Probabilistic systems

a place where categories meet probability

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Outline

- Introduction - probabilistic systems and coalgebras

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- Bisimilarity - the strong end of the spectrum

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- Trace semantics - the weak end of the spectrum

  (newer result)

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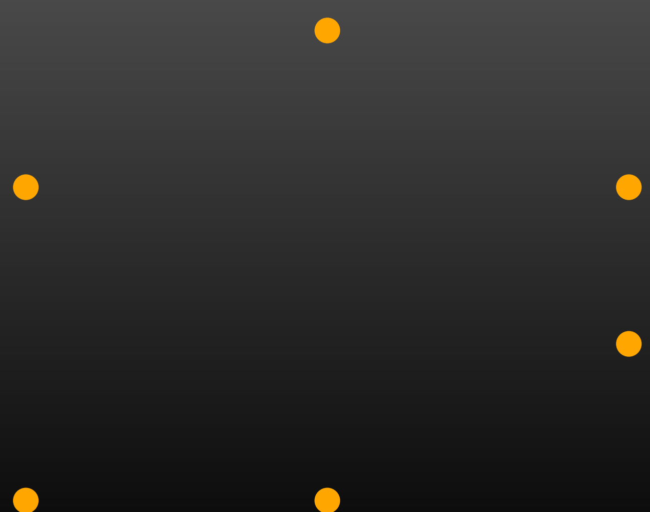

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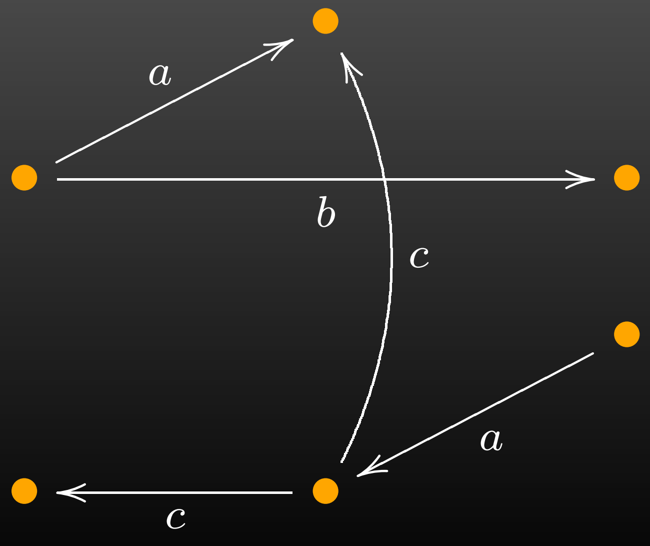

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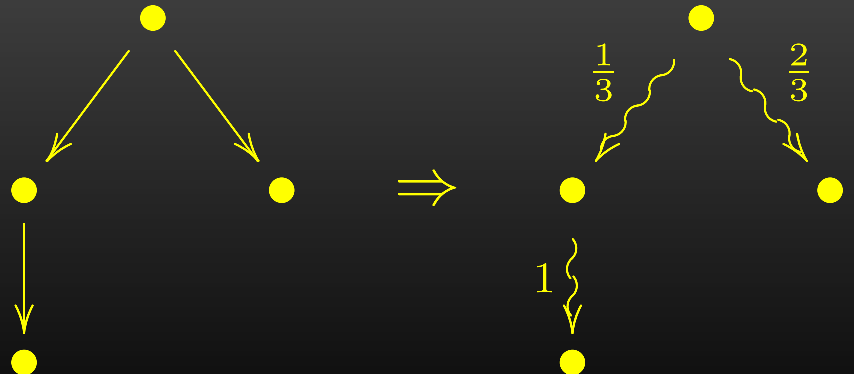

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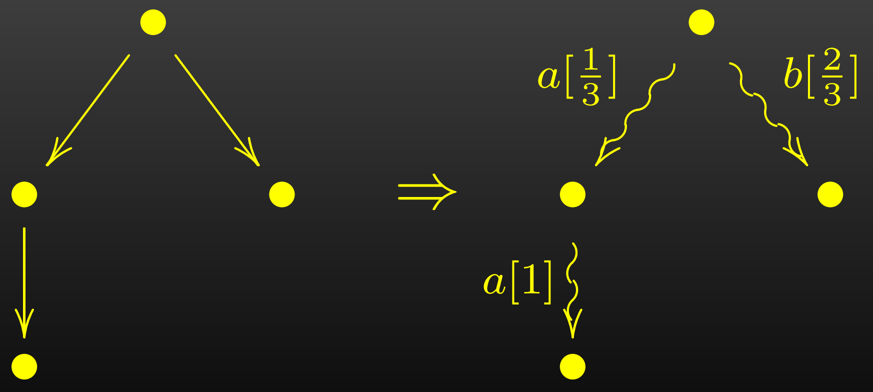

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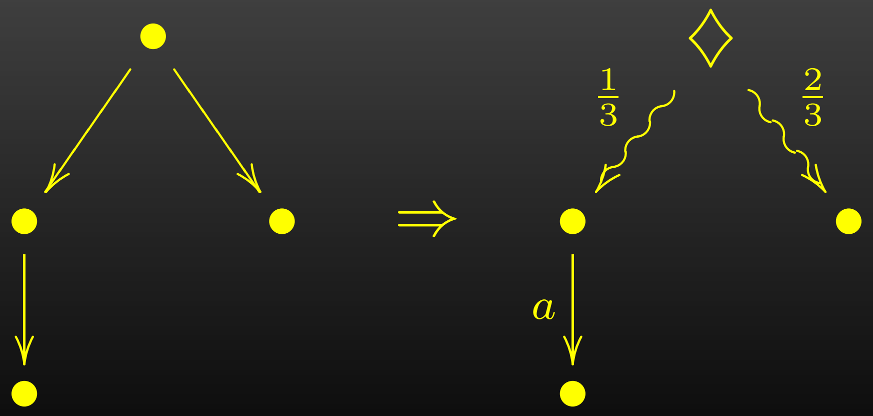

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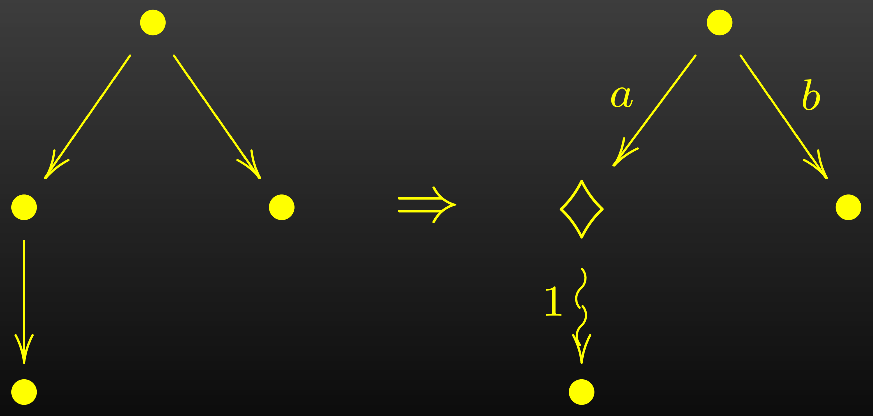

