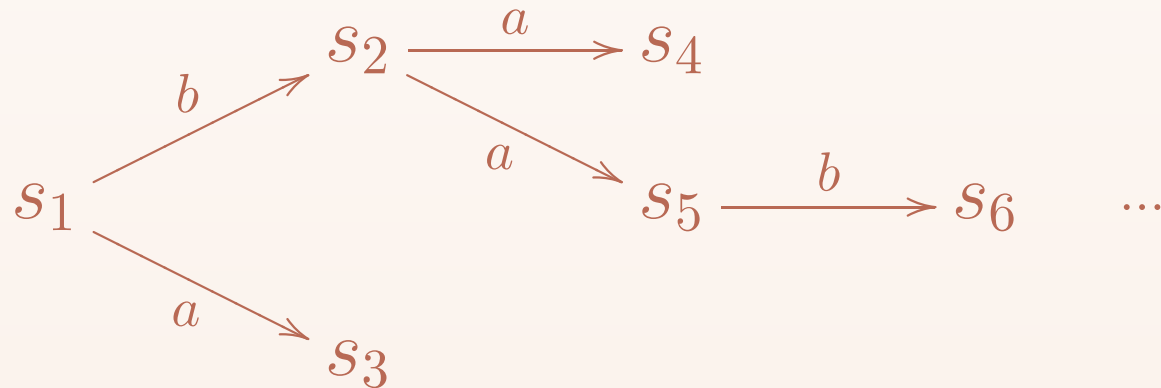
of the functor $\mathcal{F} = \mathcal{P}^A$

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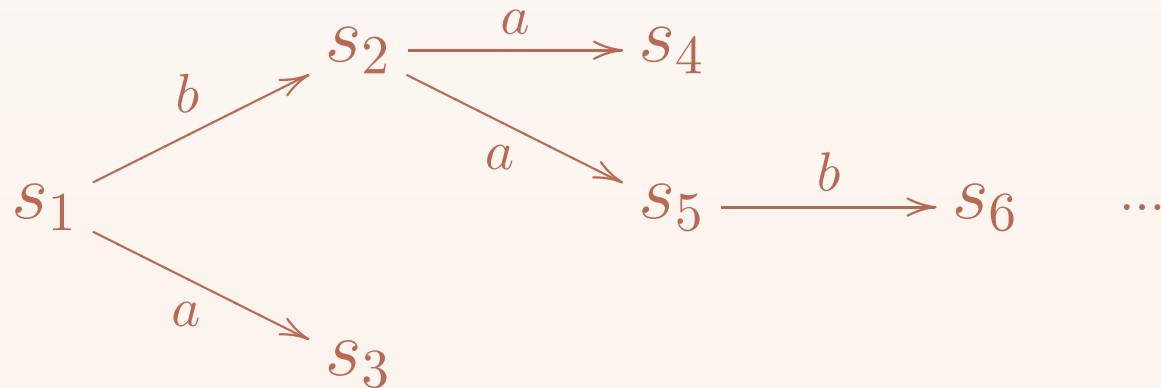


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Example:



Note: $\mathcal{P}^A \cong \mathcal{P}(A \times \mathcal{I})$

Concrete bisimulation for LTS

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$$s \xrightarrow{a} s' \Rightarrow (\exists t') t \xrightarrow{a} t', \langle s', t' \rangle \in R \text{ and}$$

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$s \approx t$ - there exists a concrete bisimulation
 R with $\langle s, t \rangle \in R$

Coalgebraic bisimulation

A *bisimulation* between two \mathcal{F} -coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *span*

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$$\begin{array}{ccccc} S & \xleftarrow{r_1} & R & \xrightarrow{r_2} & T \\ \alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\ \mathcal{F}S & \xleftarrow{\mathcal{F}r_1} & \mathcal{F}R & \xrightarrow{\mathcal{F}r_2} & \mathcal{F}T \end{array}$$

Concrete vs coalgebraic (LTS)

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(coalgebras of type $\mathcal{F} = \mathcal{P}^A$)

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sRt - there exists u with $r_1(u) = s, r_2(u) = t$

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known: $s \approx t$ if and only if $s \sim t$

Introduction of probabilities

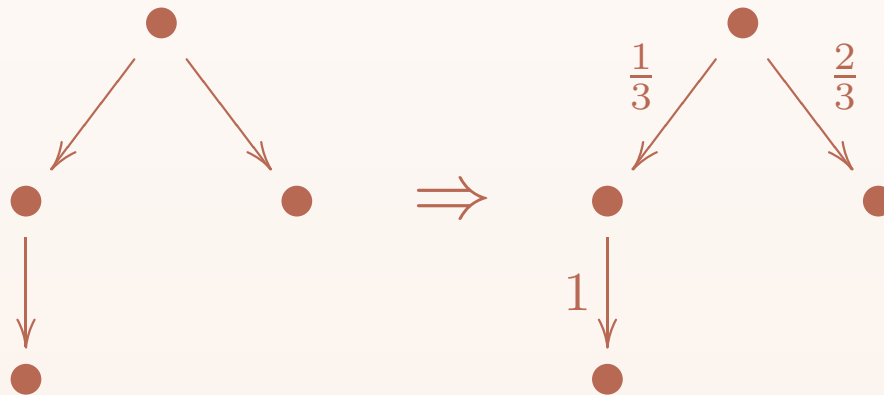
There are many ways to do it ...

Examples:

