Syntax and Semantics for Nondeterminism and Probability

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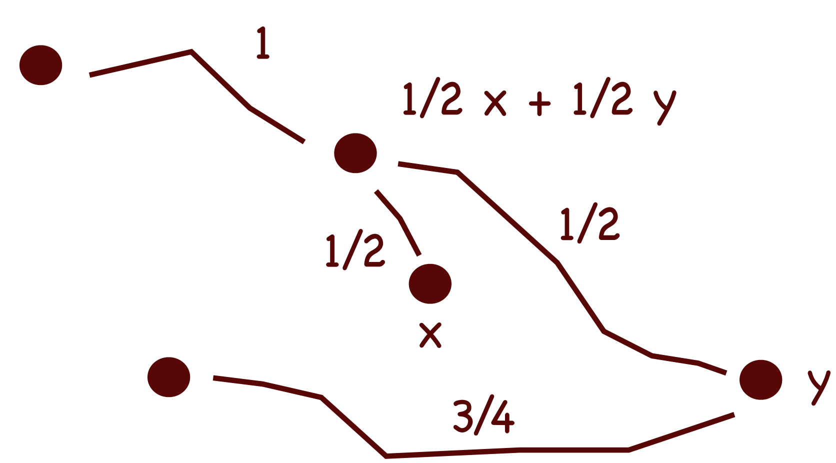
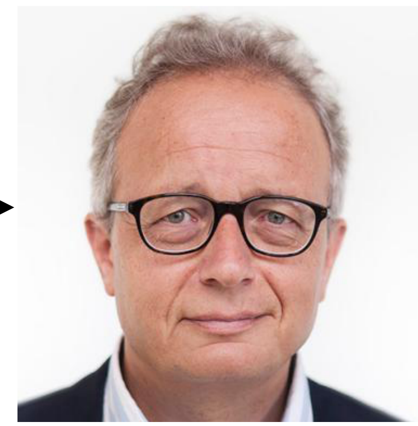


Helmut Veith Workshop, Turracher Höhe, 19.3.19

Joint work with

Ichiro Hasuo
The University of Tokyo



Bart Jacobs
Radboud University

Alexandra Silva
UCL

Harald Woracek
TU WIEN

Filippo Bonchi
IN SUPREMÆ DIGNITATIS 1343

Valeria Vignudelli
ENS DE LYON

I will tell you:

Mathematical framework based on category theory for state-based systems semantics

1. Just the absolute basics of coalgebra

for nondeterministic/ probabilistic… systems

systems with algebraic effects

2. (Trace) semantics via determinisation…

3. …enabled by algebraic structure

syntax

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

generic notion of behavioural equivalence (bisimilarity)