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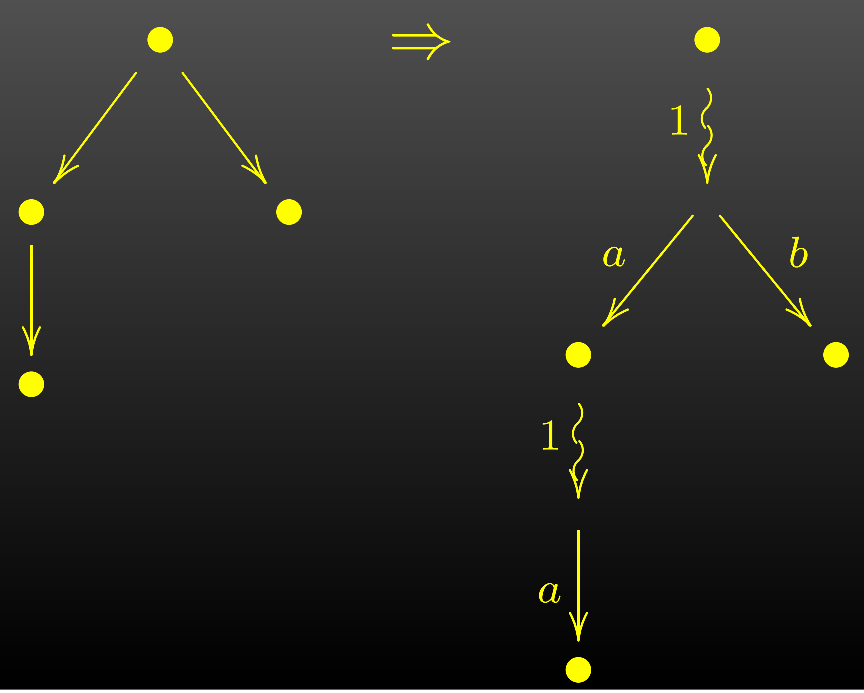

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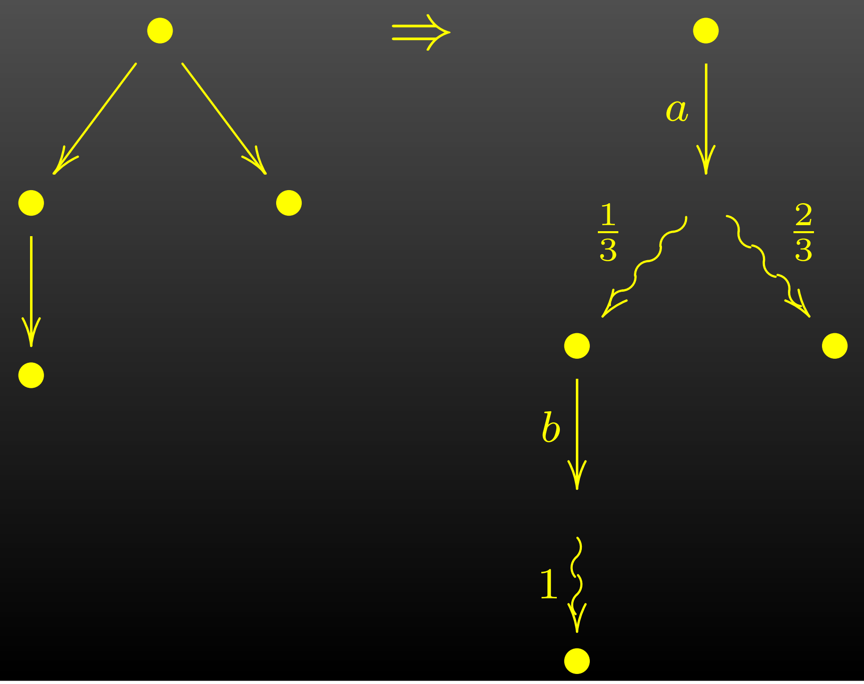

Coalgebras

are an elegant generalization of transition systems with

states + transitions

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

- based on category theory
- provide a uniform way of treating transition systems
- provide general notions and results e.g. a generic notion of bisimulation

Examples

A TS is a pair $\langle S, \alpha : S \to \mathcal{P}S \rangle$

!! coalgebra of the powerset functor $\mathcal{P}$

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An LTS is a pair $\langle S, \alpha : S \to \mathcal{P}S^A \rangle$

!!! coalgebra of the functor $\mathcal{P}^A$

Note: $\mathcal{P}^A \cong \mathcal{P}(A \times \_\,)$

More examples

Thanks to the probability distribution functor $\mathcal{D}$

$$\mathcal{D}S = \{\mu : S \to [0,1], \mu[S] = 1\}, \quad \mu[X] = \sum_{s \in X} \mu(x)$$

$$\mathcal{D}f : \mathcal{D}S \to \mathcal{D}T, \quad \mathcal{D}f(\mu)(t) = \mu[f^{-1}(\{t\})]$$

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the probabilistic systems are also coalgebras ... of functors built by the following syntax

$$\mathcal{F} ::= \_ \mid A \mid \mathcal{P} \mid \mathcal{D} \mid \mathcal{G} + \mathcal{H} \mid \mathcal{G} \times \mathcal{H} \mid \mathcal{G}^A \mid \mathcal{G} \circ \mathcal{H}$$

reactive, generative

evolve from LTS - functor $\mathcal{P}(A \times \_) \cong \mathcal{P}^A$

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note: in the probabilistic case

$$(\mathcal{D} + 1)^A \not\cong \mathcal{D}(A \times \_) + 1$$

Probabilistic system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
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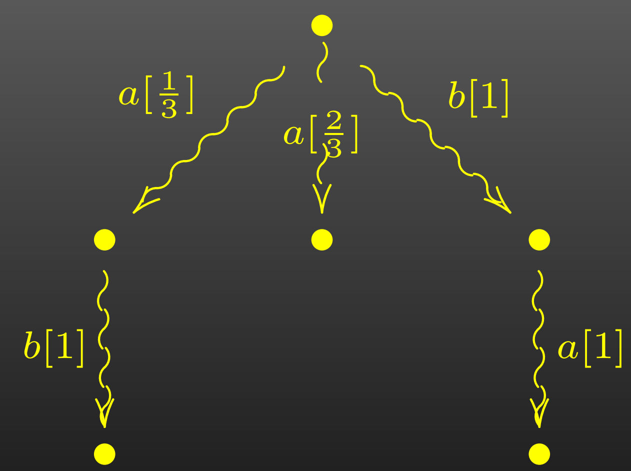

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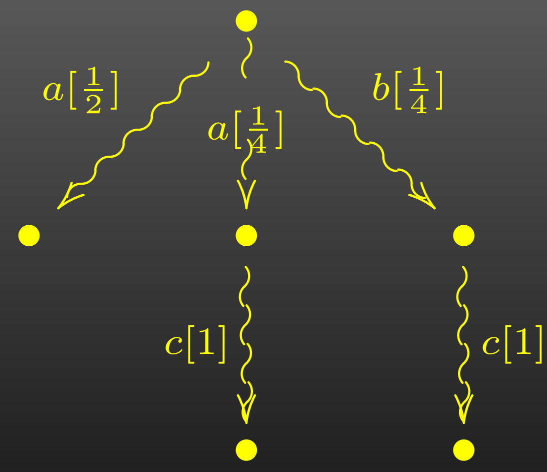

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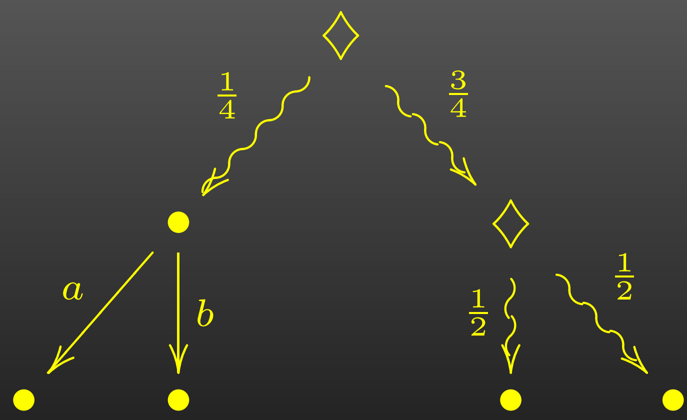

