# Proper Semirings and Proper Convex Functors

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joint work with

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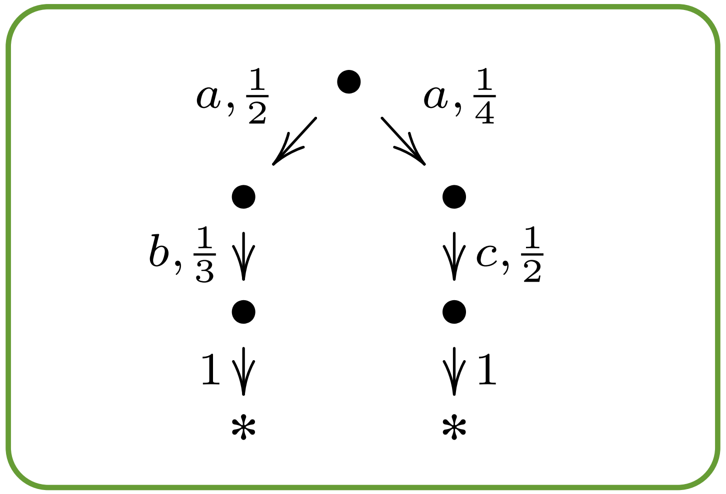
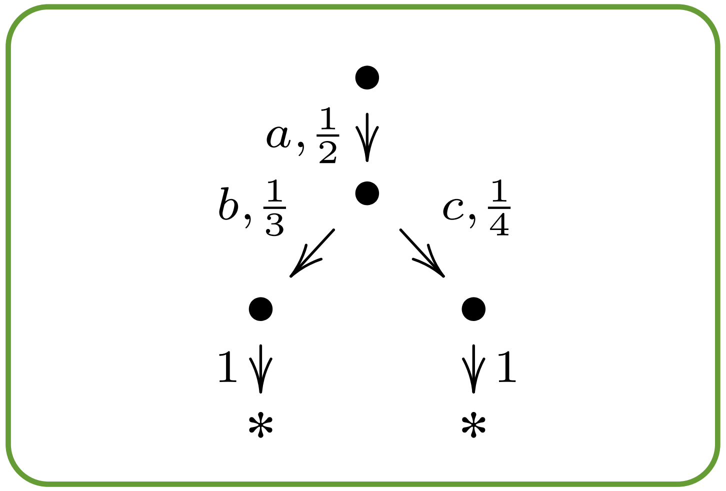
# Trace axioms for PTS

Silva&S 2011

Example



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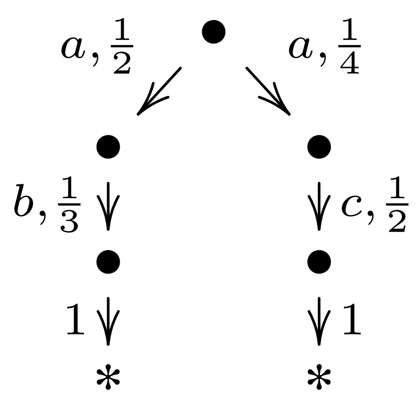


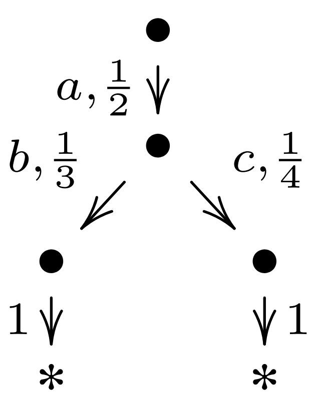


$$\left(\frac{1}{2}\cdot a\cdot\frac{1}{3}\cdot b\cdot 1\cdot *\right)\oplus\left(\frac{1}{4}\cdot a\cdot\frac{1}{2}\cdot b\cdot 1\cdot *\right)\equiv\frac{1}{2}\cdot a\cdot\left(\frac{1}{3}\cdot b\cdot 1\cdot *\oplus\frac{1}{4}\cdot c\cdot 1\cdot *\right)$$

# Trace axioms for PTS

Silva&S 2011

Axioms for bisimilarity

+

$$p \cdot a \cdot (p_1 \mathsf{E}_1 \oplus p_2 \mathsf{E}_2) \equiv p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \quad (D)$$

soundness and completeness

in positive convex algebras

# The quest for completeness

Inspired lot of new research:

- Congruences of convex algebras

S&Woracek'15

f.p. = f.g.
for (positive)
convex algebras

if f.p. = f.g. and ★ then completeness

does not hold

- Proper functors

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Now

our axiomatisation is complete since a functor F* (on positive convex algebras) is proper

# Semirings, functors

Semiring $\mathbb{S} = (S, +, 0, \cdot, 1)$

two monoids with a zero and distributive laws

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But also positive convex algebras and two functors for WA with weights in positive convex algebras.

# Proper semirings, proper functors

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A semiring is proper iff the semiring functor on $\mathbb{S}$-SMOD is.

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the semiring functor

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Why study proper semirings ?

If a semiring is proper and effectively representable, then equivalence of weighted automata is decidable, for finite alphabets.

# Proper semirings

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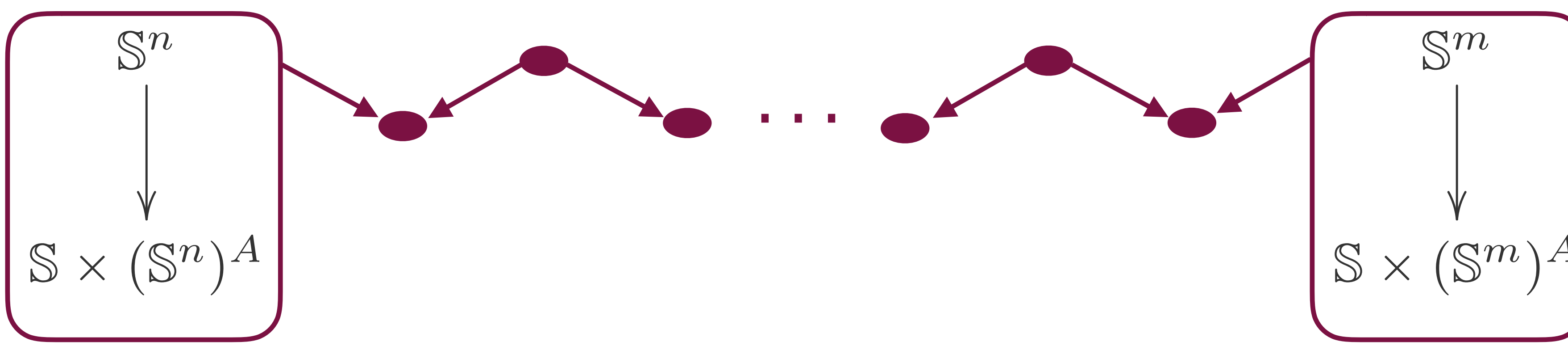

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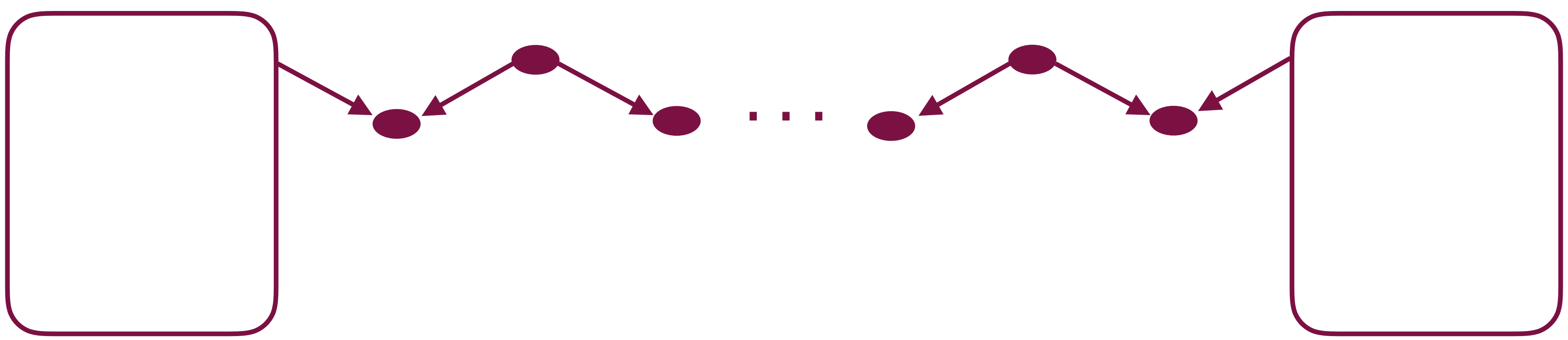