$\approx$

states

object in the base category ***C***

type

functor on the base category ***C***

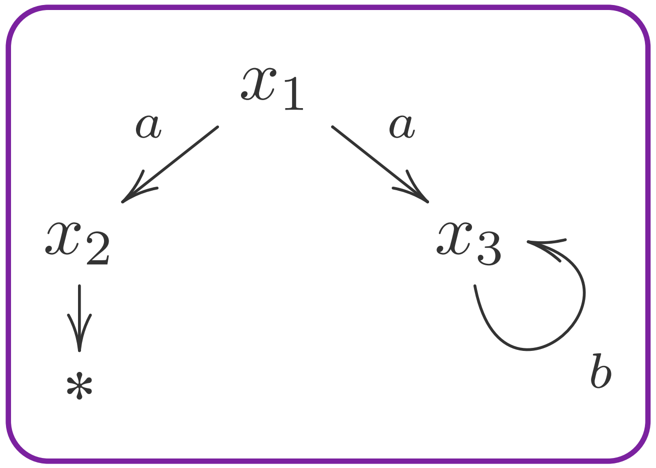
form a category too

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

Examples

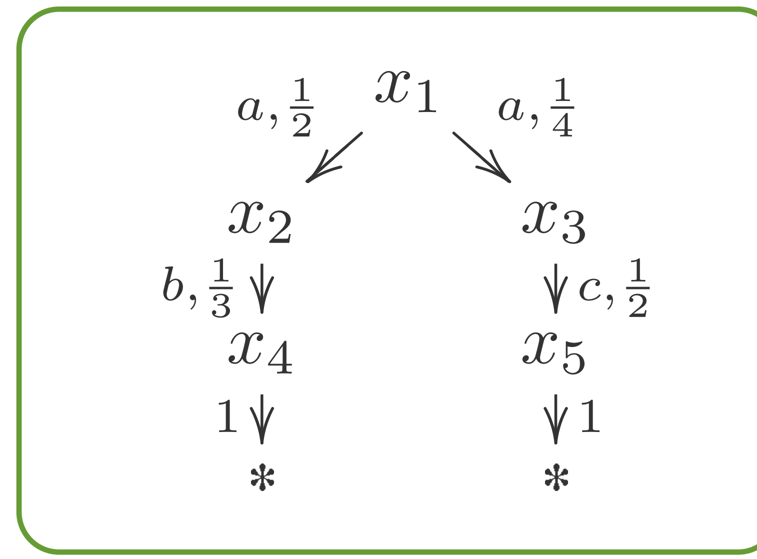
NFA

$X \to 2 \times (\mathcal{P}X)^A$



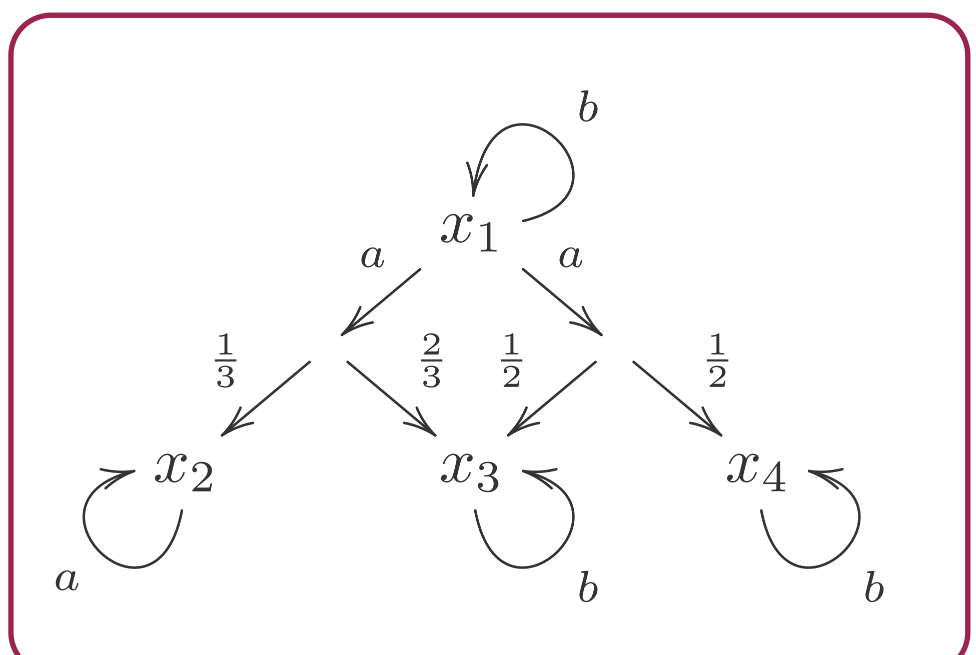
Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$



Simple PA

$X \to ? \times (\mathcal{P}\mathcal{D}X)^A$



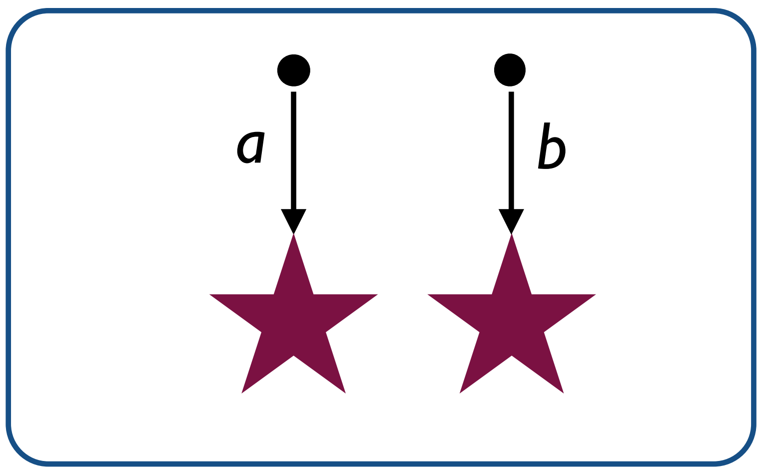
systems with nondeterminism and probability