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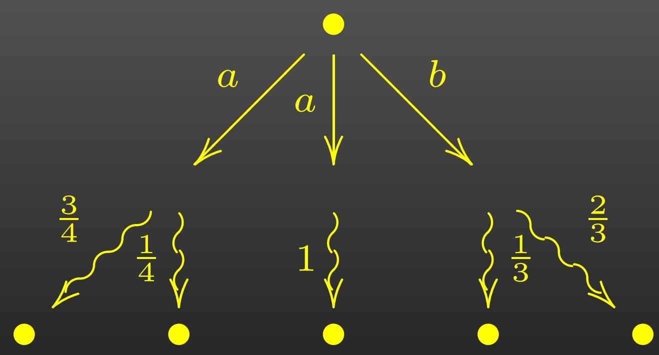

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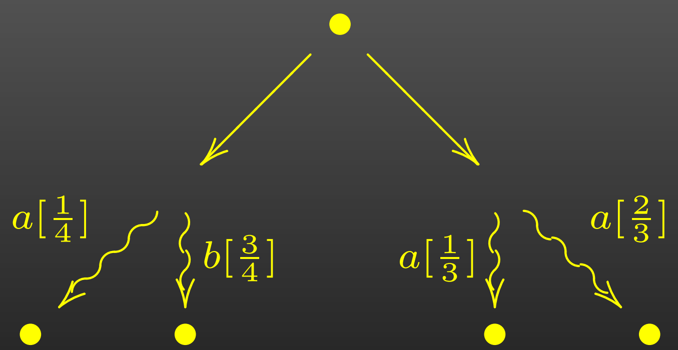
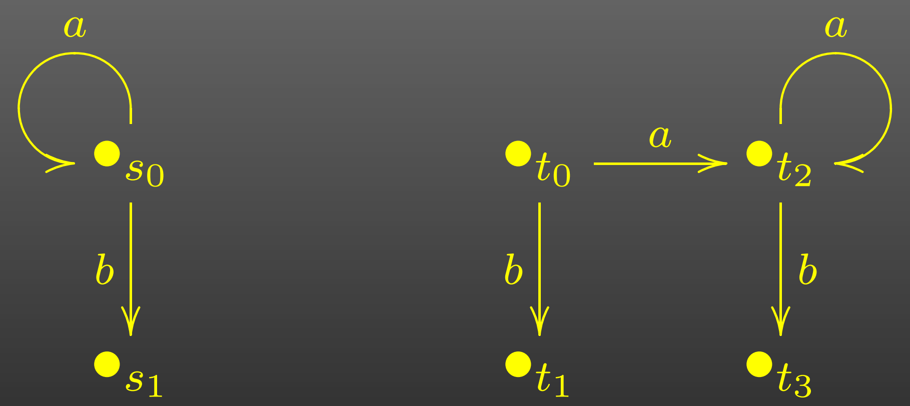


Bisimulation - LTS

Consider the LTS



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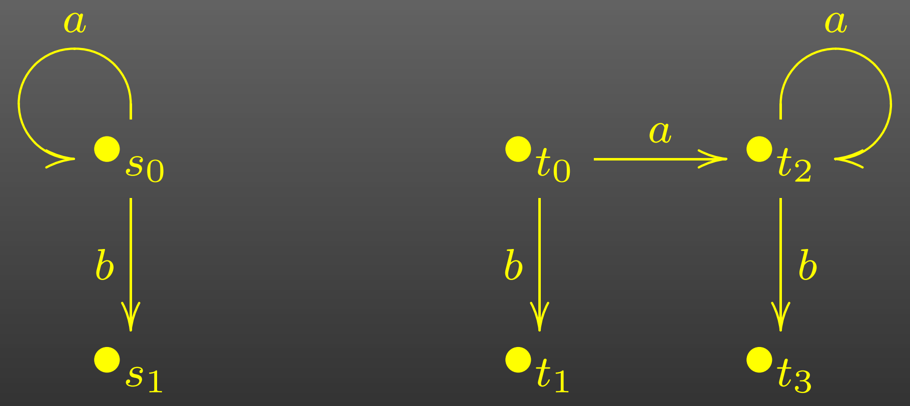

The states s_0 and t_0 are bisimilar since there is a bisimulation R relating them...

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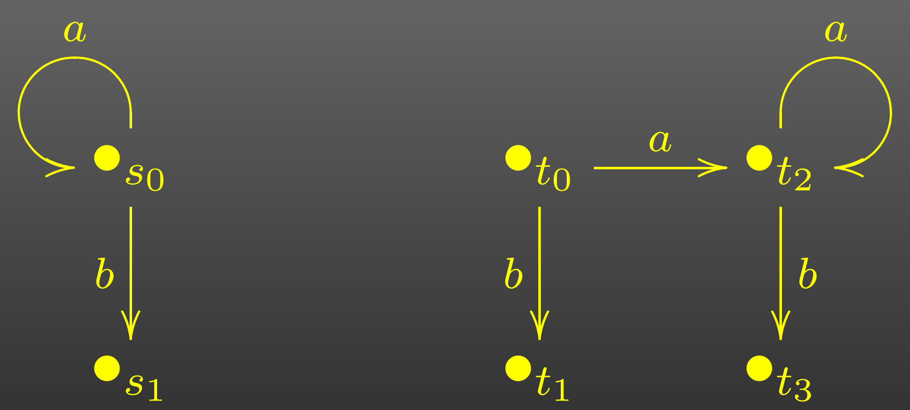


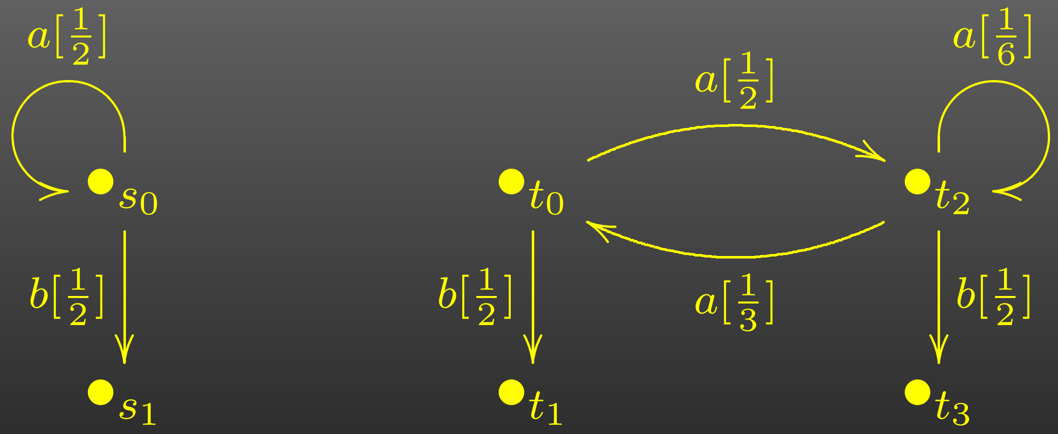
Transfer condition: $\langle s, t \rangle \in R \implies$

$$s \xrightarrow{a} s' \Rightarrow (\exists t')\ t \xrightarrow{a} t',\ \langle s', t' \rangle \in R,$$

$$t \xrightarrow{a} t' \Rightarrow (\exists s')\ s \xrightarrow{a} s',\ \langle s', t' \rangle \in R$$