Introduction of probabilities

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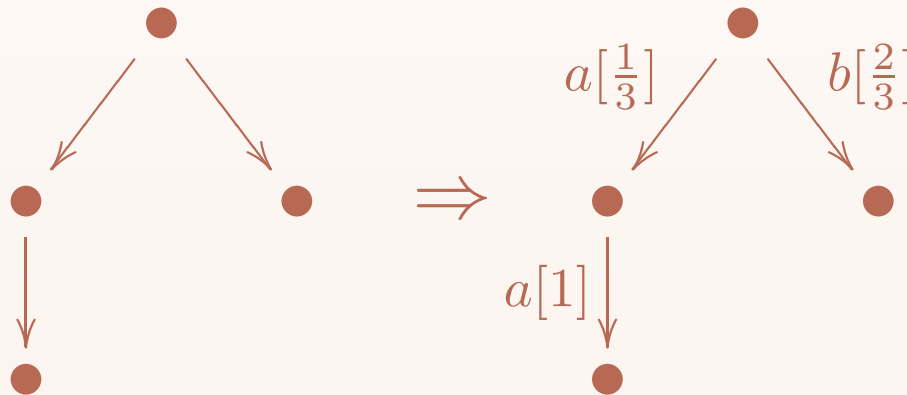
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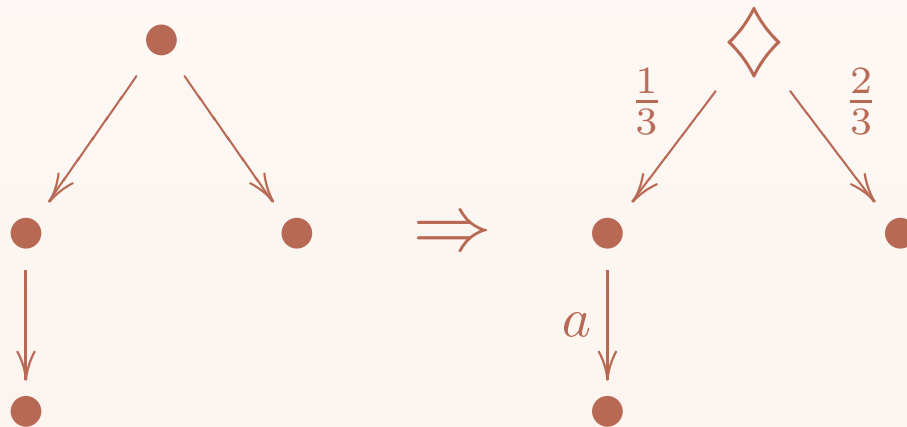
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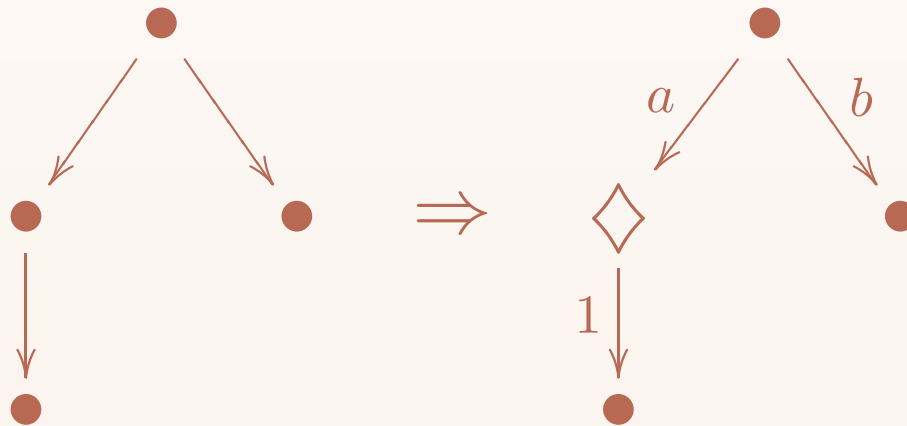
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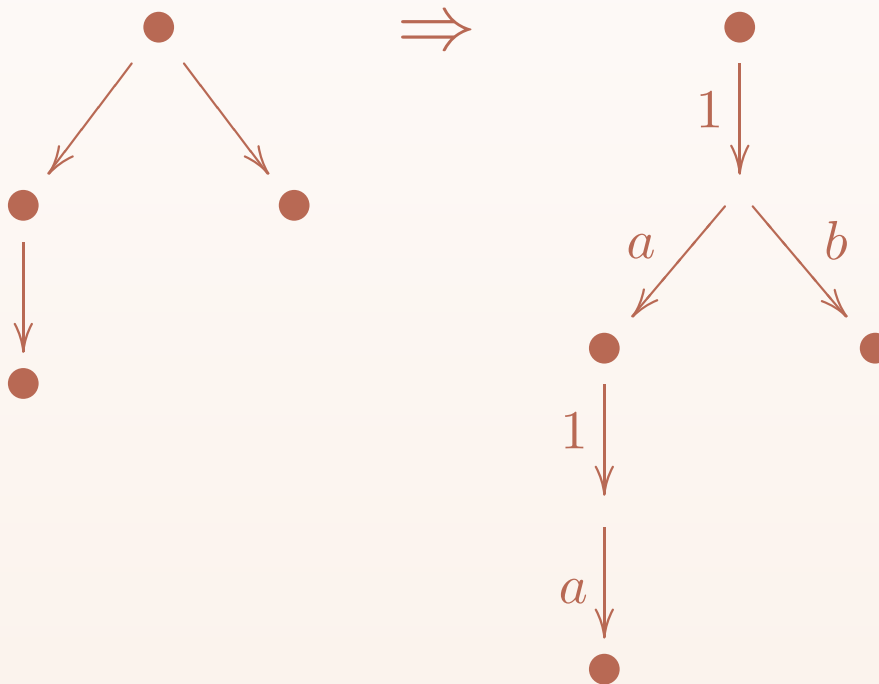
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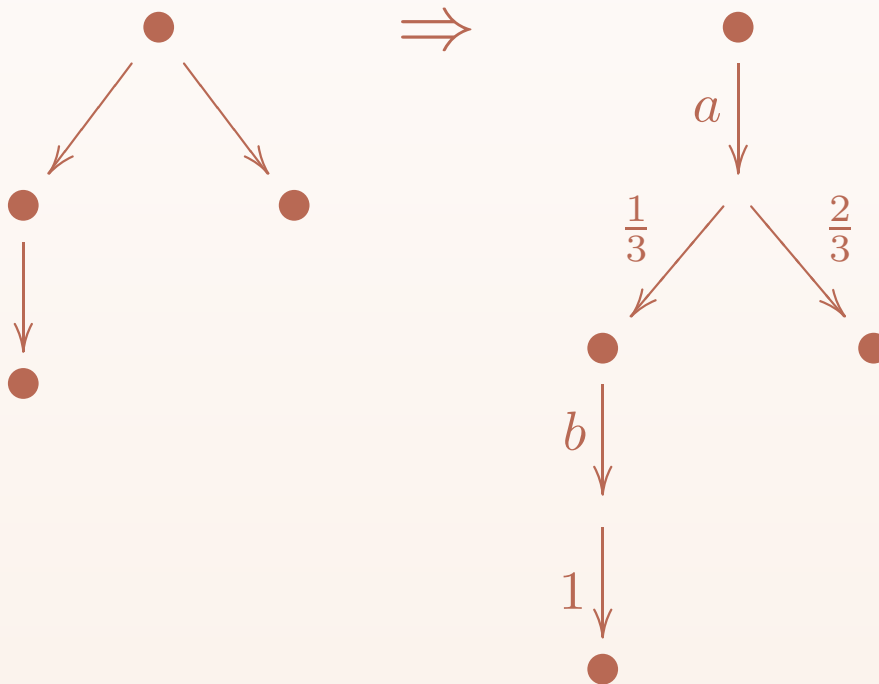
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13 types of systems - from the literature
with (or without):