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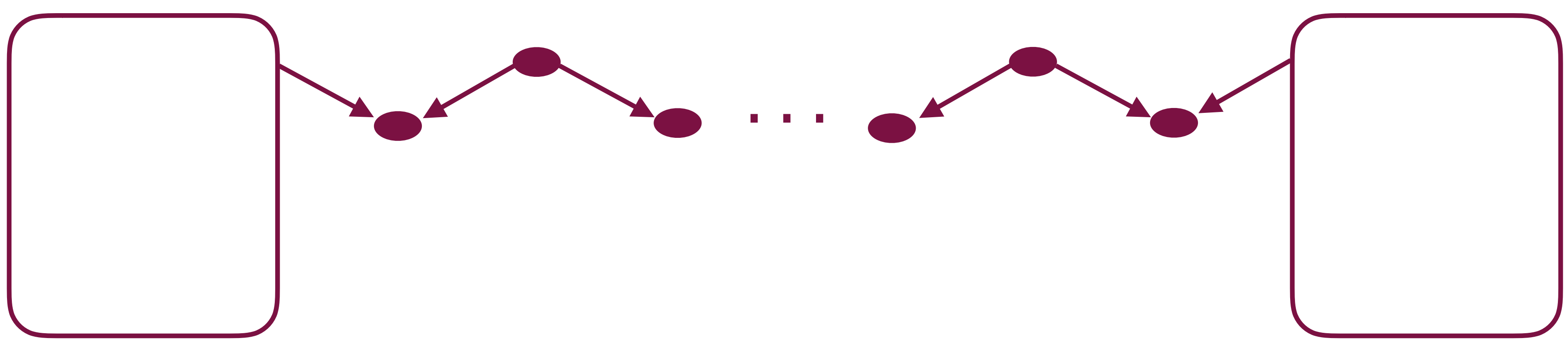

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# Proper functors

functor F on an algebraic category

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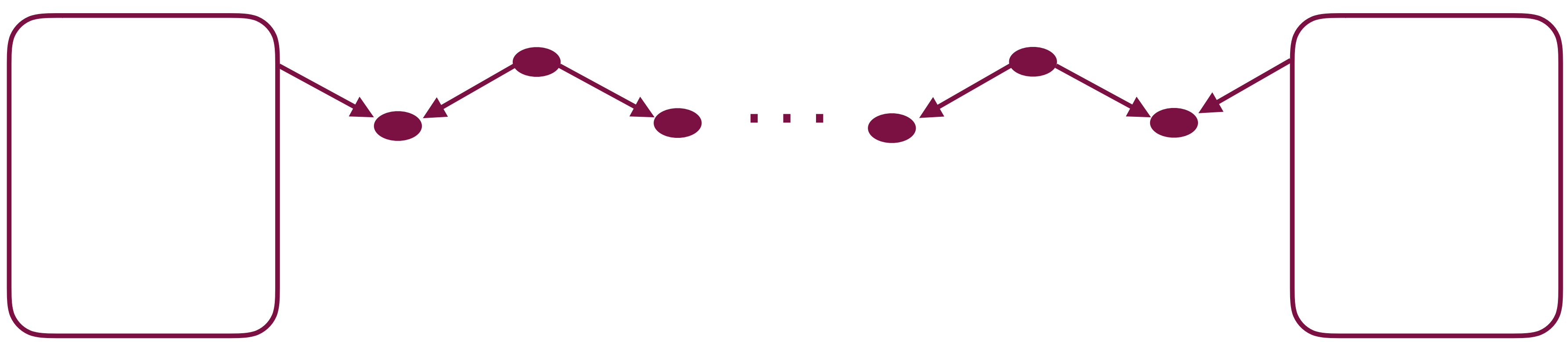

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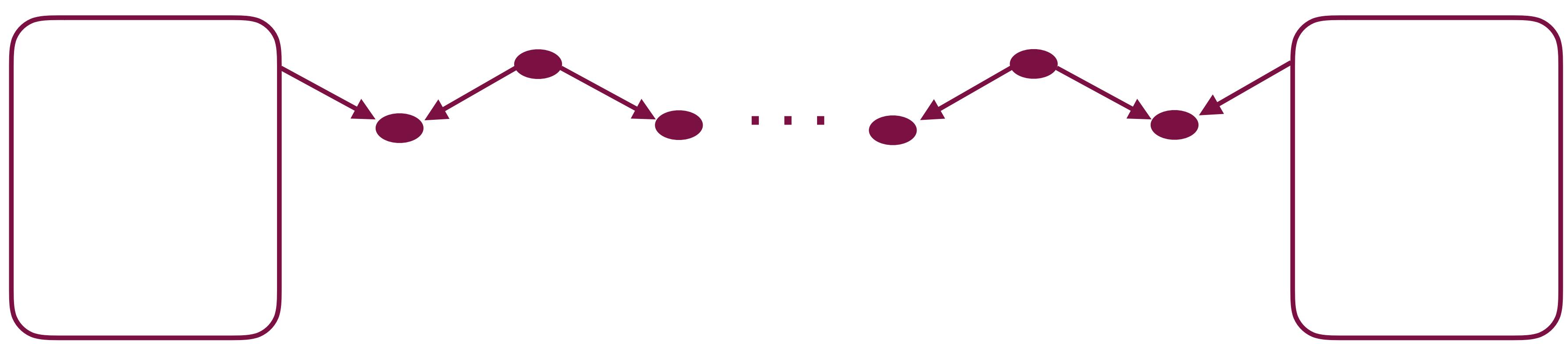

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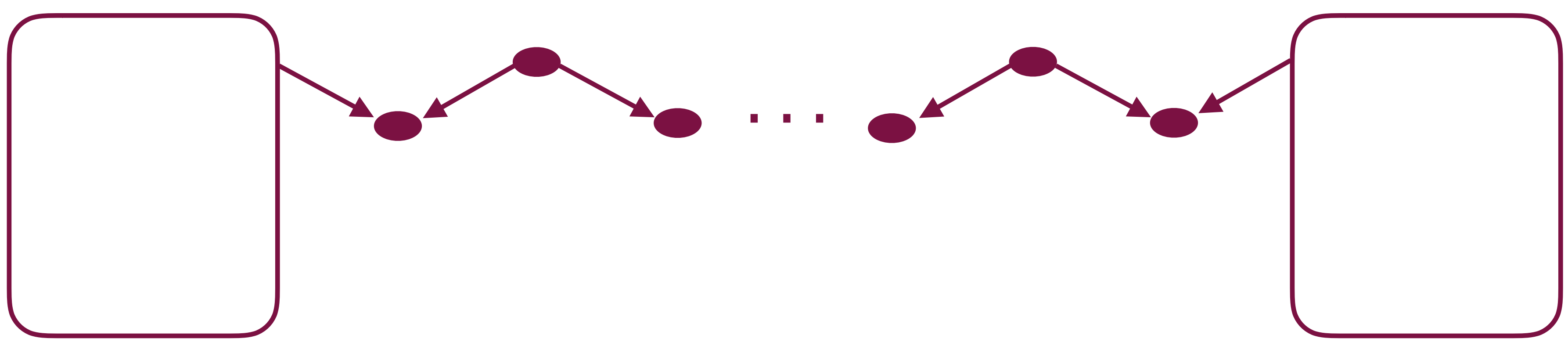

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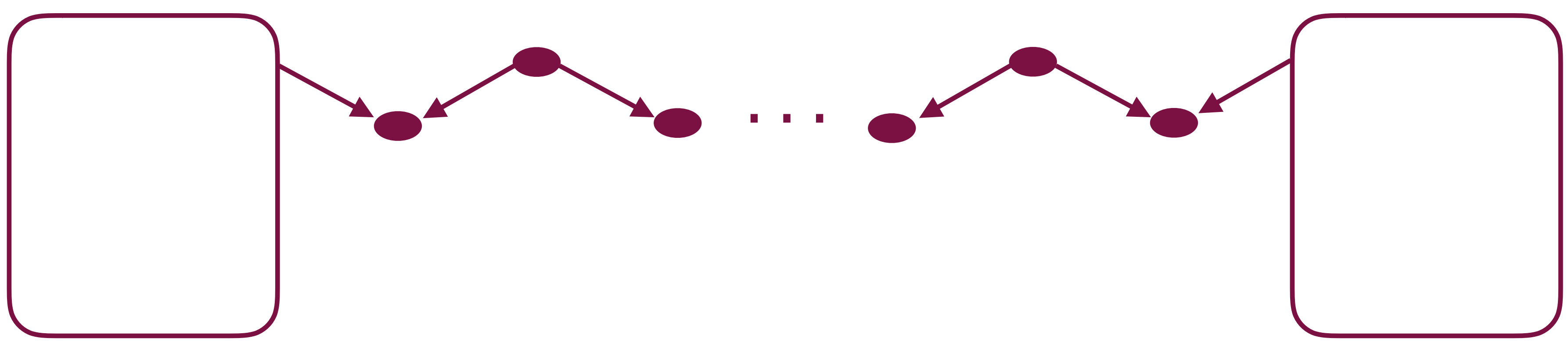

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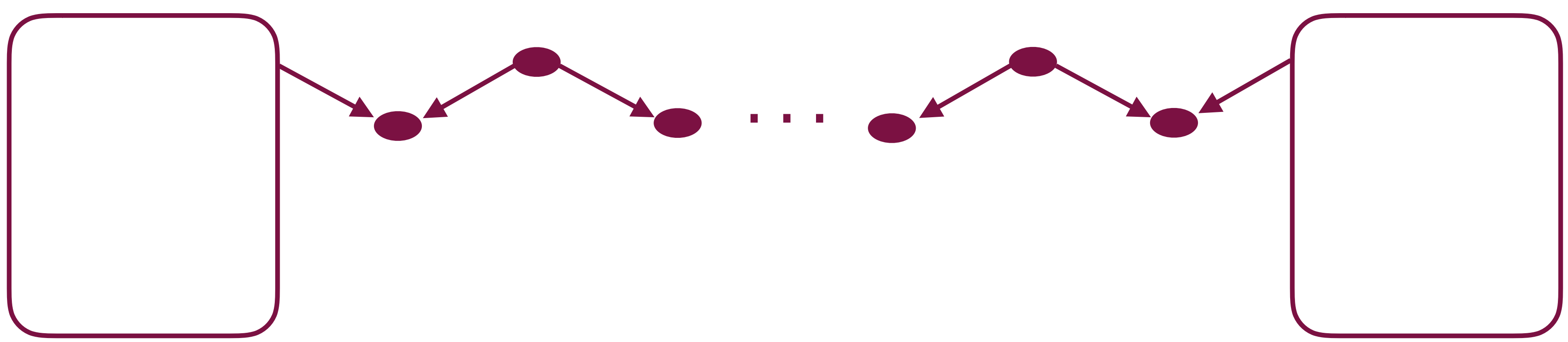