Bisimulation - generative

Consider the generative systems



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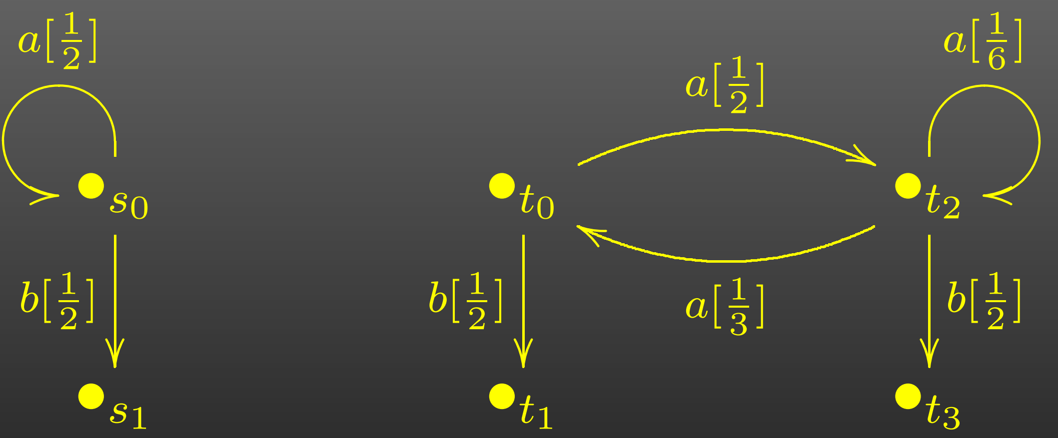
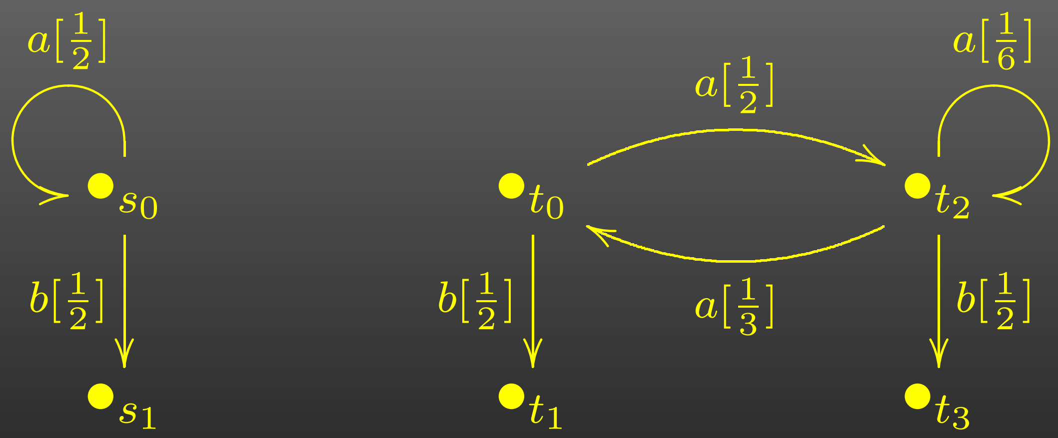


The states s_0 and t_0 are bisimilar, and so are s_0 and t_2, since there is a bisimulation R relating them...

Bisimulation - generative

Consider the generative systems

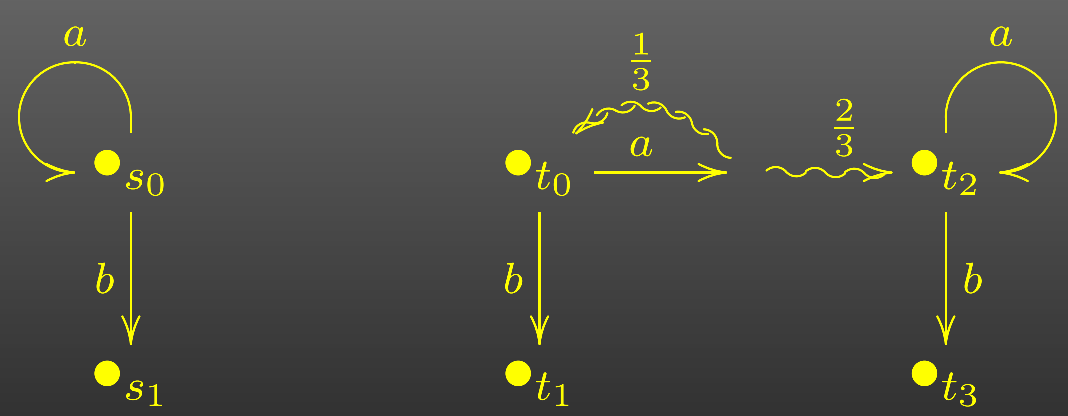


Transfer condition: $\langle s, t \rangle \in R \quad \Longrightarrow$

$$s \rightsquigarrow \mu \Rightarrow (\exists \mu')\ t \rightsquigarrow \mu',\ \mu \equiv_{R,A} \mu'$$

Bisimulation - simple Segala

Consider the simple Segala systems



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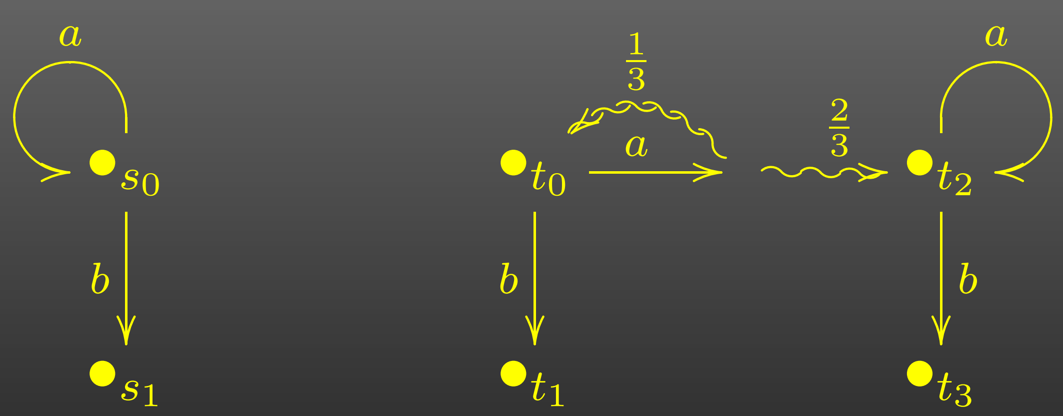

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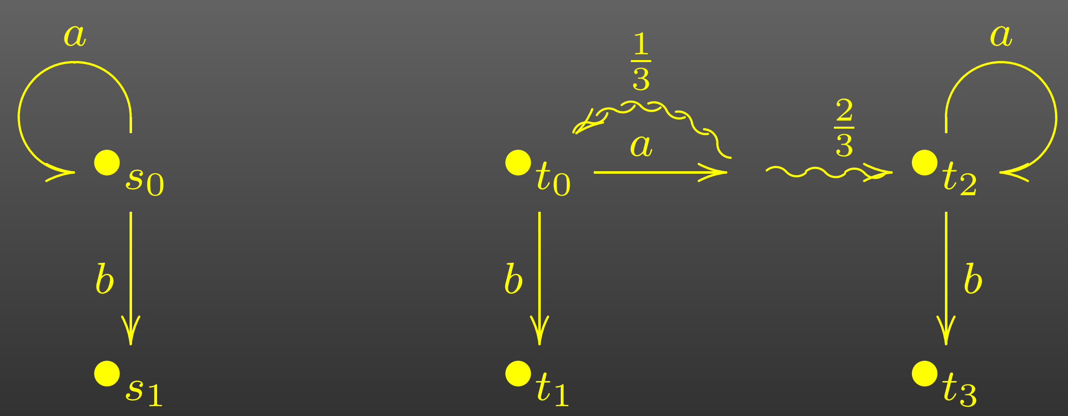

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Coalgebraic bisimulation

A bisimulation between

$$\langle S, \alpha : S \to \mathcal{F}S \rangle \text{ and } \langle T, \beta : S \to \mathcal{F}S \rangle$$

is $R \subseteq S \times T$ such that $\exists\, \gamma$:

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$$\begin{array}{ccccc}
S & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & T \\
\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \beta} \\
\mathcal{F}S & \xleftarrow[\mathcal{F}\pi_1]{} & \mathcal{F}R & \xrightarrow[\mathcal{F}\pi_2]{} & \mathcal{F}T
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$$\langle \alpha(s), \beta(t) \rangle \in \mathrm{Rel}(\mathcal{F})(R)$$

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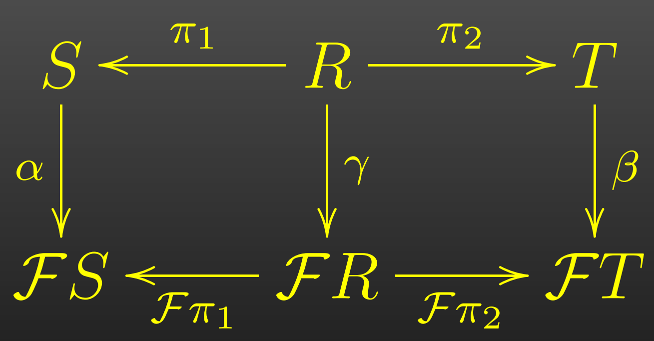
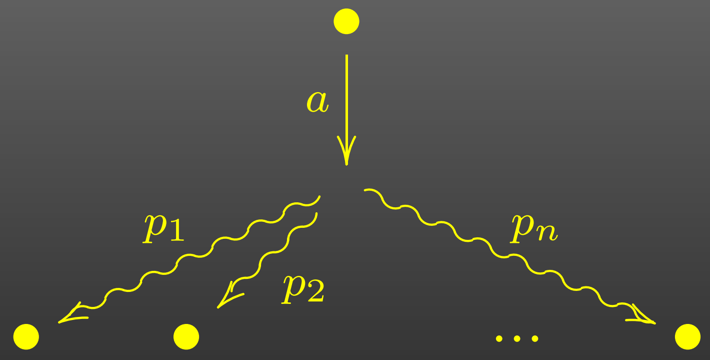


Theorem: Coalgebraic and concrete bisimilarity coincide !

Expressiveness

simple Segala system $\longrightarrow$ Segala system



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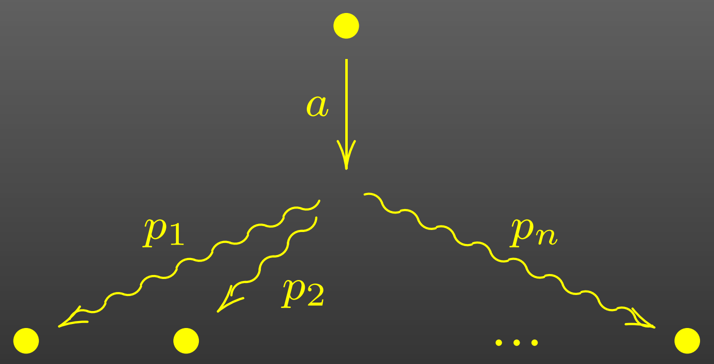

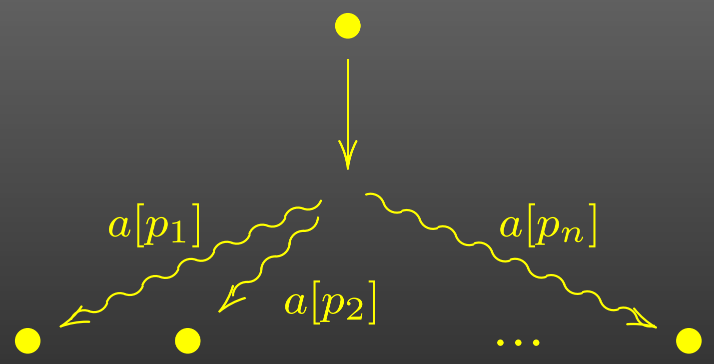

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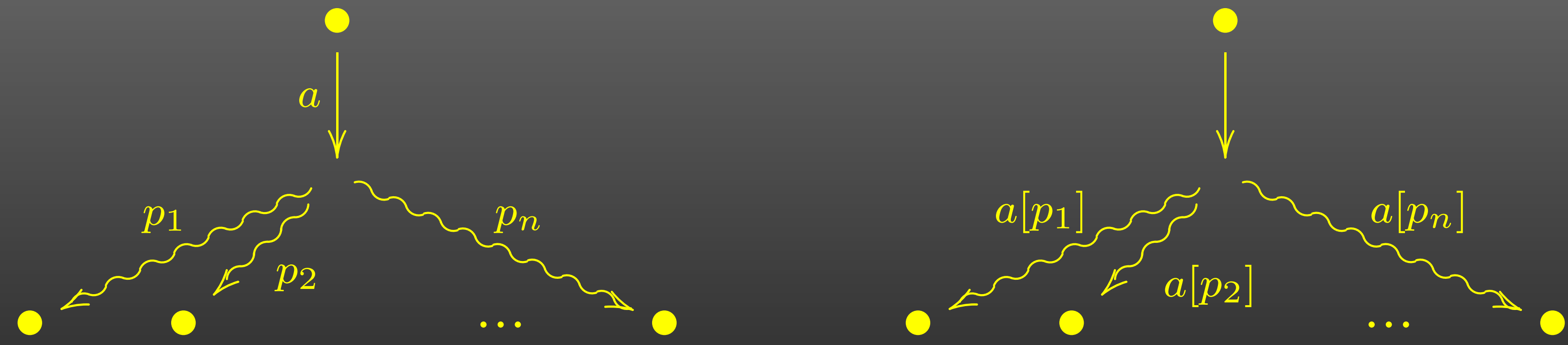

When do we consider one type of systems more expressive than another?

Comparison criterion

$\mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$

if there is a mapping $\langle S, \alpha : S \to \mathcal{F}S \rangle \overset{\mathcal{T}}{\mapsto} \langle S, \tilde{\alpha} : S \to \mathcal{G}S \rangle$ that preserves and reflects bisimilarity

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$$s_{\langle S,\alpha \rangle} \sim t_{\langle T,\beta \rangle} \iff s_{\mathcal{T}\langle S,\alpha \rangle} \sim t_{\mathcal{T}\langle T,\beta \rangle}$$

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Theorem: An injective natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ is sufficient for $\mathsf{Coalg}_{\mathcal{F}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}$

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Theorem: An injective natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ is sufficient for $\mathsf{Coalg}_{\mathcal{F}} \to \mathsf{Coalg}_{\mathcal{G}}$

proof via cocongruences - behavioral equivalence

Example

Indeed $\mathbf{SSeg} \to \mathbf{Seg}$ since $\mathcal{P}(A \times \mathcal{D}) \overset{\mathcal{P}\tau}{\Rightarrow} \mathcal{P}\mathcal{D}(A \times \_)$ is injective for