- action labels
- nondeterminism
- probabilities

System types

The (probabilistic) models of systems we consider are coalgebras

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for a functor \mathcal{F} built by the following syntax

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for a functor \mathcal{F} built by the following syntax

$$\mathcal{F} ::= \mathcal{C} \mid \mathcal{I} \mid \mathcal{P} \mid \mathcal{D}_\omega \mid \mathcal{F} + \mathcal{F} \mid \mathcal{F} \times \mathcal{F} \mid \mathcal{F}^{\mathcal{C}} \mid \mathcal{F}\mathcal{F}$$

Reactive and generative systems

evolve from LTS - functor $\mathcal{P}^A \cong \mathcal{P}(A \times \mathcal{I})$

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functor $(\mathcal{D}_\omega + 1)(A \times \mathcal{I}) = \mathcal{D}_\omega(A \times \mathcal{I}) + 1$

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Reactive systems - input type

functor $(\mathcal{D}_\omega + 1)^A$

Generative systems - output type

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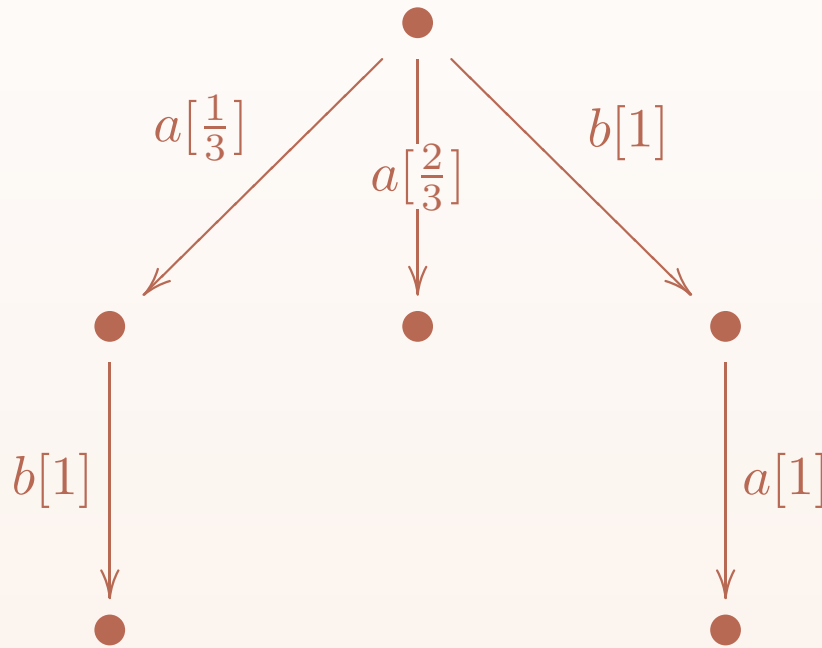
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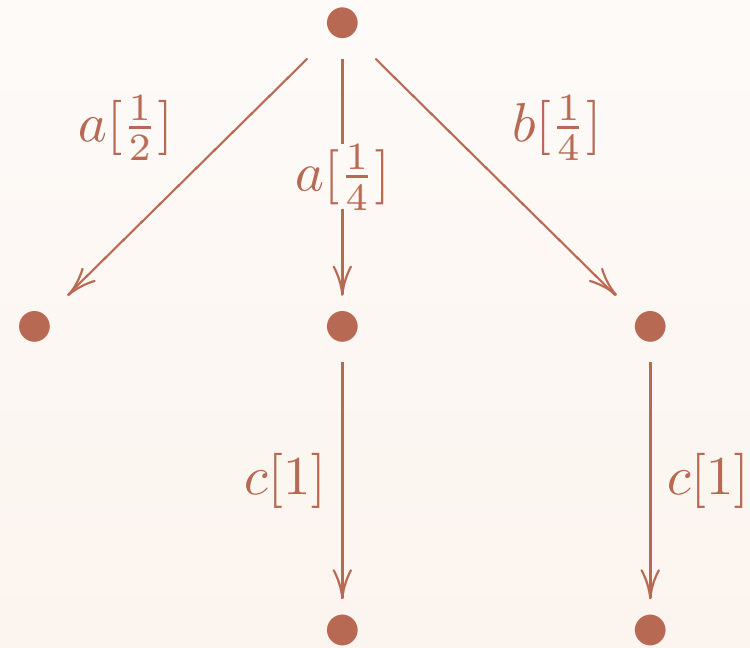
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Reactive and generative systems

Example:



Reactive system



Generative system

Bisimulation - generative systems

$\langle S, \alpha \rangle, \langle T, \beta \rangle$ - generative systems

$R \subseteq S \times T$ is a concrete bisimulation if for all $a \in A$, for all $\langle s, t \rangle \in R$ with $\mu = \alpha(s)$, $\nu = \beta(t)$ and every component C of R :