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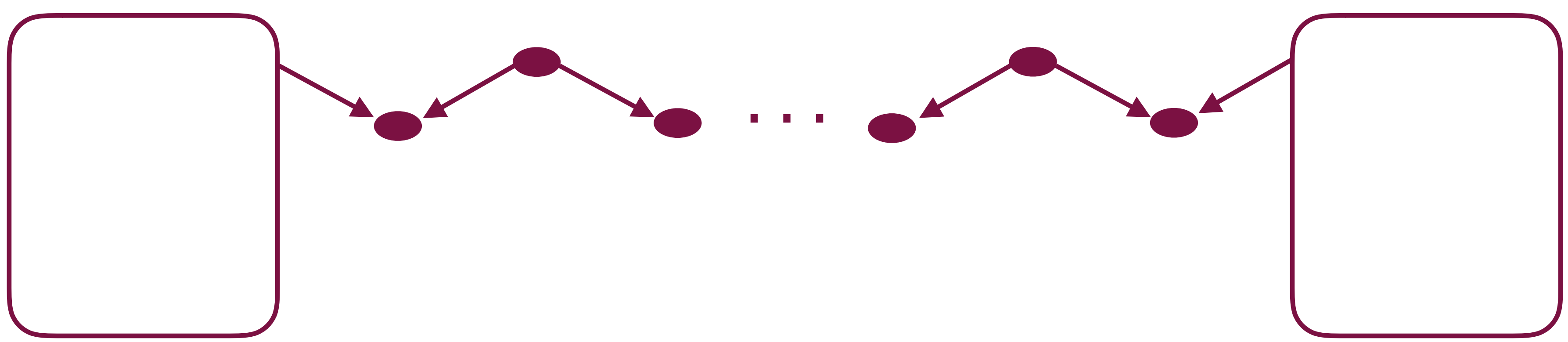

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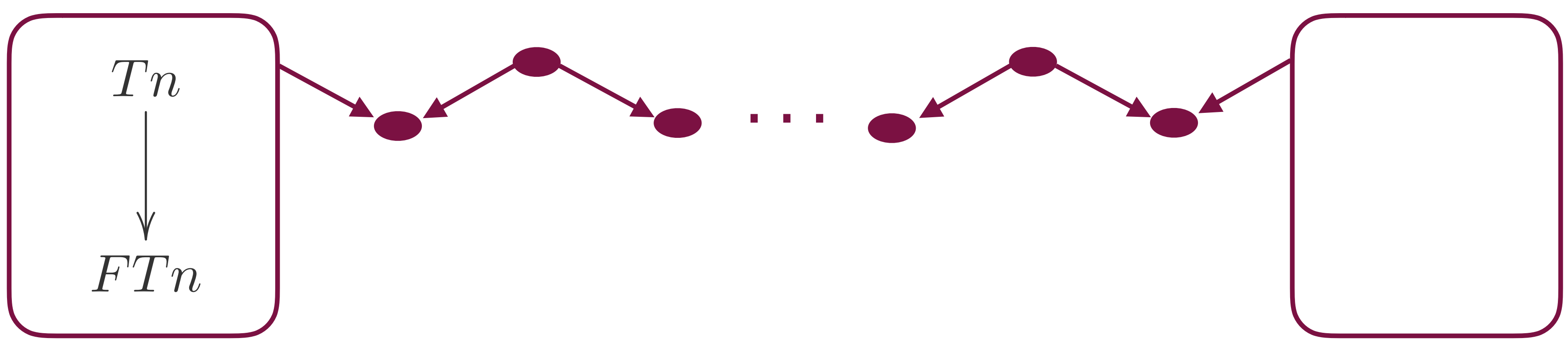

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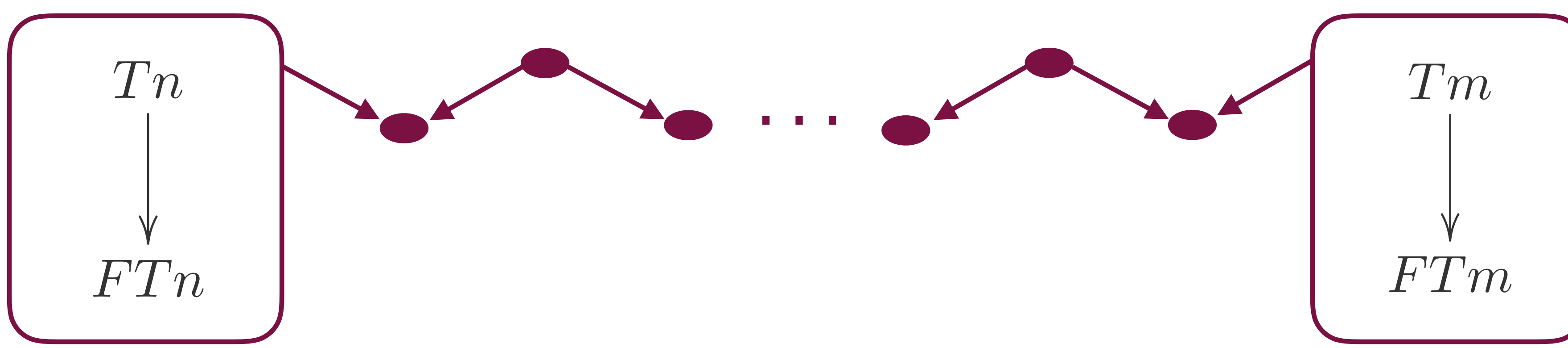

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A functor F on an algebraic category $\mathtt{Set}^T$, for a finitary monad T, is proper iff for every two behaviourally equivalent states x ≡ y in F-coalgebras with f.f.g. carriers, there is a zigzag of F-coalgebras whose all nodes have f.f.g. carriers that relates them.

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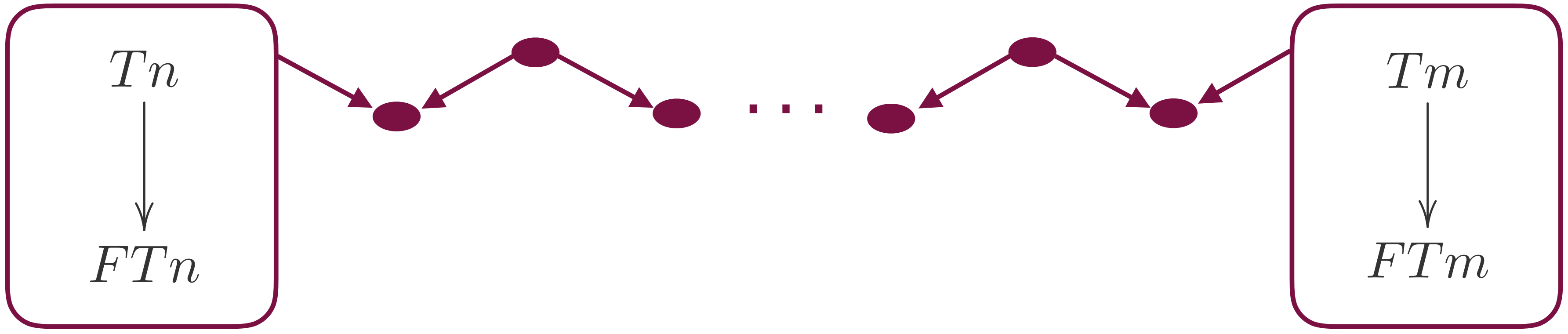

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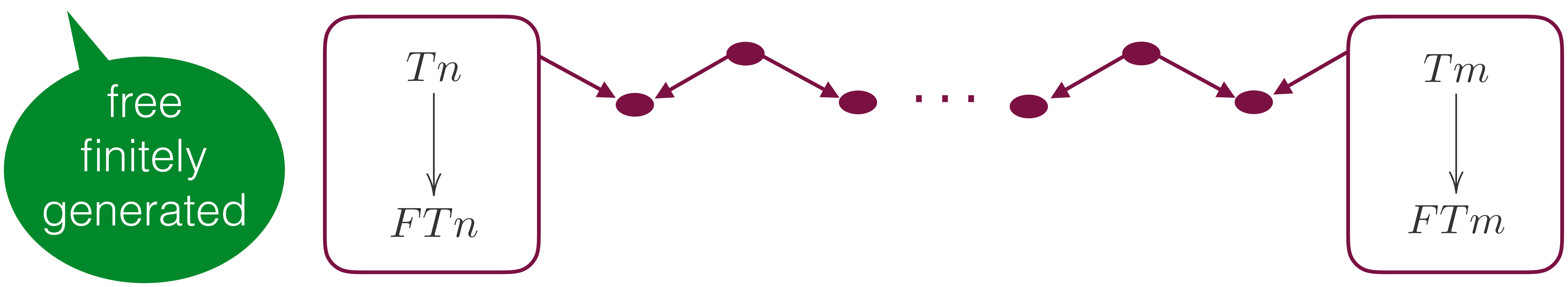