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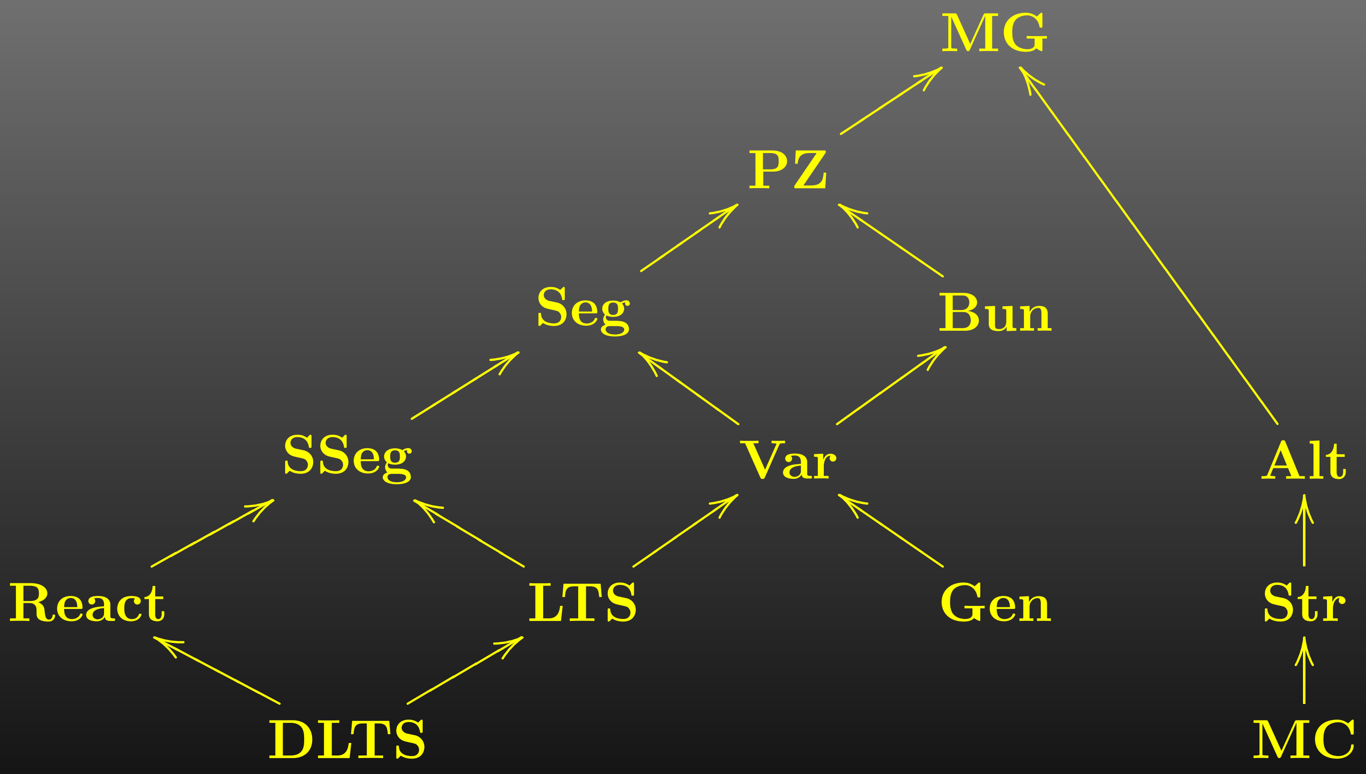
and δ_a is Dirac distribution for a

The hierarchy...



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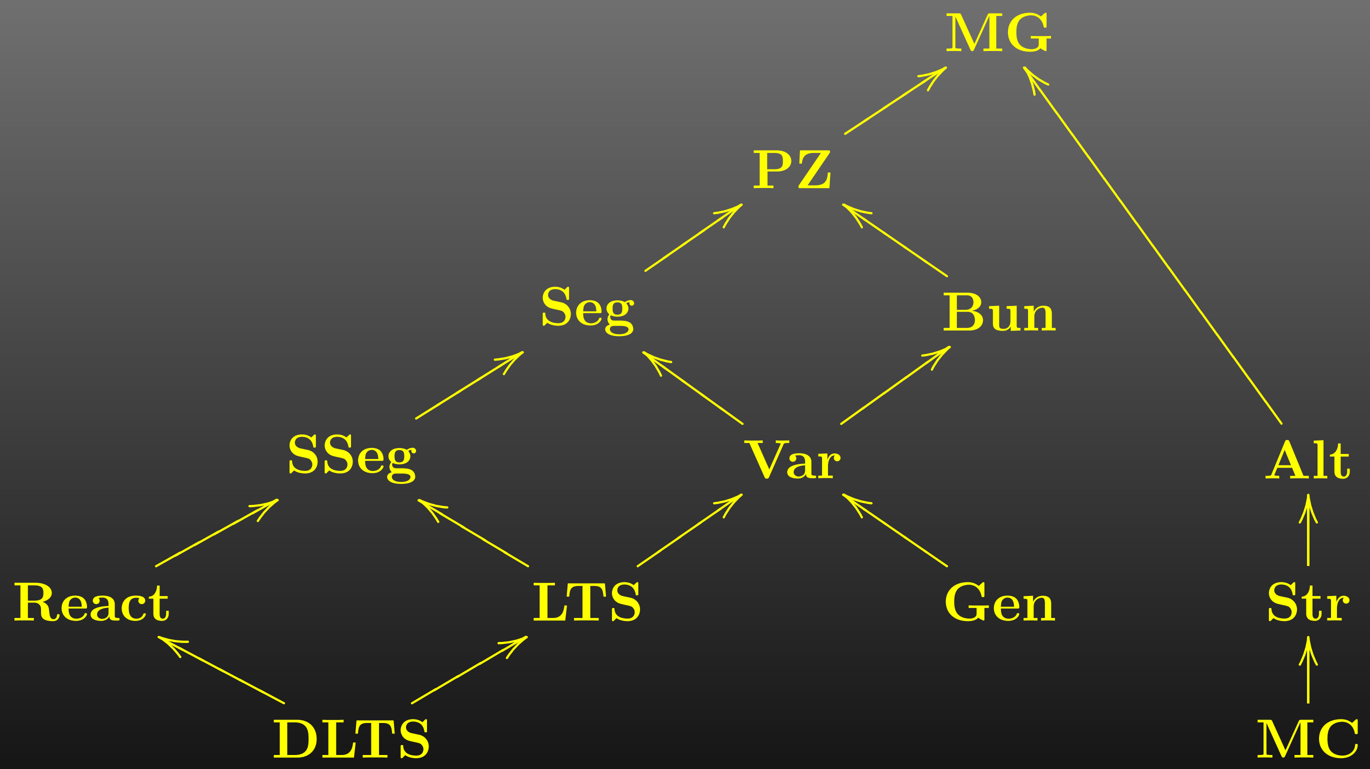


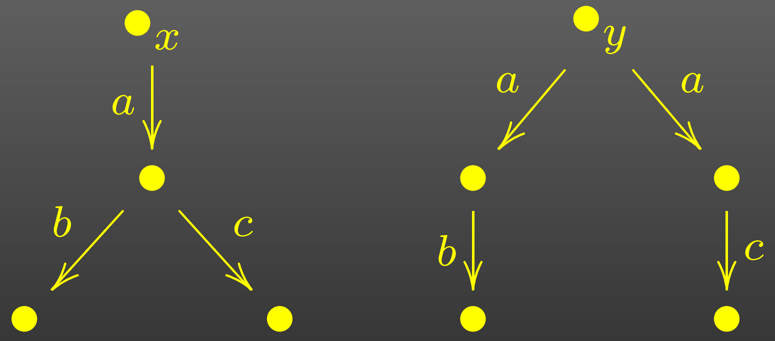
* Falk Bartels, AS, Erik de Vink

LT/BT spectrum

Bisimilarity is not the only semantics...

LT/BT spectrum

Are these non-deterministic systems equal ?