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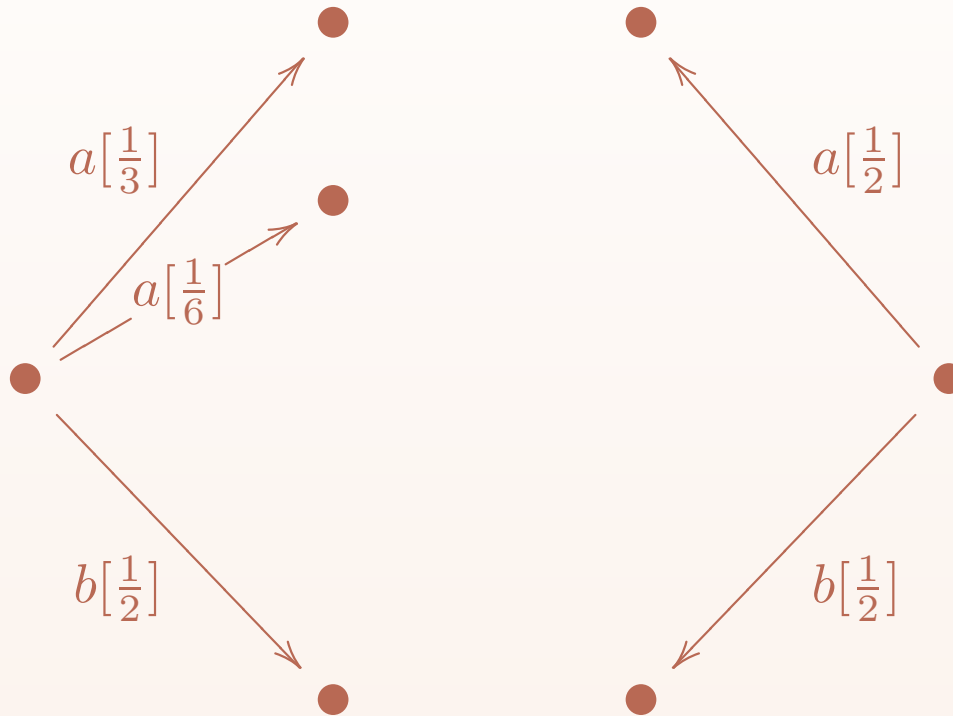
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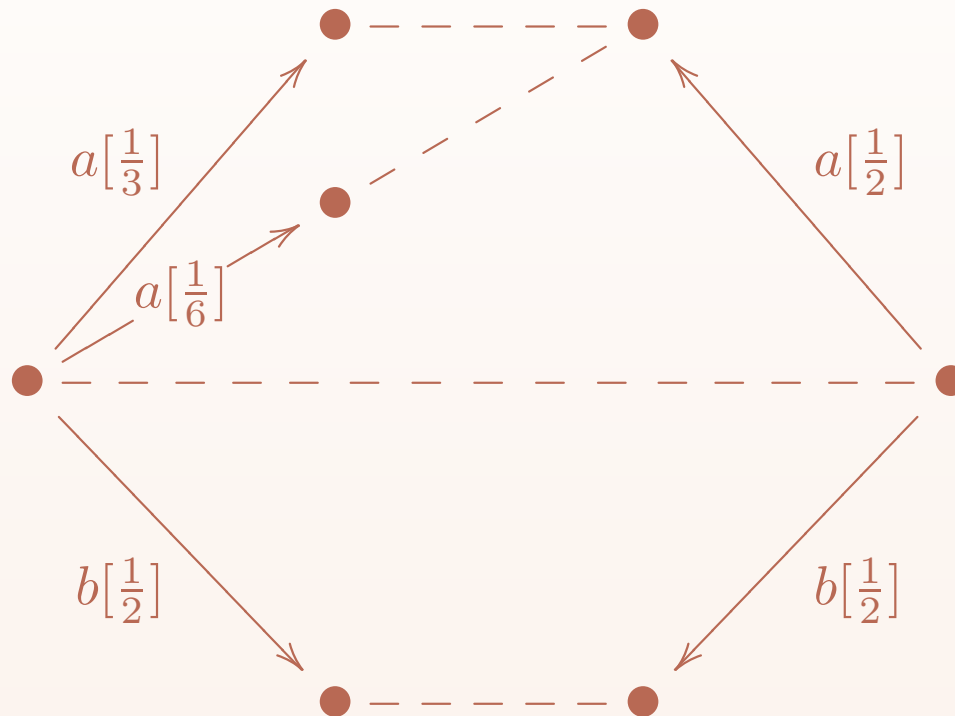
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Property:

$s \approx t$ if and only if $s \sim t$

Systems with distinction of states

Alternating systems - functor $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$

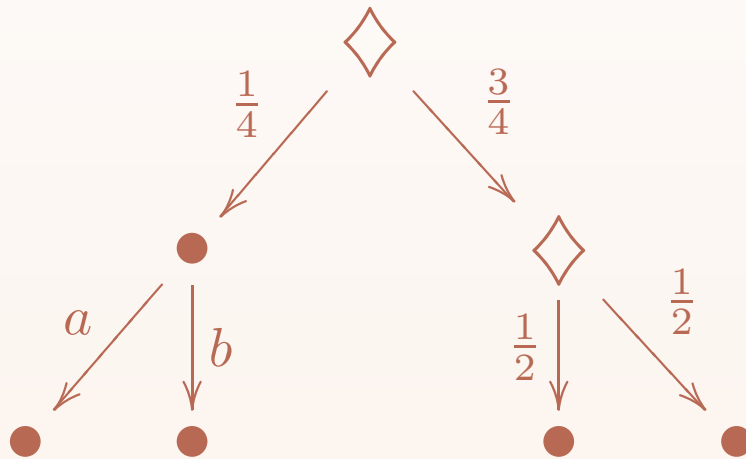
Vardi systems - functor $\mathcal{D}_\omega(A \times \mathcal{I}) + \mathcal{P}(A \times \mathcal{I})$

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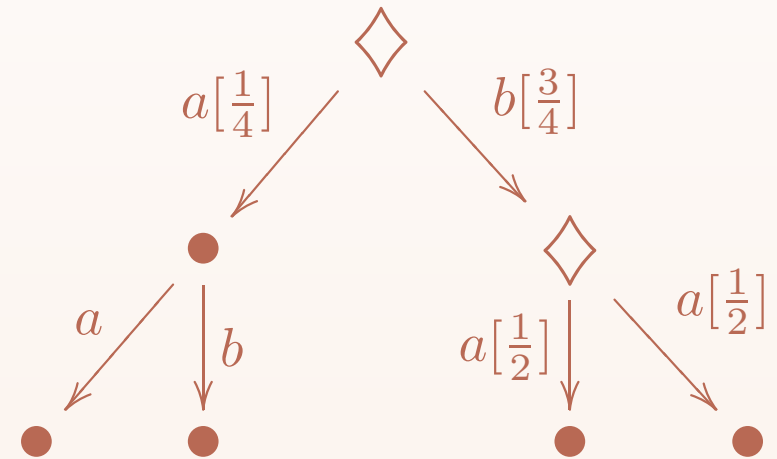
Alternating systems - functor $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$

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Example:



Alternating system



Vardi system

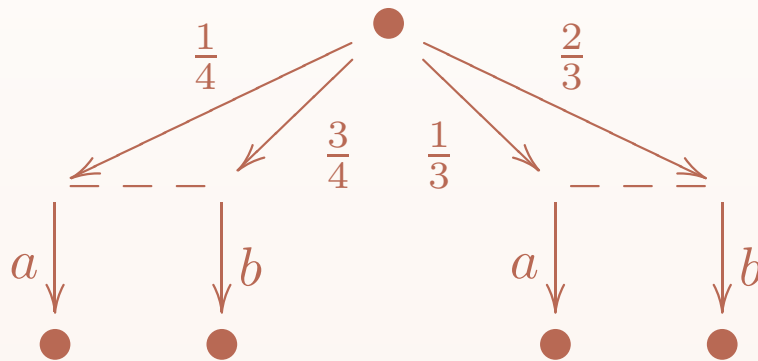
Structured transition function

Segala systems - functor $\mathcal{PD}_\omega(A \times \mathcal{I})$

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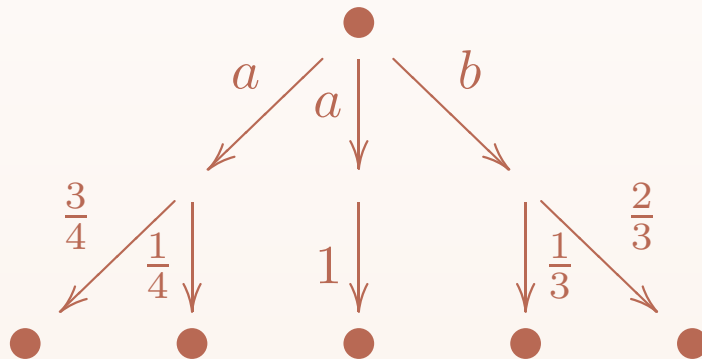
simple Segala systems - functor $\mathcal{P}(A \times \mathcal{D}_\omega)$

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Bundle systems - functor $\mathcal{D}_\omega\mathcal{P}(A \times \mathcal{I})$

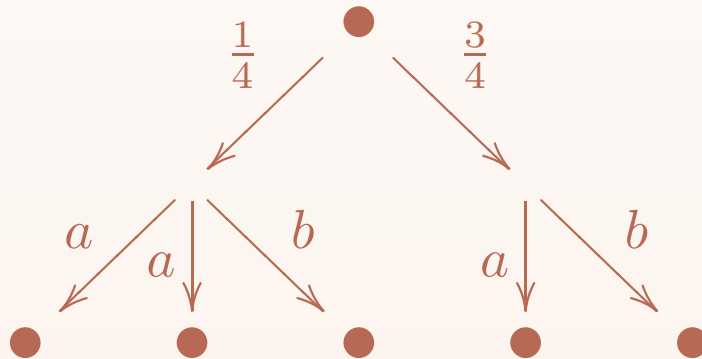
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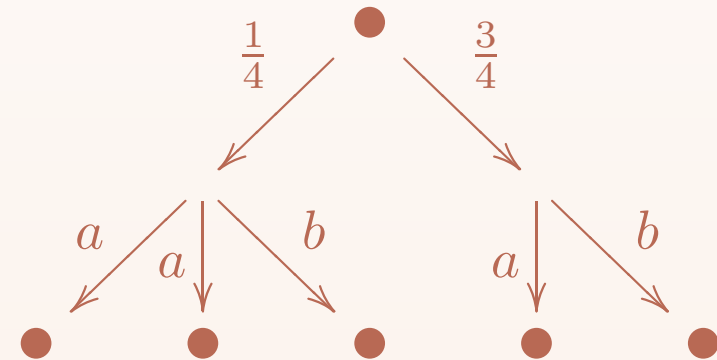
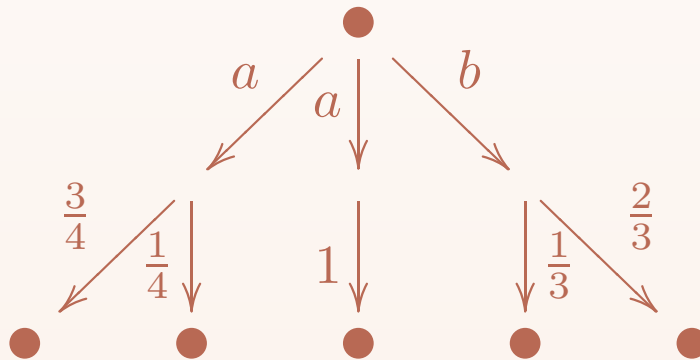
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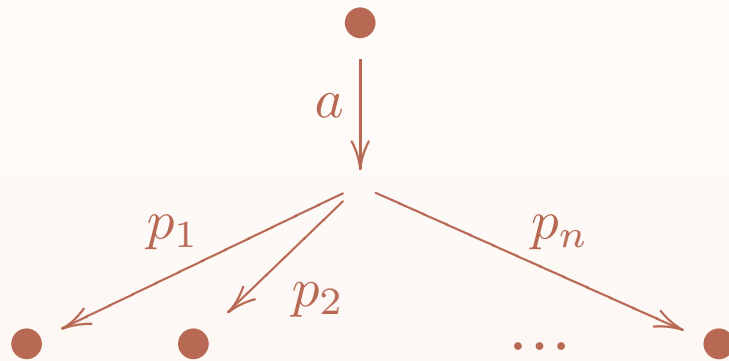
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Simple Segala system \perp Bundle system



