Syntax and Semantics for Nondeterminism and Probability

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1/2 x + 1/2 y

Helmut Veith Workshop, Turracher Höhe, 19.3.19
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I will tell you:

1. Just the absolute basics of coalgebra
2. (Trace) semantics via determinisation...
3. ...enabled by algebraic structure

Mathematical framework based on category theory for state-based systems semantics

for nondeterministic/probabilistic... systems

systems with algebraic effects

syntax
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

\[ X \xrightarrow{c} FX \]

generic notion of behavioural equivalence (bisimilarity)

\[ \simeq \]

states

type

object in the base category \( C \)

functor on the base category \( C \)

form a category too

\[ \text{CoAlg}_C(F) \]
Examples

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Rabin PA

\[ X \rightarrow [0, 1] \times (\mathcal{D}_{\leq 1}X)^A \]

Simple PA

\[ X \rightarrow ? \times (\mathcal{PD}X)^A \]

systems with nondeterminism and probability
In general

Systems

\[ X \to (MX)^A \]

Labelled Transition Systems

with M-effects

Automata

\[ X \to O \times (MX)^A \]

with M-effects

with observations in \( O \)
For a monad $M$

**NFA**

$X \rightarrow 2 \times (\mathcal{P}X)^A$

- $M = \mathcal{P}$ for nondeterminism
- Powerset, subsets

**Rabin PA**

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

- $M = \mathcal{D}$ for probability
- Distributions

**Simple PA**

$X \rightarrow ? \times (\mathcal{PD}X)^A$

- $M = \mathcal{PD}$ for nondeterminism and probability

Providing algebraic effects

$\mu: TT \Rightarrow T$

$\eta: Id \Rightarrow T$
For a monad $M$

**NFA**

$X \rightarrow 2 \times (\mathcal{P}X)^A$

$M = \mathcal{P}$

for nondeterminism

Powerset, subsets

**Rabin PA**

$X \rightarrow [0,1] \times (\mathcal{D}_{\leq 1}X)^A$

$M = \mathcal{D}$

for probability

Distributions

**Simple PA**

$X \rightarrow ? \times (\mathcal{C}X)^A$

$M = \mathcal{C}$

for nondeterminism and probability!

Convex subsets of distributions

$\mu: TT \Rightarrow T$

$\eta: Id \Rightarrow T$

providing algebraic effects
Semantics

NFA = LTS + termination

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

Are the (top states of the) following systems equivalent?

- no, they are not wrt. bisimilarity
- yes, they are wrt. trace equivalence as

\[ \text{tr}(x_1) = \text{tr}(y_1) = \{ab, ac\} \]

\[ \text{tr}: X \rightarrow \mathcal{P}(A^*) \]
Semantics

Rabin PA

\[ X \to [0,1] \times (\mathcal{D}_{\leq 1}X)^A \]

Are the (top states of the) following systems equivalent?

- different wrt. bisimilarity
- equivalent wrt. trace equivalence as

\[ \text{tr}(x_1) = \text{tr}(y_1) = \left( ab \mapsto \frac{1}{6}, ac \mapsto \frac{1}{8} \right) \]

\[ \text{tr}: X \to \mathcal{D}(A^*) \]
Trace semantics coalgebraically?

Two ideas:

1. unfold branching + transitions on words
2. trace = bisimilarity after determinisation
Trace semantics coalgebraically

Two approaches:

1. modelling in a Kleisli category
2. modelling in an Eilenberg-Moore category

We can relate (1) and (2)

Hasuo, Jacobs, S. LMCS ’07

Silva, Bonchi, Bonsangue, Rutten FSTTCS’10

Jacobs, Silva, S. JCSS’15
Traces via determinisation

Automaton with M-effects

$X \rightarrow O \times (MX)^A$

Determinisation

$MX \rightarrow O \times (MX)^A$

O has to be an M-algebra!

trace = bisimilarity after determinisation

MX carries the free M-algebra

Eilenberg-Moore algebras

Algebras for M

ideally we have a presentation

HVW’19
Eilenberg-Moore Algebras

• objects

\[ MA \downarrow^a A \]

• morphisms

\[ MA \downarrow^a A \]

\[ MB \downarrow^b B \]

satisfying

\[ A \xrightarrow{\eta} MA \]

\[ MMA \xrightarrow{\mu} MA \]

\[ MA \xrightarrow{a} A \]

\[ MA \xrightarrow{a} A \]

\[ MB \xrightarrow{b} B \]

\[ A \xrightarrow{h} B \]
Traces via determinisation

NFA

\[ X \rightarrow 2 \times (\mathcal{P}X)^A \]

DFA

\[ \mathcal{P}X \rightarrow 2 \times (\mathcal{P}X)^A \]

trace = bisimilarity after determinisation

Algebras for \( \mathcal{P} \)

finite powerset!

HVW’19
Traces via determinisation

Rabin PA

\[ X \rightarrow [0,1] \times \mathcal{D}_{\leq 1} X^A \]

- \( a, \frac{1}{2} \)
- \( \downarrow x_1 \)
- \( x_2 \)
- \( b, \frac{1}{3} \)
- \( \downarrow \)
- \( x_4 \)
- \( 1 \)
- \( \star \)
- \( a, \frac{1}{4} \)
- \( \downarrow x_3 \)
- \( x_5 \)
- \( \downarrow \)
- \( 1 \)
- \( \star \)

DFA

\[ \mathcal{D}_{\leq 1} X \rightarrow [0,1] \times \mathcal{D}_{\leq 1} X^A \]

- \( \frac{1}{2} x_2 + \frac{1}{4} x_3 \)
- \( \downarrow a \)
- \( \frac{1}{6} x_4 \)
- \( \downarrow b \)
- \( \frac{1}{8} \)
- \( \star \)
- \( \frac{1}{8} \)
- \( \star \)

trace = bisimilarity after determinisation

Algebras for \( \mathcal{D}_{\leq 1} \)

(positive) convex algebras

finitely supported (sub)distributions!
Traces via determinisation

Simple PA
\[ \xrightarrow{} ? \times (\mathcal{E}X)^A \]

DFA
\[ \mathcal{E}X \rightarrow ? \times (\mathcal{E}X)^A \]

trace = bisimilarity after determinisation

Algebras for \( C \)

convex semilattices

finitely generated convex sets of distr...

HWW'19
Presentation for $c$

Algebras for $c$

finitely generated convex sets of distributions

$\Delta = (A, \oplus, +_p)$

$p \in (0, 1)$

convex semilattices

Bonchi, S., Vignudelli ‘19

S., Woracek ’15, ’17, ’18

semilattice

convex algebra

distributivity

\[
\begin{align*}
(x \oplus y) \oplus z & \overset{(A)}{=} x \oplus (y \oplus z) \\
x \oplus y & \overset{(C)}{=} y \oplus x \\
x \oplus x & \overset{(I)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x +_q y) +_p z & \overset{(A_p)}{=} x +_pq (y +_{\frac{p(1-q)}{1-pq}} z) \\
x +_p y & \overset{(C_p)}{=} y +_{1-p} x \\
x +_p x & \overset{(I_p)}{=} x
\end{align*}
\]

\[
\begin{align*}
(x \oplus y) +_p z & \overset{(D)}{=} (x +_p z) \oplus (y +_p z)
\end{align*}
\]

Ana Sokolova
Three things to take home:

1. Semantics via determinisation is easy for systems / automata with M-effects

2. Having a presentation for M gives us syntax

3. Having the syntax makes determinisation natural!

Many general properties follow also a sound up-to context proof technique.

Combining nondeterminism and probability becomes easy.

Thank You!