In general

Systems

$X \to (MX)^A$

Labelled Transition Systems

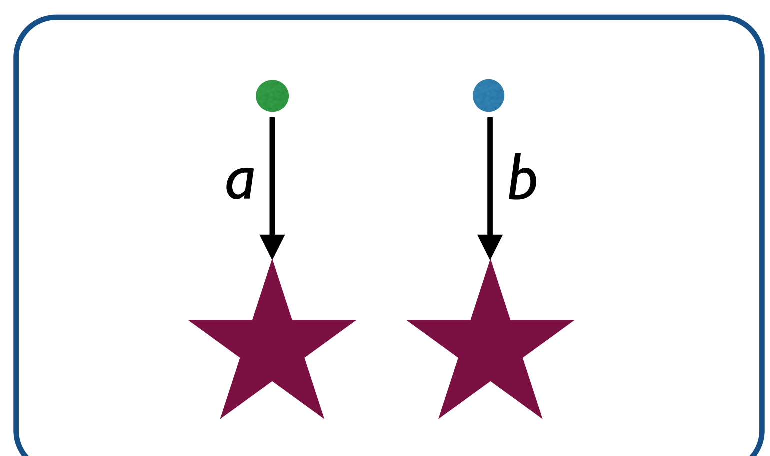


with M-effects

Automata

$X \to O \times (MX)^A$

with observations in O



with M-effects

For a monad M

providing algebraic effects

$\mu \colon TT \Rightarrow T$

$\eta \colon Id \Rightarrow T$

NFA

$X \to 2 \times (\mathcal{P}X)^A$

M = $\mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

M = $\mathcal{D}$
for probability

Distributions

Simple PA

$X \to ? \times (\mathcal{PD}X)^A$

M = $\mathcal{PD}$???
for nondeterminism
and probability

For a monad M

providing algebraic effects

$\mu : TT \Rightarrow T$

$\eta : Id \Rightarrow T$

NFA

$X \to 2 \times (\mathcal{P}X)^A$

M = $\mathcal{P}$
for nondeterminism

Powerset, subsets

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

M = $\mathcal{D}$
for probability

Distributions

Simple PA

$X \to ? \times (\mathcal{C}X)^A$

M = $\mathcal{C}$
for nondeterminism and probability !

Convex subsets of distributions

Semantics

NFA = LTS + termination

$$X \to 2 \times (\mathcal{P}X)^A$$

Are the (top states of the) following systems equivalent?

$$\begin{tikzcd}
 & x_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
x_2 \arrow[d, "b"'] & & x_3 \arrow[d, "c"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
\qquad
\begin{tikzcd}
 & y_1 \arrow[d, "a"'] & \\
 & y_2 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
y_3 \arrow[d] & & y_4 \arrow[d] \\
* & & *
\end{tikzcd}$$

- no, they are not wrt. bisimilarity
- yes, they are wrt. trace equivalence as

$$\mathrm{tr}(x_1) = \mathrm{tr}(y_1) = \{ab, ac\}$$

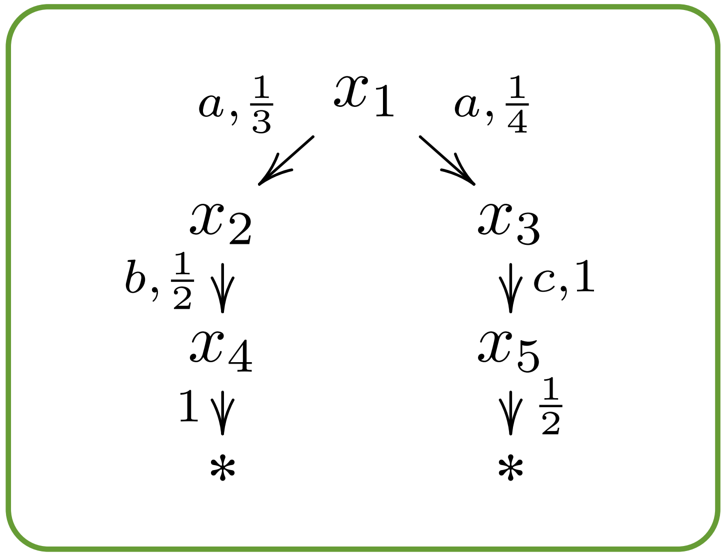
$$\mathrm{tr}\colon X \to \mathcal{P}(A^*)$$