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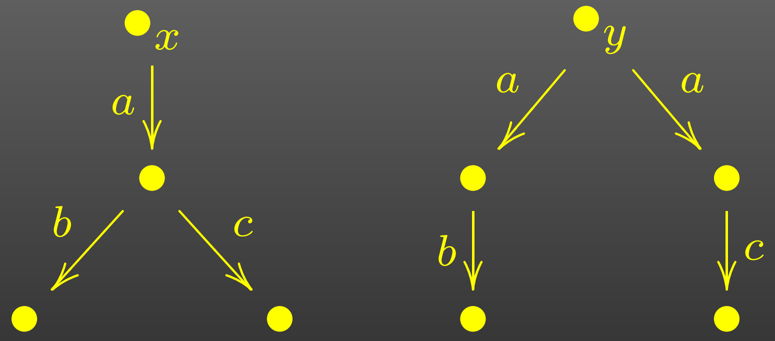
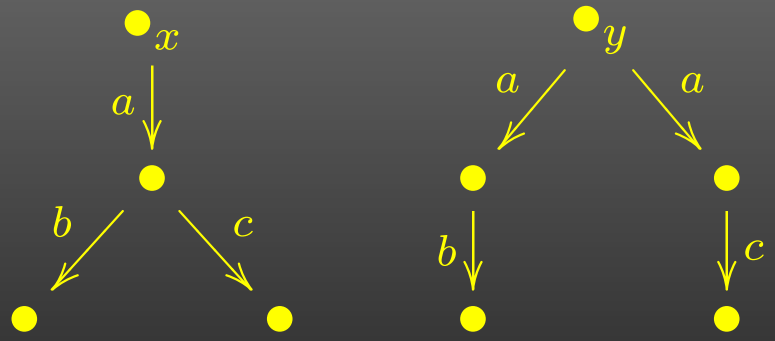


x and y are:

- different wrt. bisimilarity

LT/BT spectrum

Are these non-deterministic systems equal ?



x and y are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

Traces - LTS

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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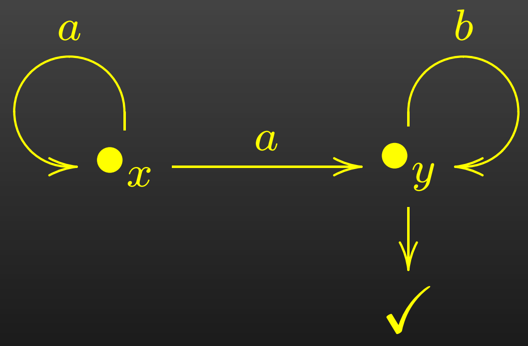

$$\mathrm{tr}(y) \;=\; b^*, \qquad \mathrm{tr}(x) \;=\; a^+ \cdot \mathrm{tr}(y) \;=\; a^+ \cdot b^*$$

Traces - generative

For generative probabilistic systems with ex. termination

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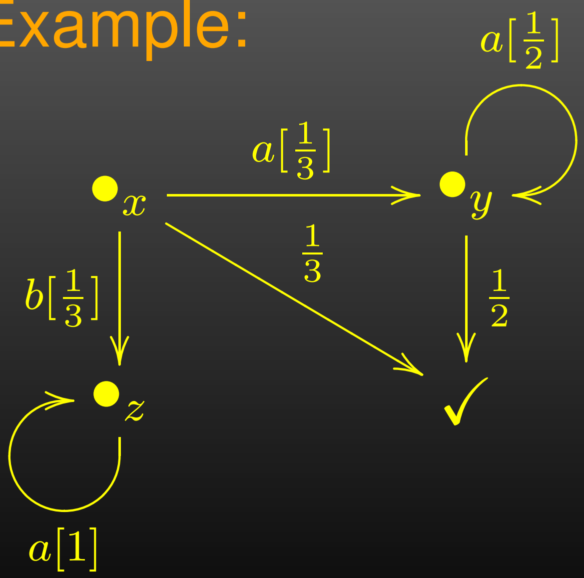

$$\mathrm{tr}(x): \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3} \cdot \frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\cdots$$

Trace of a coalgebra ?

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main idea: coinduction in a Kleisli category

Coinduction

$$\begin{array}{ccc}
\mathcal{F}X & \overset{\mathcal{F}(\mathrm{beh})}{- - - - \dashrightarrow} & \mathcal{F}Z \\
\uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \cong} \\
X & \underset{\mathrm{beh}}{- - - - \dashrightarrow} & Z
\end{array}$$

system final coalgebra

Coinduction

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\begin{array}{ccc}
\mathcal{F}X & \xrightarrow{\mathcal{F}(\mathrm{beh})} & \mathcal{F}Z \\
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\end{array}
$$

system final coalgebra

- finality = $\exists!$(morphism for any $\mathcal{F}$- coalgebra)
- beh gives the behavior of the system
- this yields final coalgebra semantics

Coinduction

$$\begin{array}{ccc} \mathcal{F}X & \xrightarrow{\mathcal{F}(\text{beh})} & \mathcal{F}Z \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle \cong} \\ X & \xrightarrow[\text{beh}]{} & Z \end{array}$$

system final coalgebra

- f.c.s. in **Sets** = bisimilarity
- f.c.s. in a Kleisli category = trace semantics

Types of systems

For trace semantics systems are suitably modelled as coalgebras in $\mathbf{Sets}$

$$X \xrightarrow{c} \mathcal{T}\,\mathcal{F}X$$

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needed: distributive law $\mathcal{F}\mathcal{T} \Rightarrow \mathcal{T}\mathcal{F}$

Distributive law

is needed since branching is irrelevant:

LTS with $\checkmark$ - $\mathcal{PF} = \mathcal{P}(1 + A \times \_)$

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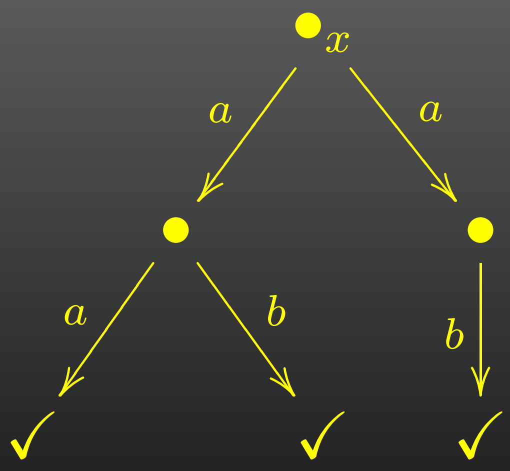

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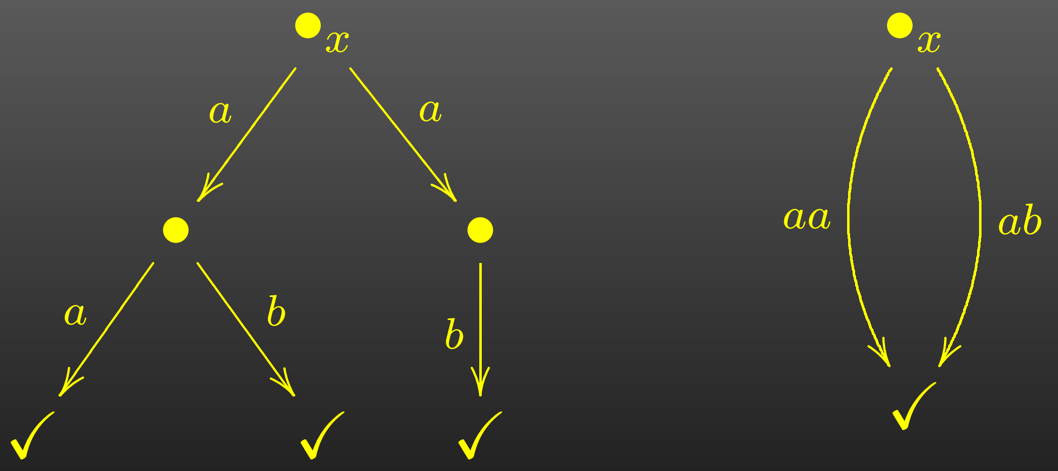