Proper functors enable “easy” completeness proofs of axiomatizations of expression languages…

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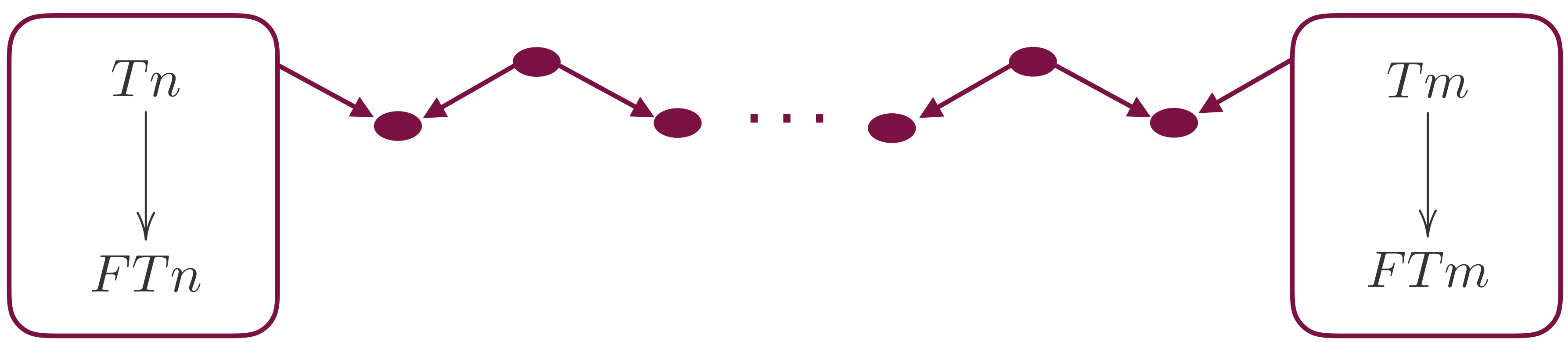


Proper functors enable “easy” completeness proofs of axiomatizations of expression languages…

proving properness is difficult

# Previous results

Proper:

- Boolean semiring — Bloom & Ésik '93
- Finite commutative ordered semirings — Ésik & Kuich '01 — 7
- Eucledian domains, skew fields — Béal & Lombardy & Sakarovich '05 — 2
- $\mathbb{N}, \mathbb{B}, \mathbb{Z}$, skew fields — Béal & Lombardy & Sakarovich '05 — 1
- Noetherian semirings, commutative rings, finite semirings — Ésik & Maletti '10 — 1

Improper:

- Tropical semiring — Ésik & Maletti '10

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these are all known (im)proper semirings

Improper:

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# Here

Framework for proving properness

**Instantiate it on known semirings**

- Noetherian 1
- Naturals $\mathbb{N}$ 1

**Prove new semirings proper**

- Non-negative rationals $\mathbb{Q}_+$ 1
- Non-negative reals $\mathbb{R}_+$ 1

**Prove new convex functors proper**

- $[0,1] \times (-)^A$ 1
- F*, a subfunctor of the above 3

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- F*, a subfunctor of the above (3)

on positive convex algebras

$\mathtt{Set}^{\mathcal{D}}$

# The proof method

Noetherian

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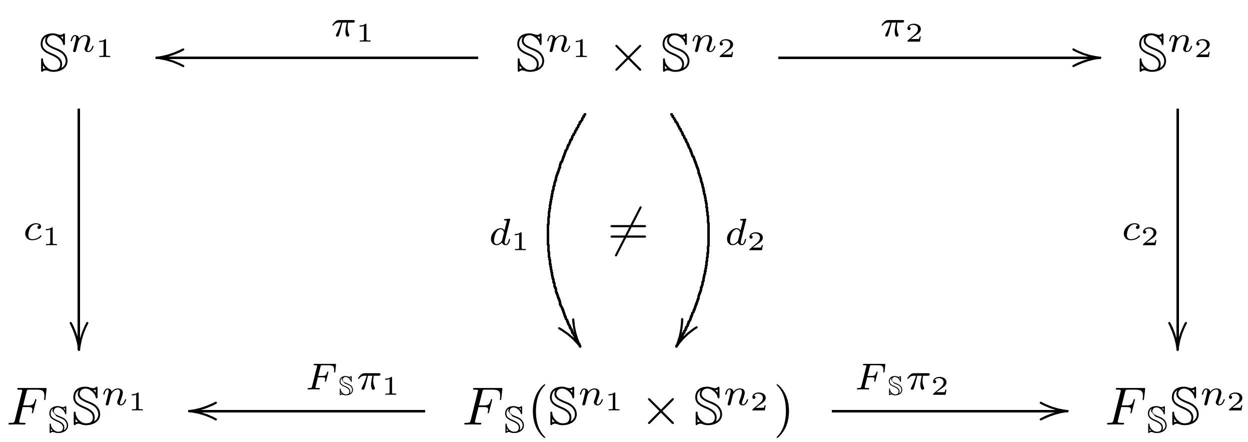

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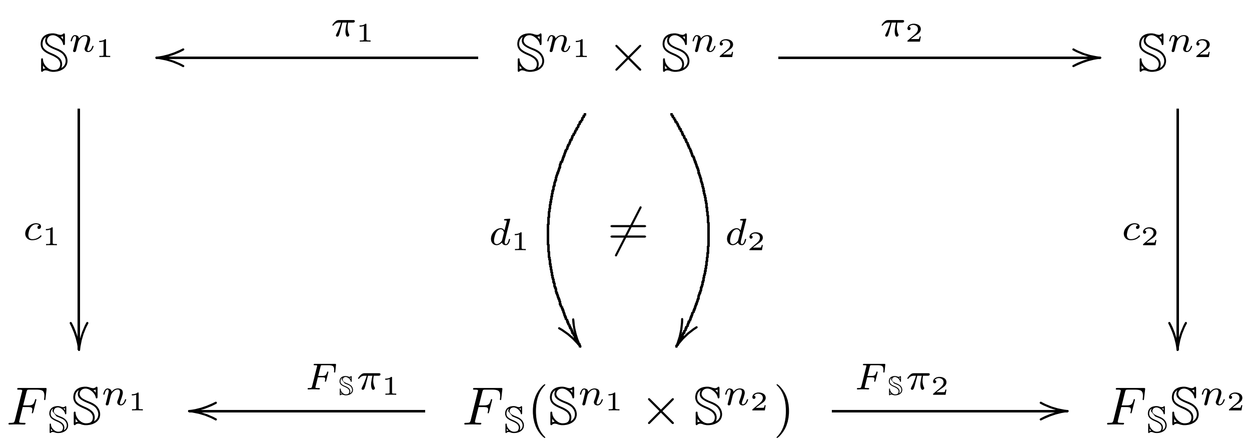


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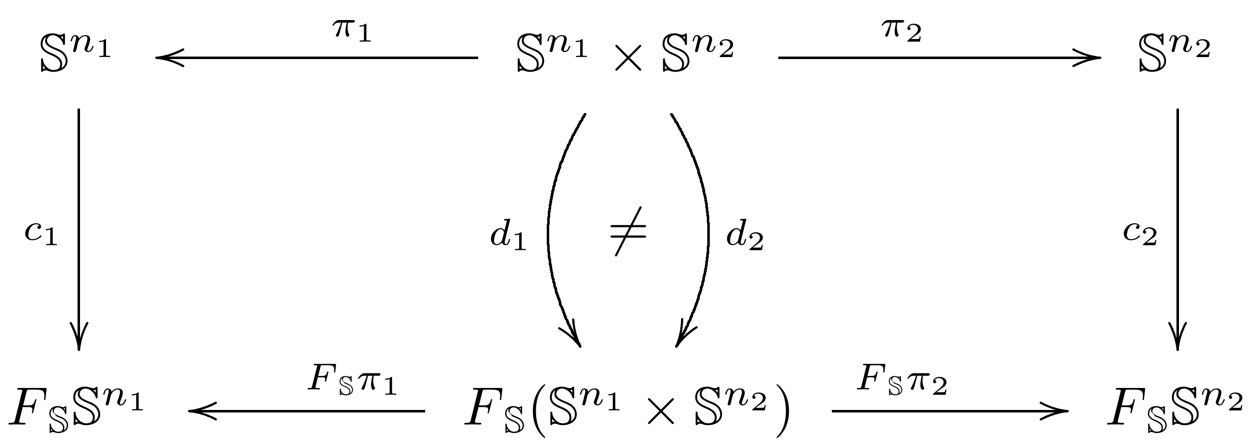