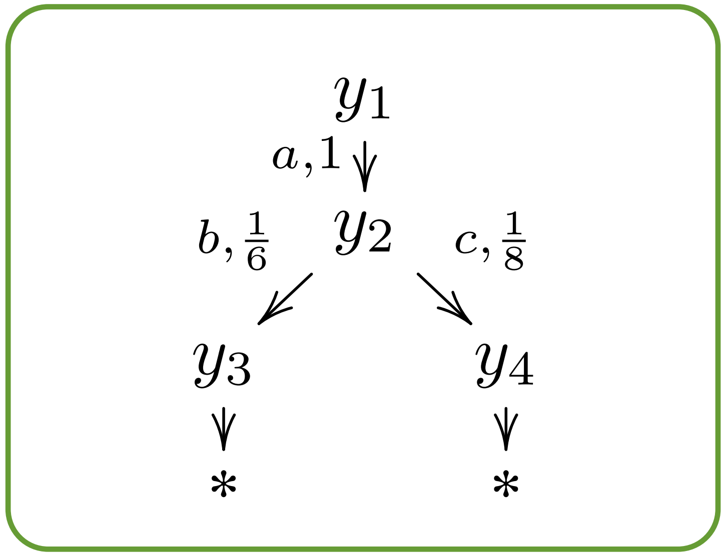
Semantics

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

Are the (top states of the) following systems equivalent?





- different wrt. bisimilarity
- equivalent wrt. trace equivalence as

$$\mathrm{tr}(x_1) = \mathrm{tr}(y_1) = \left(ab \mapsto \frac{1}{6}, ac \mapsto \frac{1}{8}\right)$$

$$\mathrm{tr}\colon X \to \mathcal{D}(A^*)$$

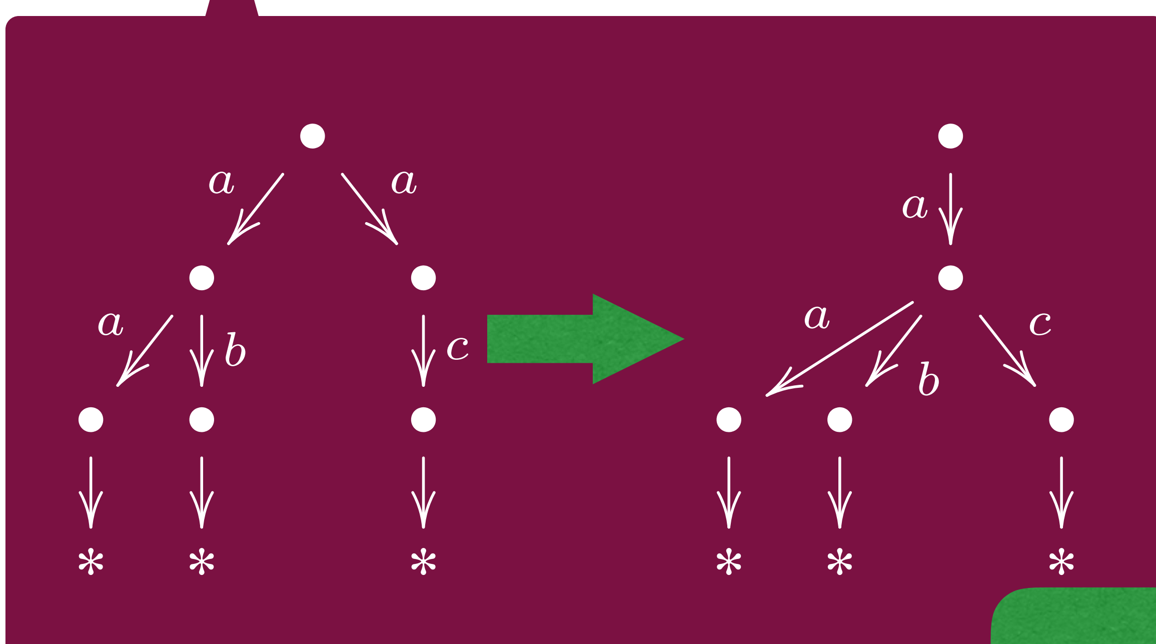
Trace semantics coalgebraically?