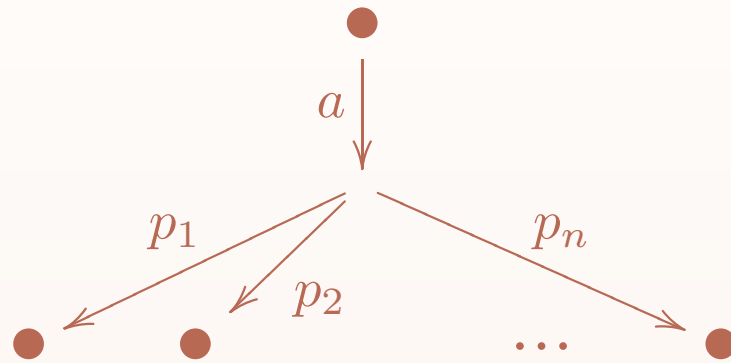
An intuitive translation

simple Segala system \rightarrow Segala system

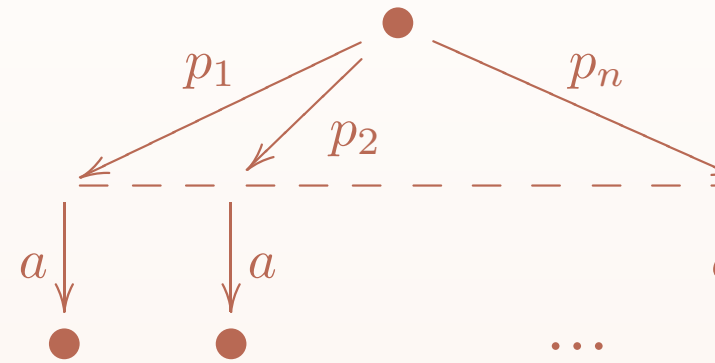


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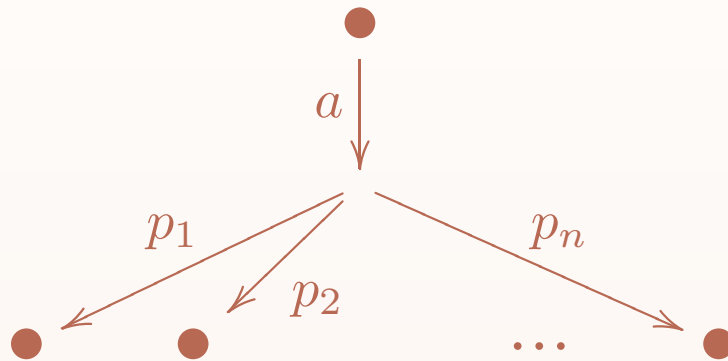


Segala system



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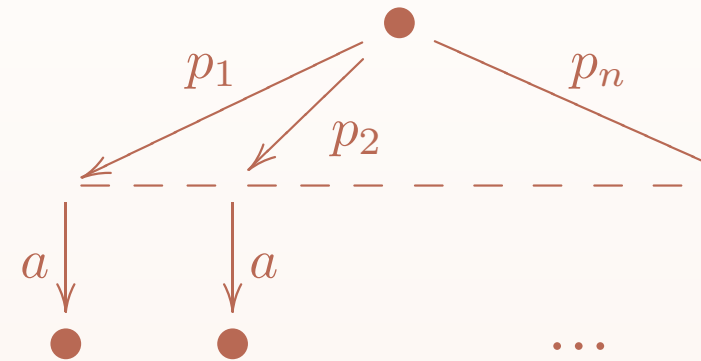


$\langle S, \alpha \rangle$

$$\alpha : S \rightarrow \mathcal{P}(A \times \mathcal{D}_\omega)$$

$$\alpha(s) = \{(a_i, \mu_i) \mid i \in I\}$$

Segala system



$\langle S, \alpha' \rangle$

$$\alpha' : S \rightarrow \mathcal{PD}_\omega(A \times S)$$

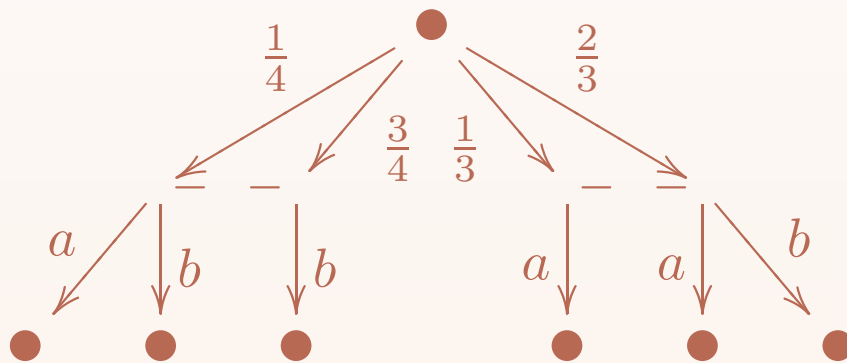
$$\alpha'(s) = \{\mu'_i \mid i \in I\}$$

$$\mu'_i(a_i, s') = \mu_i(s')$$

Complicated system types

Pnueli-Zuck systems - functor $\mathcal{PD}_\omega\mathcal{P}(A \times \mathcal{I})$

Example:

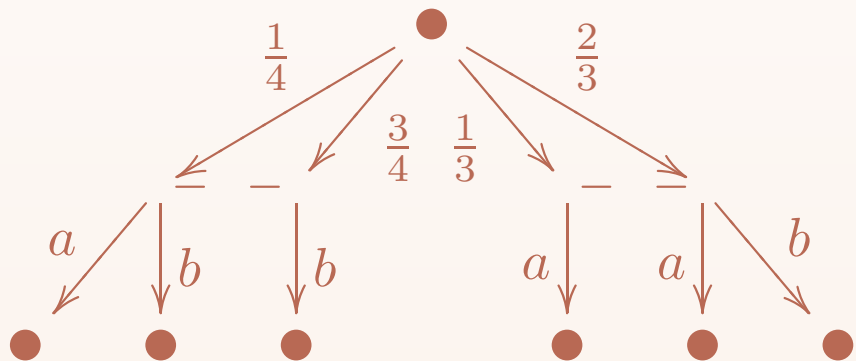


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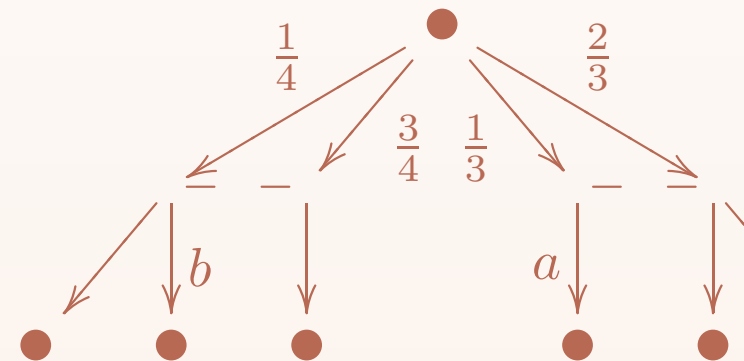
Pnueli-Zuck systems - functor $\mathcal{PD}_\omega \mathcal{P}(A \times \mathcal{I})$

most general systems - functor $\mathcal{PD}_\omega \mathcal{P}(A \times \mathcal{I} + \mathcal{I})$

Example:



Pnueli-Zuck system



most general system

Expressiveness

When do we consider one type of system more expressive than another?