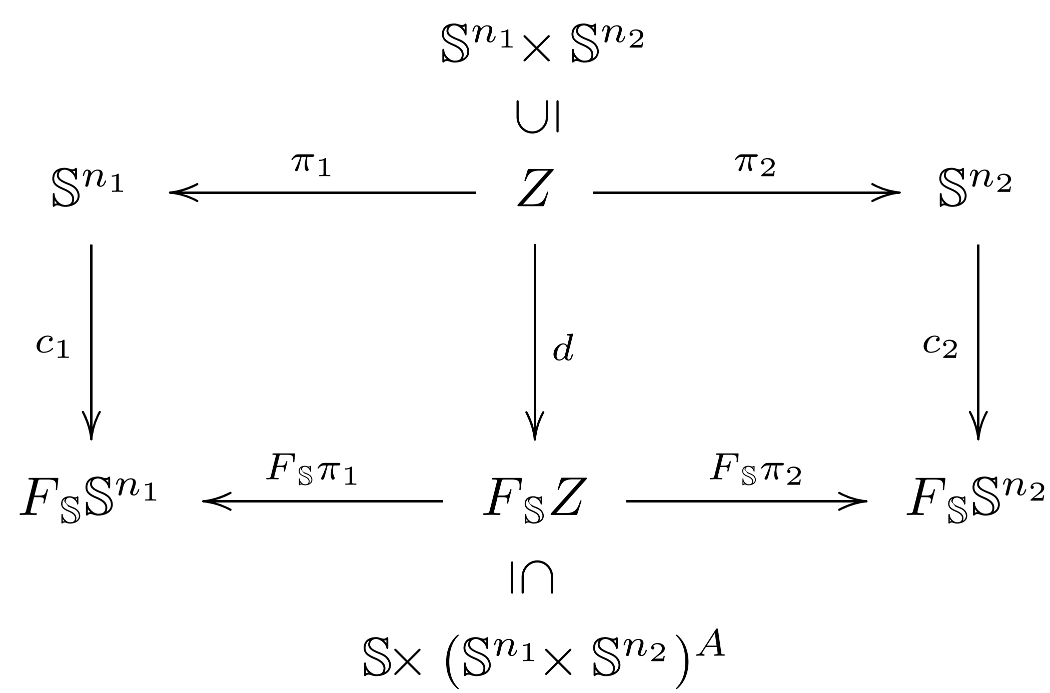

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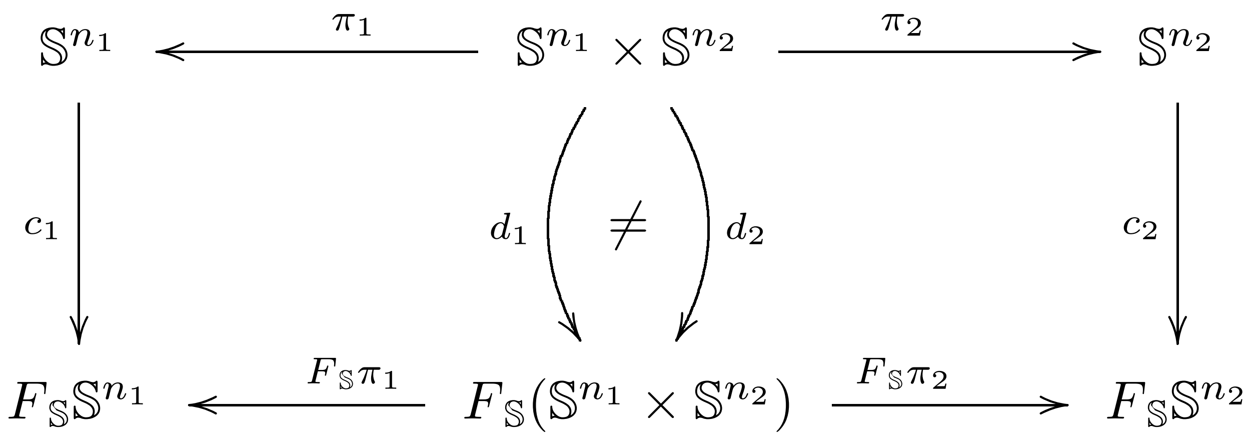


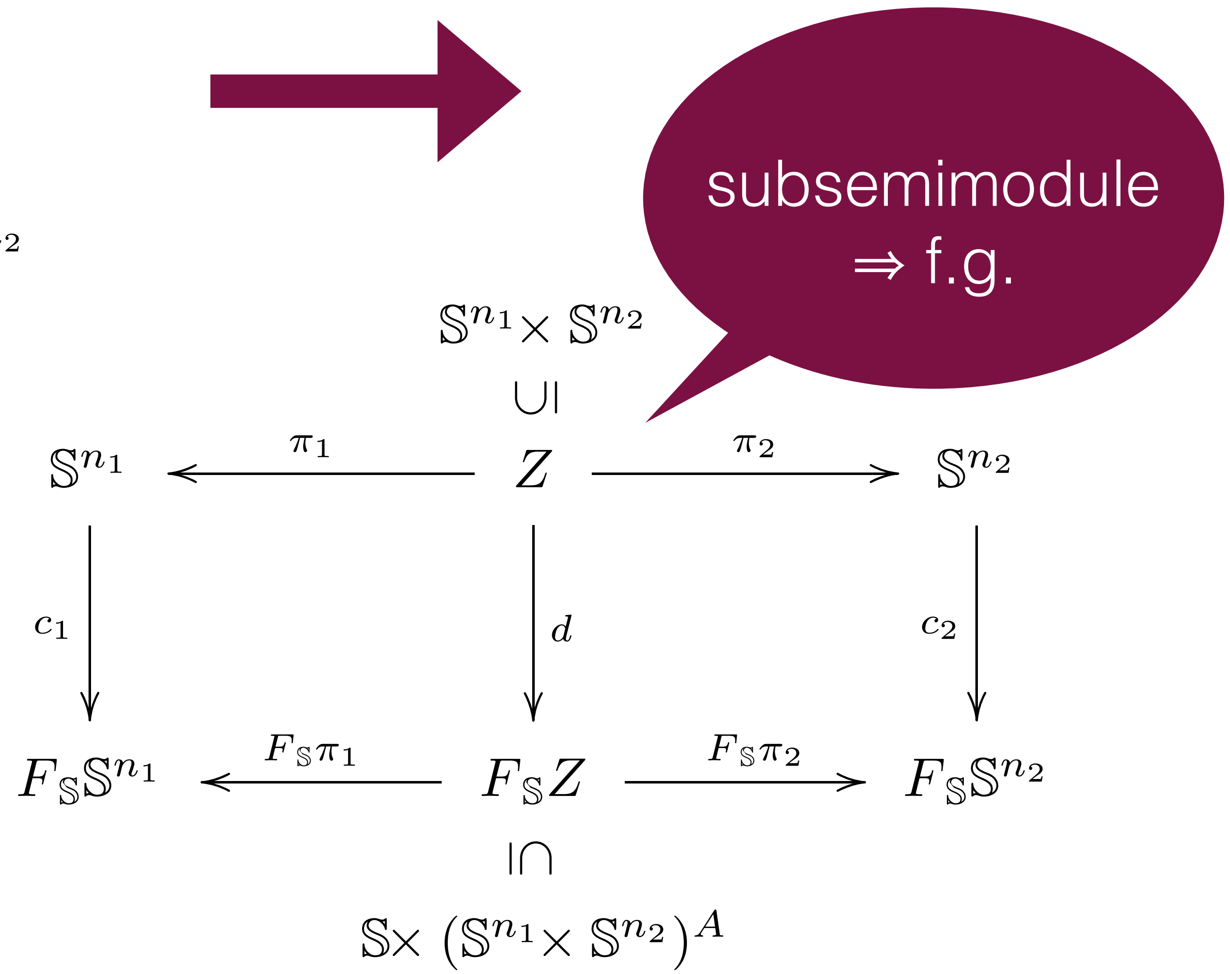

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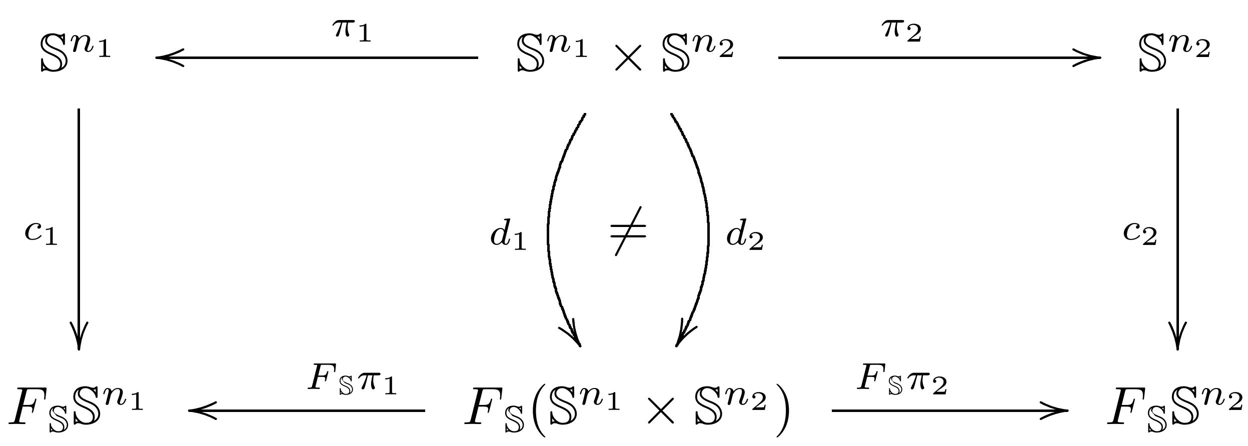
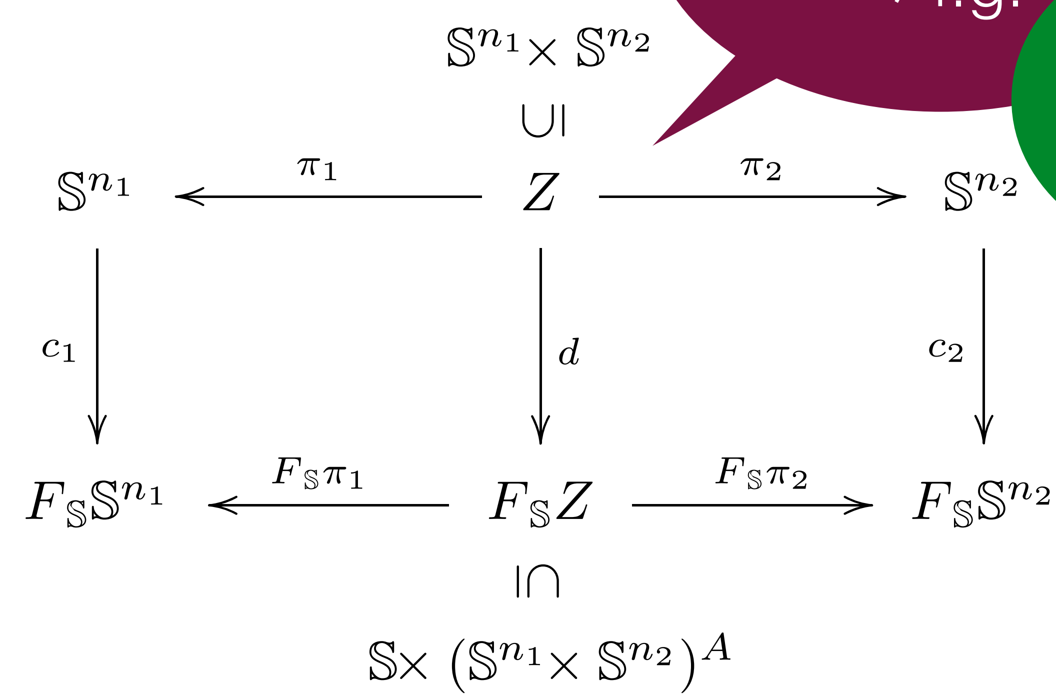