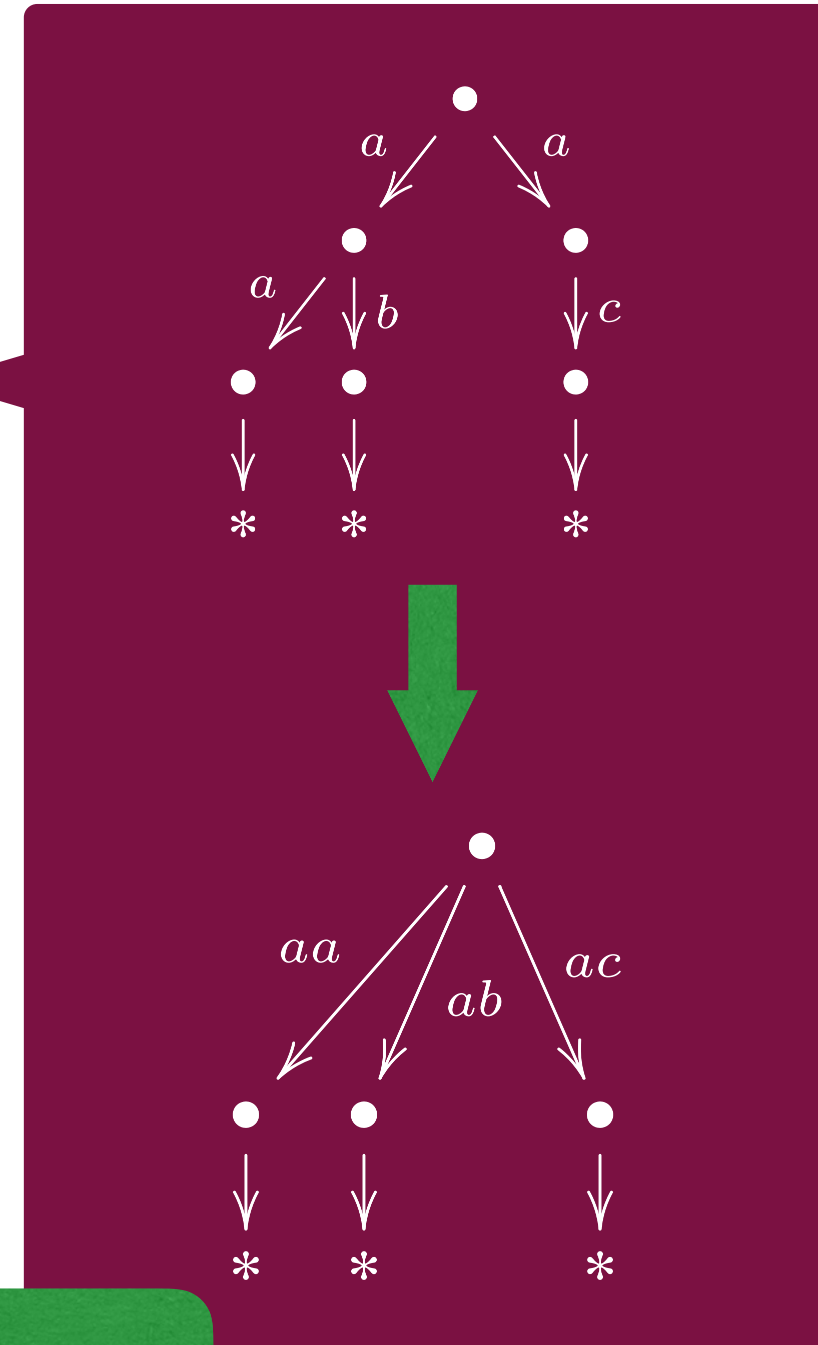
NFA / LTS

Two ideas:

(1) unfold branching + transitions on words

(2) trace = bisimilarity after determinisation





monads !

Trace semantics coalgebraically

Two approaches:

(1) modelling in a Kleisli category

(2) modelling in an Eilenberg-Moore category

Hasuo, Jacobs, S. LMCS '07

Silva, Bonchi, Bonsangue, Rutten FSTTCS'10

algebras of a monad M

Jacobs, Silva, S. JCSS'15

we can relate (1) and (2)

Traces via determinisation

Automaton with M-effects

$X \to O \times (MX)^A$

O has to be an M-algebra !