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Example:

LTS (functor: $\mathcal{P}(A \times \mathcal{I})$)

are clearly less expressive than

Alternating Systems (functor: $\mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I})$):

Any LTS can be viewed as an Alternating System that never uses the option to do a probabilistic step.

Expressiveness (2)

Our approach:

Systems of type \mathcal{F} are at most as expressive as systems of type \mathcal{G} , if there is a mapping

$$\mathcal{T} : \text{Coalg}_{\mathcal{F}} \rightarrow \text{Coalg}_{\mathcal{G}}$$

with

$$\langle S, \alpha \rangle \xrightarrow{\mathcal{T}} \langle S, \tilde{\alpha} \rangle$$

that *preserves* and *reflects* bisimilarity:

$$s_{\langle S, \alpha \rangle} \sim t_{\langle T, \beta \rangle} \iff s_{\mathcal{T}\langle S, \alpha \rangle} \sim t_{\mathcal{T}\langle T, \beta \rangle}$$

Translation of coalgebras

Note that in the LTS vs. Alternating Systems example there exists a natural transformation

$$\iota_r : \mathcal{P}(A \times \mathcal{I}) \Rightarrow \mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I}).$$

Translation of coalgebras

Note that in the LTS vs. Alternating Systems example there exists a natural transformation

$$\iota_r : \mathcal{P}(A \times \mathcal{I}) \Rightarrow \mathcal{D}_\omega + \mathcal{P}(A \times \mathcal{I}).$$

Generally, a natural transformation $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ gives rise to a translation of coalgebras $\mathcal{T}_\tau : \text{Coalg}_{\mathcal{F}} \rightarrow \text{Coalg}_{\mathcal{G}}$ as follows:

$$\begin{array}{ccc} \begin{array}{c} S \\ \downarrow \alpha \\ \mathcal{F}S \end{array} & \xrightarrow{\mathcal{T}_\tau} & \begin{array}{c} S \\ \downarrow \alpha \\ \mathcal{F}S \\ \downarrow \tau_S \\ \mathcal{G}S \end{array} \end{array}$$

Preservation of bisimulations

The translation \mathcal{T}_τ preserves bisimulations:

A bisimulation $\langle R, r_1, r_2 \rangle$ between
 $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$

$$\begin{array}{ccccc} S & \xleftarrow{r_1} & R & \xrightarrow{r_2} & T \\ \alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\ \mathcal{F}S & \xleftarrow{\mathcal{F}r_1} & \mathcal{F}R & \xrightarrow{\mathcal{F}r_2} & \mathcal{F}T \end{array}$$

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 S & \xleftarrow{r_1} & R & \xrightarrow{r_2} & T \\
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 \tau_S \downarrow & \text{nat. } \tau & \downarrow \tau_R & \text{nat. } \tau & \downarrow \tau_T \\
 \mathcal{G}S & \xleftarrow{\mathcal{G}r_1} & \mathcal{G}R & \xrightarrow{\mathcal{G}r_2} & \mathcal{G}T
 \end{array}$$

is a bisimulation between $\mathcal{T}_\tau \langle S, \alpha \rangle$ and $\mathcal{T}_\tau \langle T, \beta \rangle$ as well.

Reflection of bisimilarity

But: \mathcal{I}_τ need not reflect bisimilarity.

Example:

Let τ be the natural transformation

$$\widetilde{\text{supp}} : \mathcal{D}_\omega + 1 \Rightarrow \mathcal{P}$$

that *forgets* the probabilities.

Reflection of bisimilarity

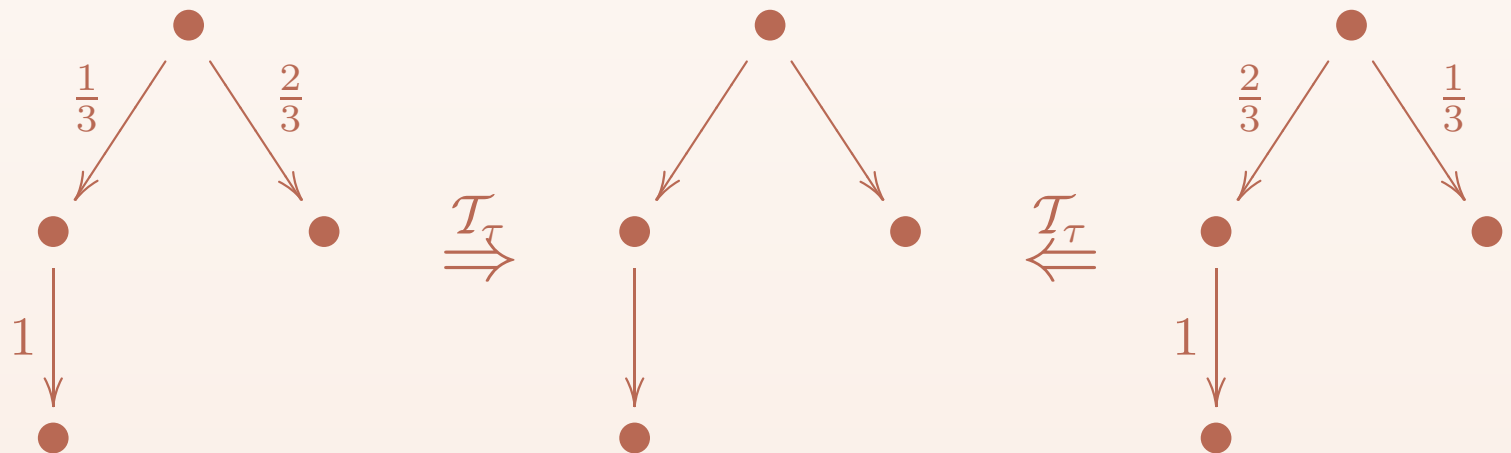
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$$\widetilde{\text{supp}} : \mathcal{D}_\omega + 1 \Rightarrow \mathcal{P}$$

that *forgets* the probabilities.



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Observation:

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On the other hand, although intuitively componentwise injectivity should be a sufficient condition for reflection of bisimilarity, a proof is not immediate.