subsemimodule ⇒ f.g.

since Noetherian



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Semiring $\mathbb{S}$, ring completion $\mathbb{E}$

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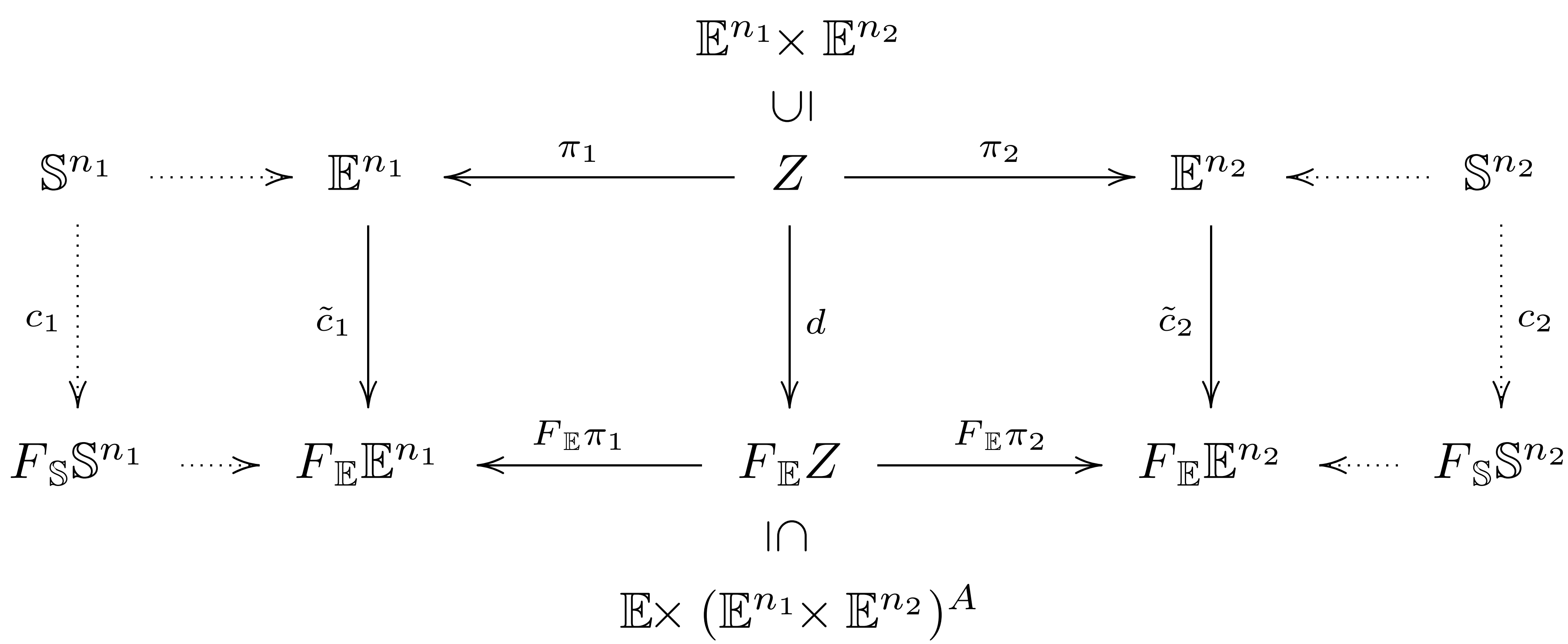

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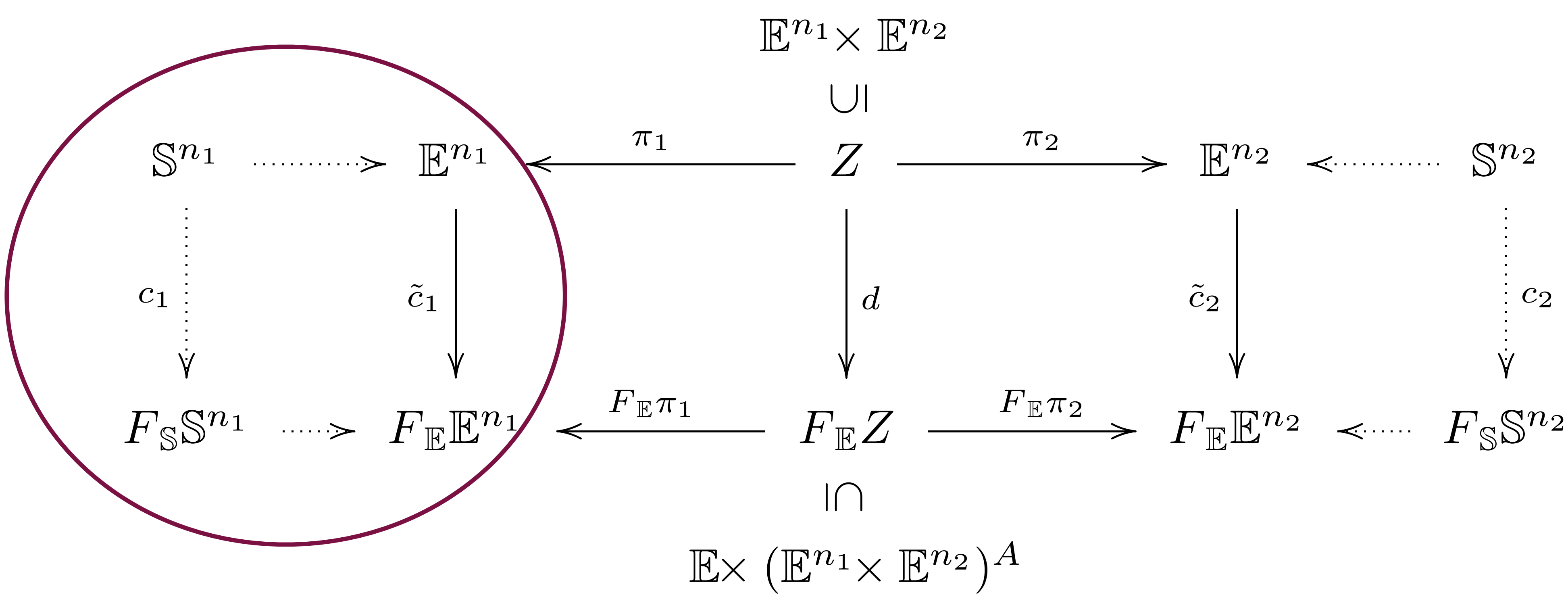