Determinisation

$MX \to O \times (MX)^A$

MX carries the free M-algebra

trace = bisimilarity after determinisation

Algebras for M

ideally we have a presentation

Eilenberg-Moore algebras

Eilenberg-Moore Algebras

abstractly

$\mathcal{EM}(M)$

- objects

$$\begin{tikzcd} MA \arrow[d, "a"] \\ A \end{tikzcd}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal] & MA \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} MMA \arrow[r, "\mu"] \arrow[d, "Ma"'] & MA \arrow[d, "a"] \\ MA \arrow[r, "a"] & A \end{tikzcd}$$

- morphisms

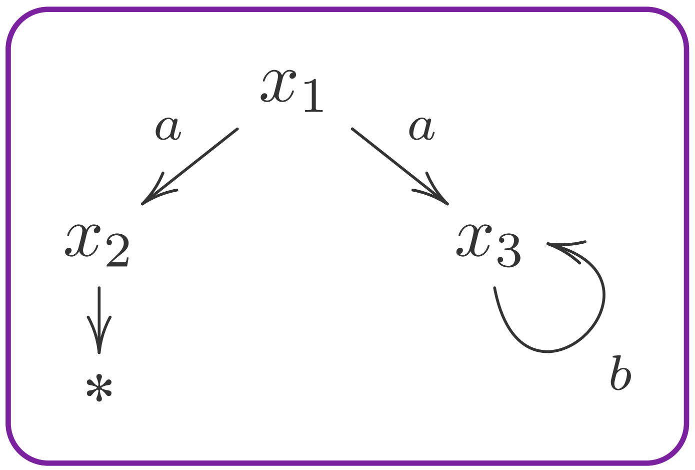
$$\begin{tikzcd} MA \arrow[d, "a"] \\ A \end{tikzcd} \xrightarrow{h} \begin{tikzcd} MB \arrow[d, "b"] \\ B \end{tikzcd}$$

$$\begin{tikzcd} MA \arrow[r, "Mh"] \arrow[d, "a"'] & MB \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

Traces via determinisation

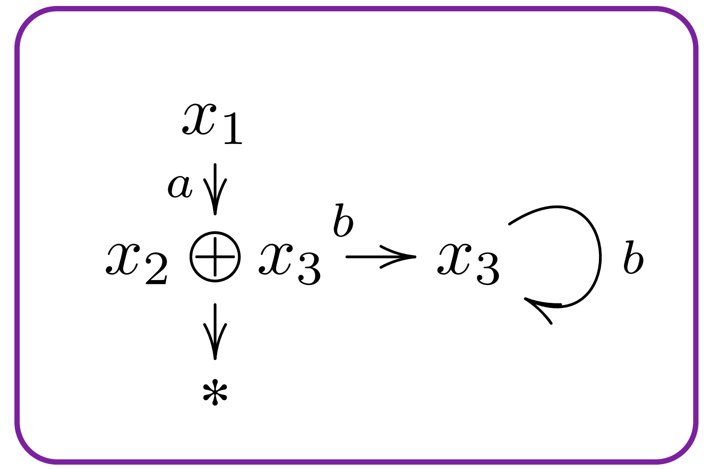
NFA

$X \to 2 \times (\mathcal{P}X)^A$



DFA

$\mathcal{P}X \to 2 \times (\mathcal{P}X)^A$



trace = bisimilarity after determinisation

Algebras for $\mathcal{P}$

join semilattices with bottom

finite powerset !

Traces via determinisation

Rabin PA

$X \to [0,1] \times (\mathcal{D}_{\leq 1} X)^A$

$$\begin{tikzcd}[column sep=small]
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{4}}"] & \\
x_2 \arrow[d, "{b,\frac{1}{3}}"'] & & x_3 \arrow[d, "{c,\frac{1}{2}}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}$$