Cocongruences

A *cocongruence* between two \mathcal{F} -coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ is a *cospan*

$$\langle Q, q_1 : S \rightarrow Q, q_2 : T \rightarrow Q \rangle$$

such that there exists a \mathcal{F} -coalgebra structure γ on Q making the diagram below commute.

$$\begin{array}{ccccc} S & \xrightarrow{q_1} & Q & \xleftarrow{q_2} & T \\ \alpha \downarrow & & \downarrow \gamma & & \downarrow \beta \\ \mathcal{F}S & \xrightarrow{\mathcal{F}q_1} & \mathcal{F}Q & \xleftarrow{\mathcal{F}q_2} & \mathcal{F}T \end{array}$$

Behavioural equivalence

Two states s and t in two \mathcal{F} -coalgebras $\langle S, \alpha \rangle$ and $\langle T, \beta \rangle$ respectively are called *behavioural equivalent* if they are identified by some cocongruence.

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Result:

If all components of the natural transformation

$$\tau : \mathcal{F} \Rightarrow \mathcal{G}$$

are injective, then \mathcal{I}_τ reflects behavioural equivalence.

Bisimilarity vs. beh. equivalence

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 \Rightarrow both notions coincide.

Corollary:

If \mathcal{F} preserves weak pullbacks and all components of $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ are injective, then \mathcal{T}_τ reflects bisimilarity.

Need of w.p. preservation

The corollary is not valid without assuming that \mathcal{F} preserves weak pullbacks.

Counter-example:

Consider the functors

$$\mathcal{F}X := \{ \langle x, y, z \rangle \in X^3 \mid |\{x, y, z\}| \leq 2 \}$$

and

$$\mathcal{G}X := X^3$$

and let $\tau : \mathcal{F} \Rightarrow \mathcal{G}$ be the set inclusion.

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\Rightarrow All functors used to define the different probabilistic system types preserve weak pullbacks.

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One expressiveness statement

To show for instance that
Generative systems

(functor: $\mathcal{F} := \mathcal{D}_\omega(A \times \mathcal{I}) + 1$)

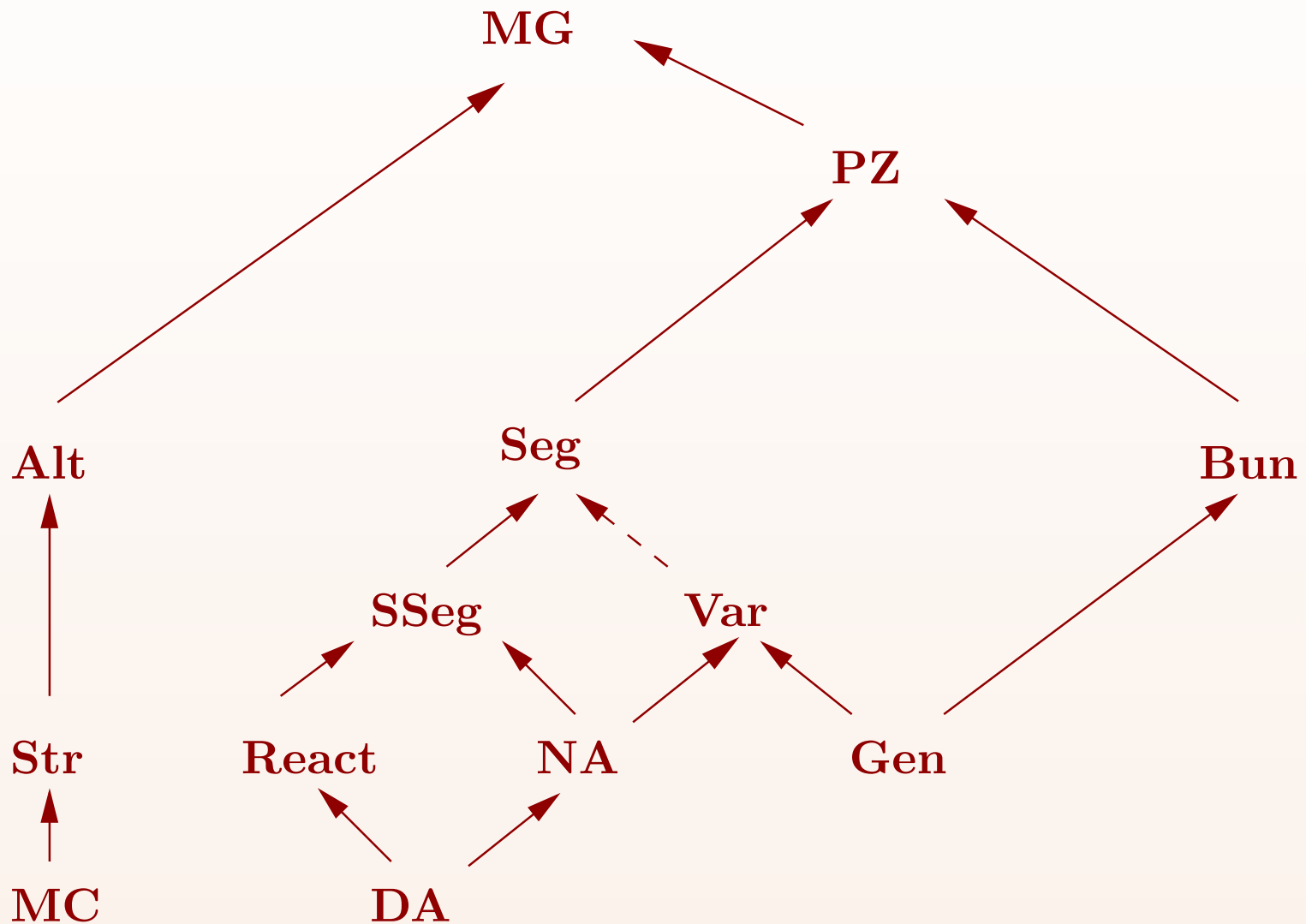
are at most as expressive as
Vardi systems

(functor: $\mathcal{G} := \mathcal{D}_\omega(A \times \mathcal{I}) + \mathcal{P}(A \times \mathcal{I})$)

we employ the natural transformation

$$\mathcal{D}_\omega(A \times \mathcal{I}) + \eta(A \times \mathcal{I}) : \mathcal{F} \Rightarrow \mathcal{G}.$$

The hierarchy of system types



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