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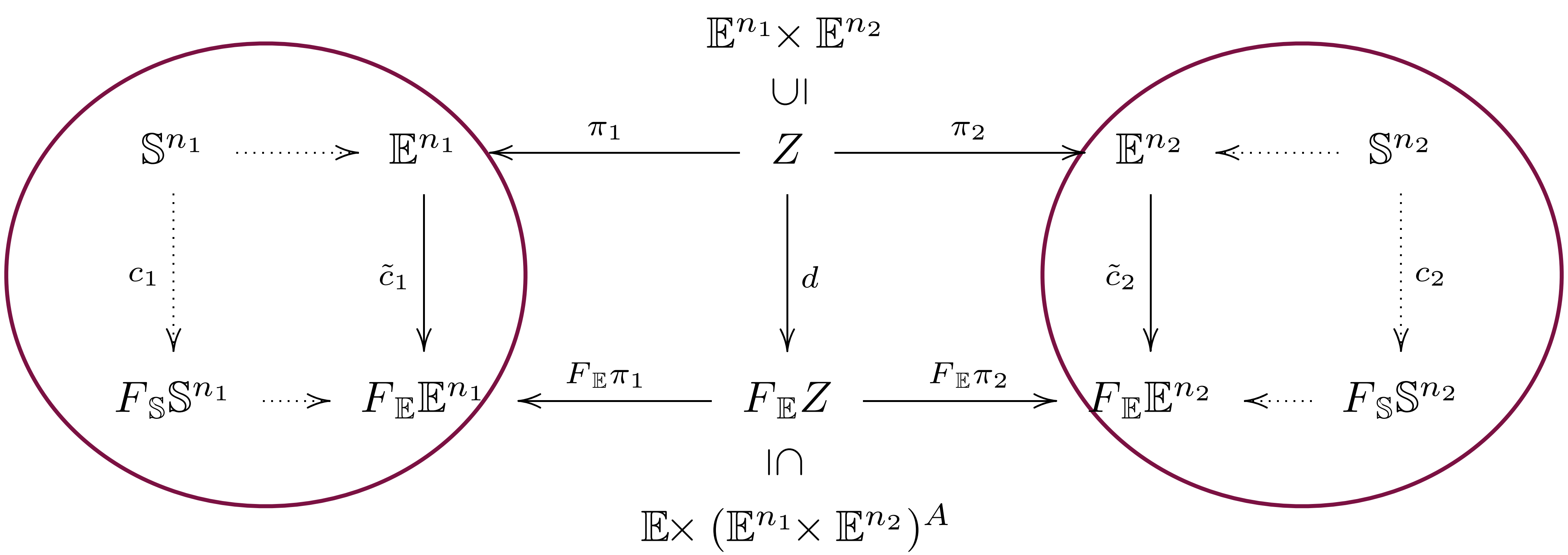

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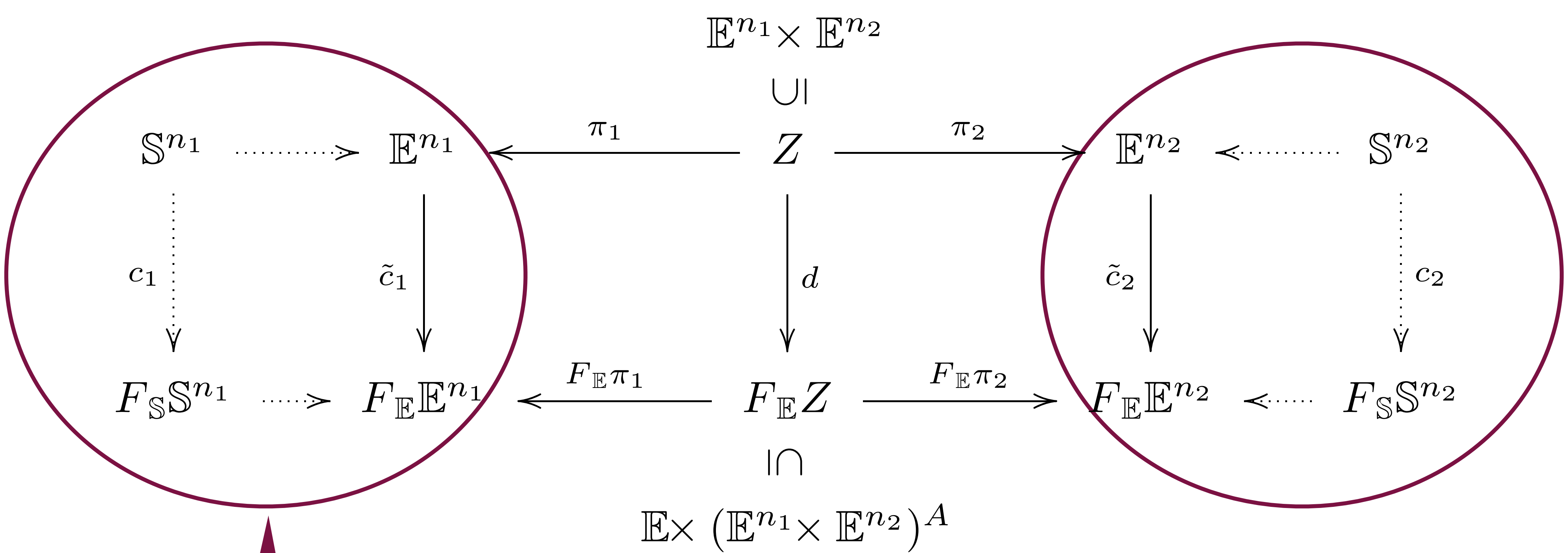


extension lemma

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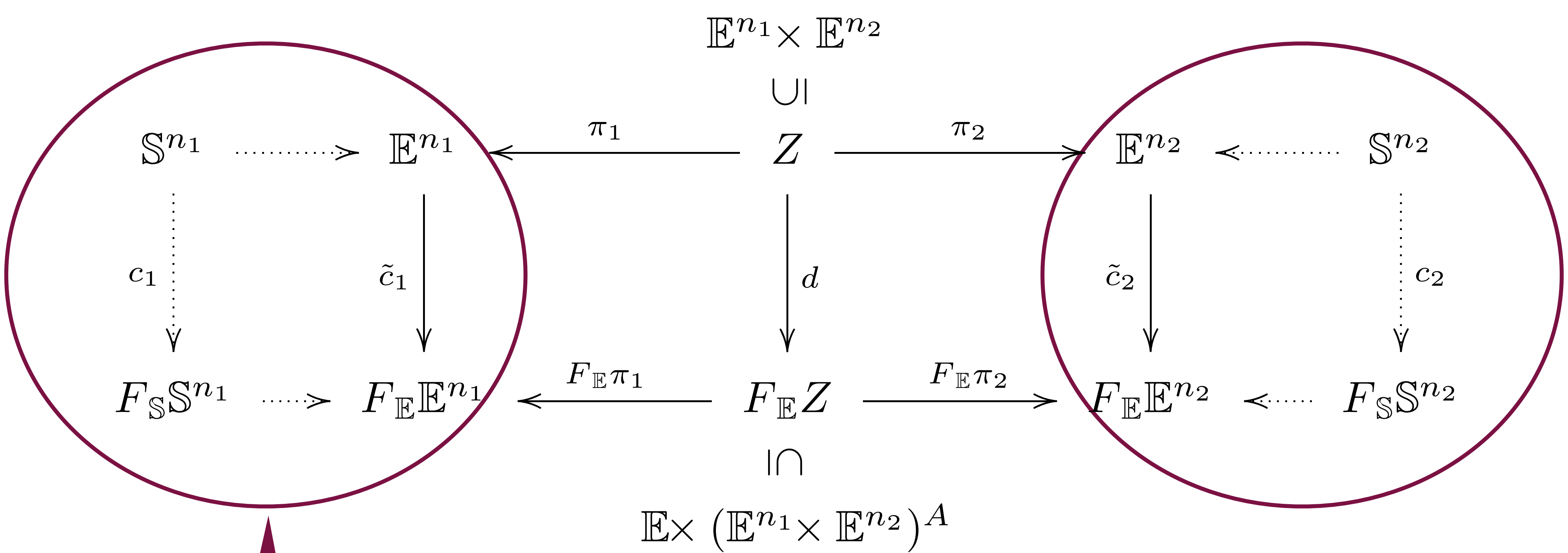


extension lemma

zigzag

$$(\mathbb{S}^{n_1}, c_1) \xleftarrow{\pi_1} (Z \cap (\mathbb{S}^{n_1} \times \mathbb{S}^{n_2}), d) \xrightarrow{\pi_2} (\mathbb{S}^{n_2}, c_2)$$

f.g. in $\mathbb{E}$-SMOD

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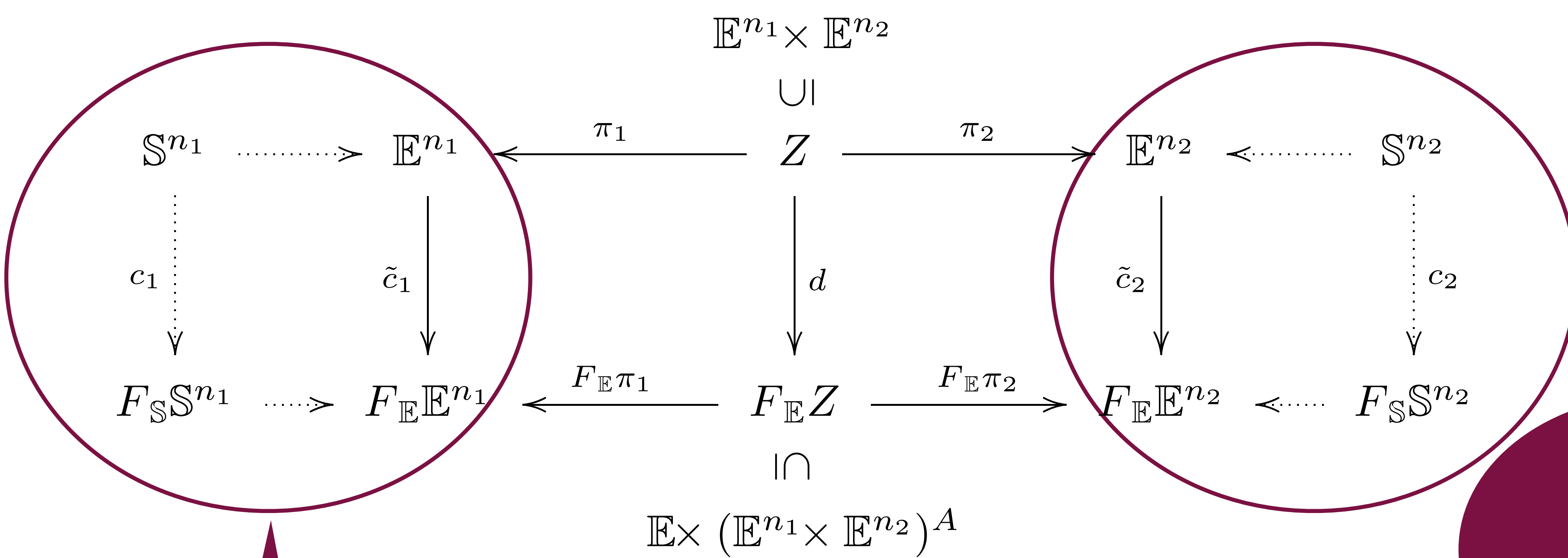