DFA

$\mathcal{D}_{\leq 1} X \to [0,1] \times (\mathcal{D}_{\leq 1} X)^A$

$$\begin{tikzcd}[column sep=small]
 & x_1 \arrow[d, "a"] & \\
 & \frac{1}{2}x_2 + \frac{1}{4}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\frac{1}{6}x_4 \arrow[d, "\frac{1}{6}"'] & & \frac{1}{8}x_5 \arrow[d, "\frac{1}{8}"] \\
* & & *
\end{tikzcd}$$

trace = bisimilarity after determinisation

Algebras for $\mathcal{D}_{(\leq 1)}$

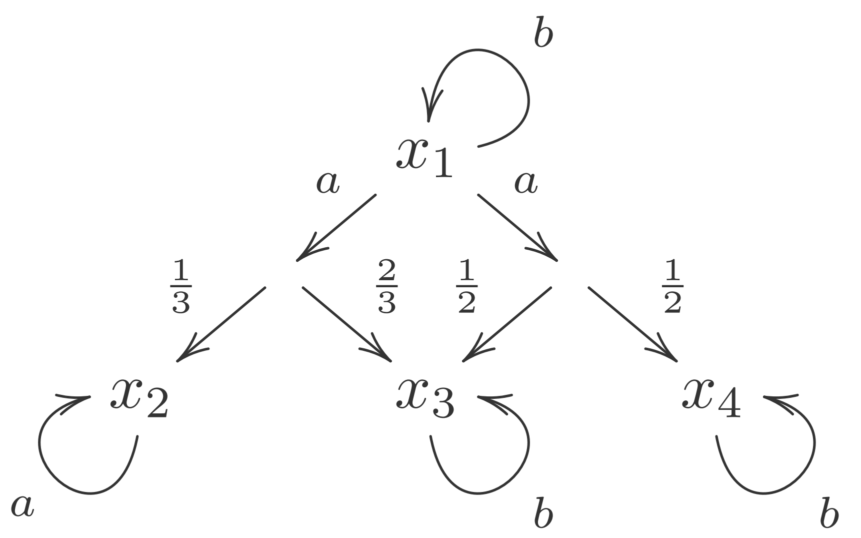
(positive) convex algebras

finitely supported (sub)distributions!

Traces via determinisation

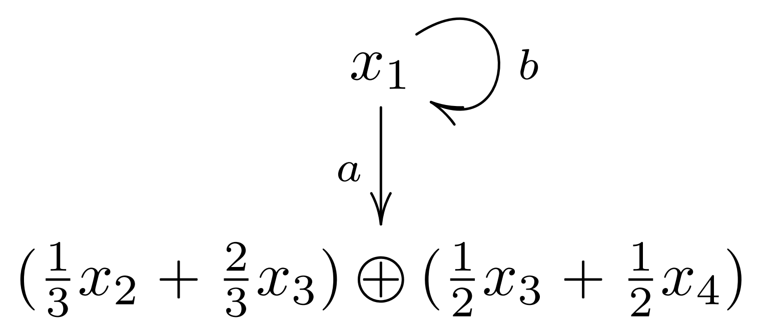
Simple PA

$X \to ? \times (\mathcal{C}X)^A$



DFA

$\mathcal{C}X \to ? \times (\mathcal{C}X)^A$



trace = bisimilarity after determinisation

Algebras for C

convex semilattices

finitely generated convex sets of distr…

Presentation for $\mathscr{C}$

Algebras for $\mathscr{C}$

convex semilattices

finitely generated convex sets of distr…

Bonchi, S., Vignudelli '19

$$\mathbb{A} = (A, \oplus, +_p)$$

$p \in (0,1)$

$$\begin{aligned} (x \oplus y) \oplus z &\overset{(A)}{=} x \oplus (y \oplus z) \\ x \oplus y &\overset{(C)}{=} y \oplus x \\ x \oplus x &\overset{(I)}{=} x \end{aligned}$$

semilattice

S., Woracek '15, '17, '18

$$\begin{aligned} (x +_q y) +_p z &\overset{(A_p)}{=} x +_{pq} (y +_{\frac{p(1-q)}{1-pq}} z) \\ x +_p y &\overset{(C_p)}{=} y +_{1-p} x \\ x +_p x &\overset{(I_p)}{=} x \end{aligned}$$

convex algebra

$$(x \oplus y) +_p z \overset{(D)}{=} (x +_p z) \oplus (y +_p z)$$

distributivity

Many general properties follow
also a sound
up-to context
proof technique

Three things to take home:

1. Semantics via determinisation
is easy for systems / automata with M-effects

2. Having a presentation for M gives us syntax

3. Having the syntax makes determinisation natural !

combining nondeterminism and probability becomes easy

Thank You !