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# Reduction lemma

**Reduction lemma**

For a semiring $\mathbb{S}$, ring completion $\mathbb{E}$

$Z \cap (\mathbb{S}^{n_1} \times \mathbb{S}^{n_2})$ is f.g. in $\mathbb{S}$-$\mathtt{SMOD}$

| | | | |
|---|---|---|---|
| $\mathbb{S}$ | $\mathbb{N}$ | $\mathbb{Q}_+$ | $\mathbb{R}_+$ |
| $\mathbb{E}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
| $\mathbb{S}$-$\mathtt{SMOD}$ | $\mathbb{N}$-$\mathtt{SMOD}$ $(\mathtt{CMON})$ | $\mathbb{Q}_+$-$\mathtt{SMOD}$ | $\mathbb{R}_+$-$\mathtt{SMOD}$ $(\mathtt{CONE})$ |
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Hilbert 1890

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Hilbert 1890

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| $\mathbb{E}$ | $\mathbb{Z}$ | $\mathbb{Q}$ | $\mathbb{R}$ |
| $\mathbb{S}$-$\mathtt{SMOD}$ | $\mathbb{N}$-$\mathtt{SMOD}$ ($\mathtt{CMON}$) | $\mathbb{Q}_+$-$\mathtt{SMOD}$ | $\mathbb{R}_+$-$\mathtt{SMOD}$ ($\mathtt{CONE}$) |
| $\mathbb{E}$-$\mathtt{MOD}$ | $\mathbb{Z}$-$\mathtt{MOD}$ ($\mathtt{AB}$) | $\mathbb{Q}$-$\mathtt{MOD}$ ($\mathbb{Q}$-$\mathtt{VEC}$) | $\mathbb{R}$-$\mathtt{MOD}$ ($\mathbb{R}$-$\mathtt{VEC}$) |

Hilbert 1890

consequence

Minkowski 1896

# Reduction lemma

**Reduction lemma**

For a semiring $\mathbb{S}$, ring completion $\mathbb{E}$

$Z \cap (\mathbb{S}^{n_1} \times \mathbb{S}^{n_2})$ is f.g. in $\mathbb{S}$-$\mathtt{SMOD}$

| | | | |
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→ proper

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# On positive convex algebras

WA functor $F_{[0,1]} = [0,1] \times (-)^A$

and its subfunctor $F^*$

arising from trace axioms for PTS

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WA functor

$$F_{[0,1]} = [0,1] \times (-)^A$$

and its subfunctor *F**

arising from trace axioms for PTS

$$F^* X = \{(o,f) \in F_{[0,1]}(X) \mid \forall a \in A.\, \exists p_a \in [0,1].\, \exists x_a \in X.\, o + \sum_{a \in A} p_a \leqslant 1 \wedge f(a) = p_a x_a\}$$

# On positive convex algebras

For $$F_{[0,1]} = [0,1] \times (-)^A$$

The proof of properness works similarly via the extension and reduction lemma

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Minkowski 1896

for the subfunctor *F**

the proof of properness requires additionally something new

Kakutani fixpoint theorem

# To summarise

We have:

- framework for proving properness
- proofs that old and new semirings are proper
- convex WA functor and subfunctor is proper

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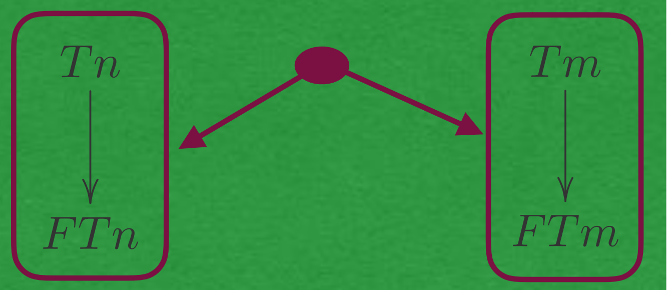

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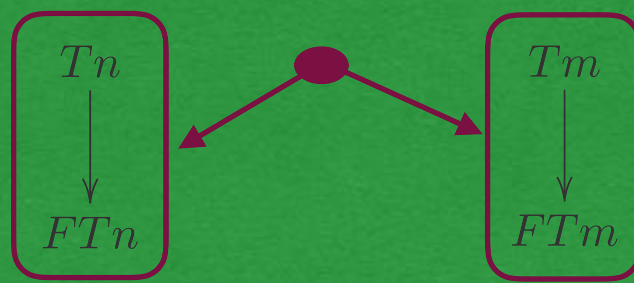


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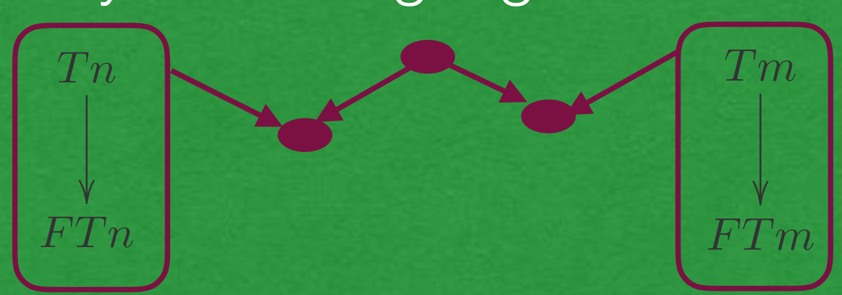

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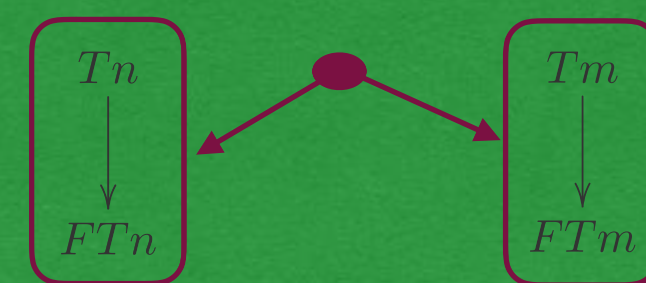


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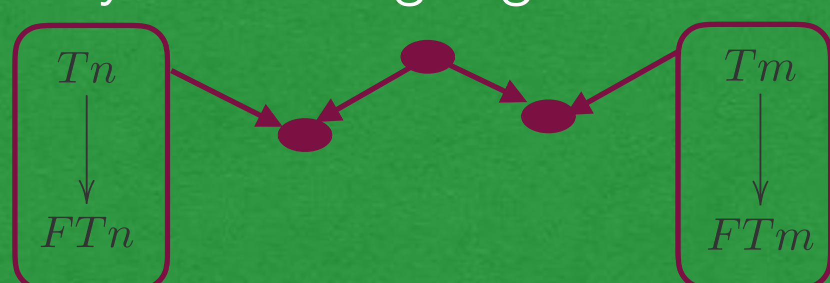


reduction lemmas via strong classical results

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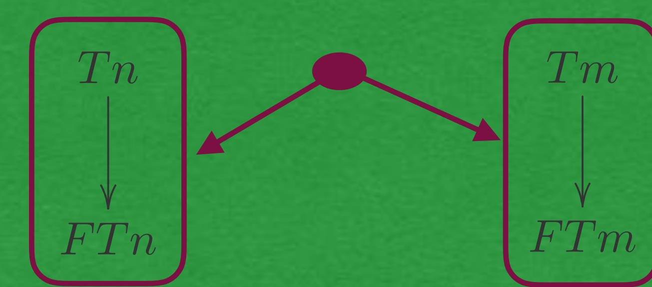


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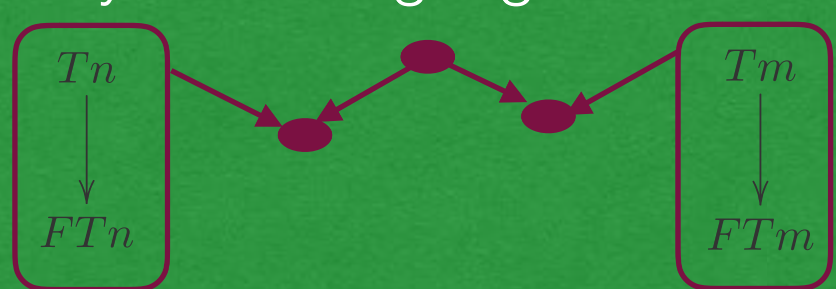


reduction lemmas via strong classical results

Thank You !