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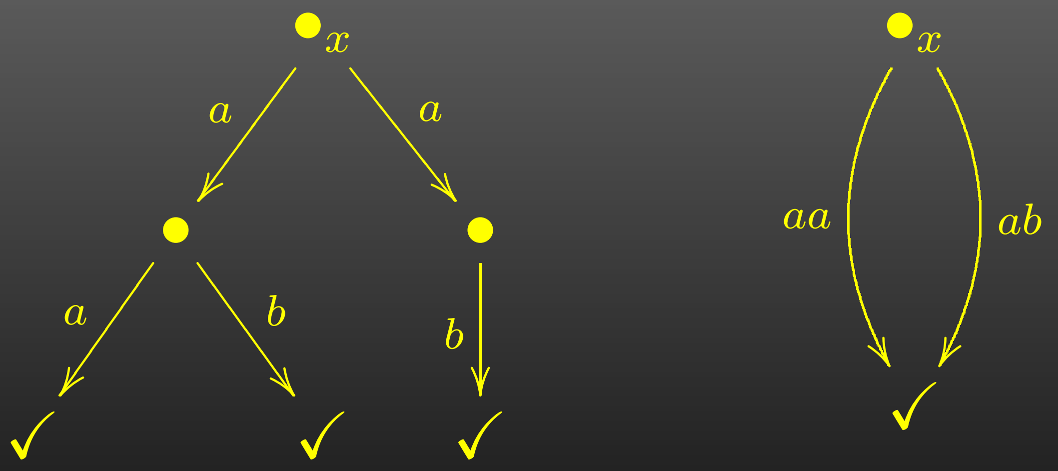

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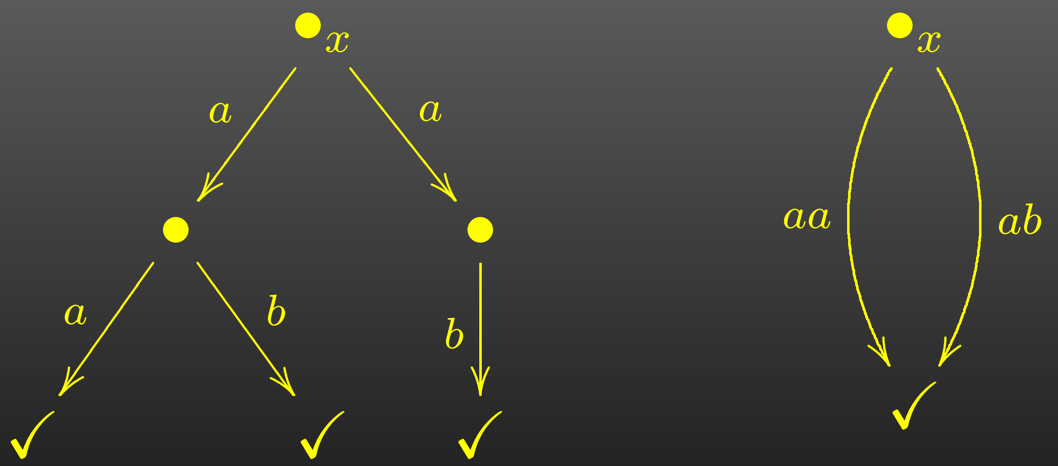

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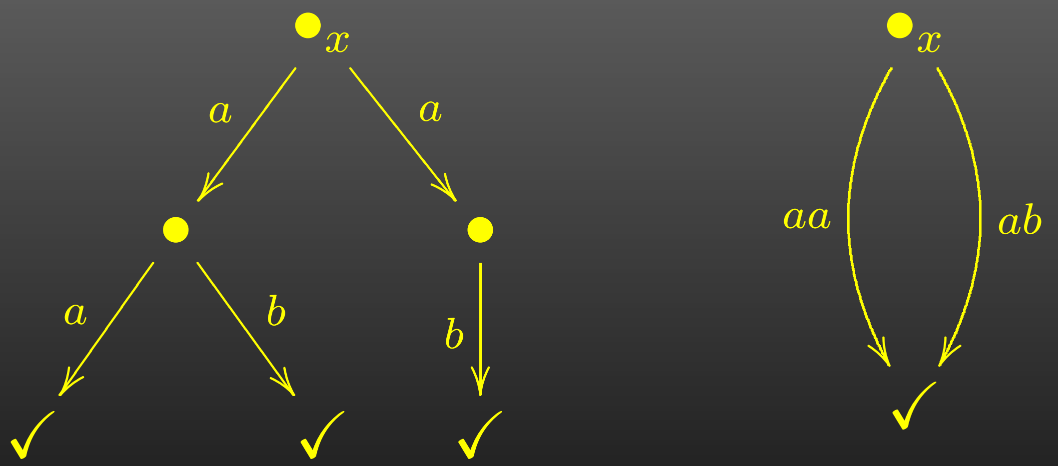

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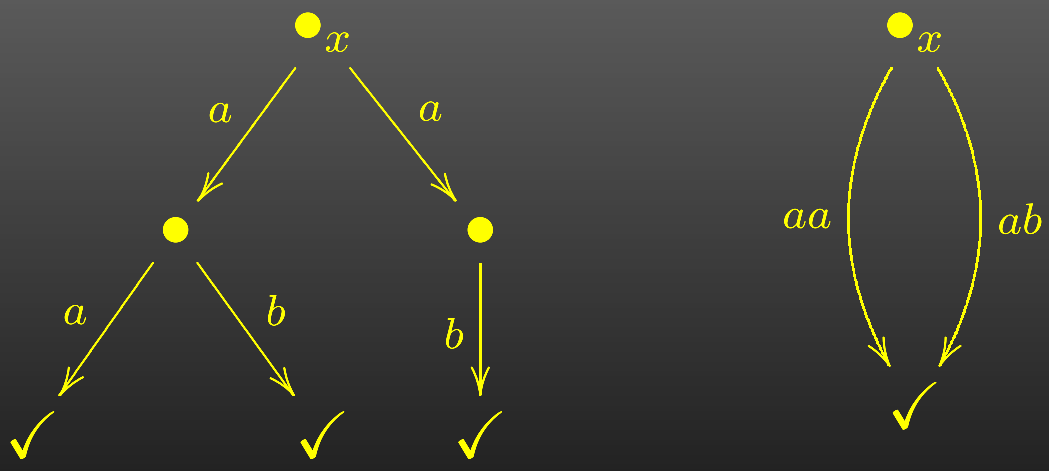

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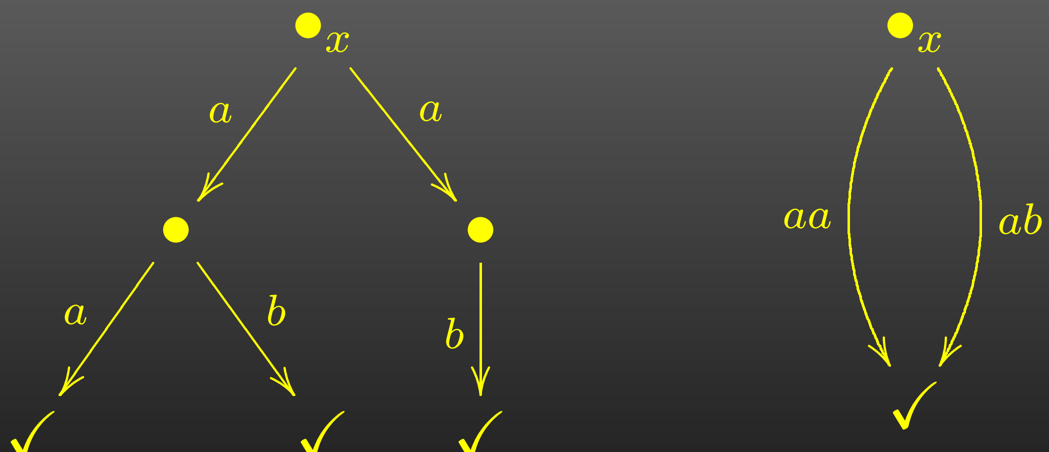

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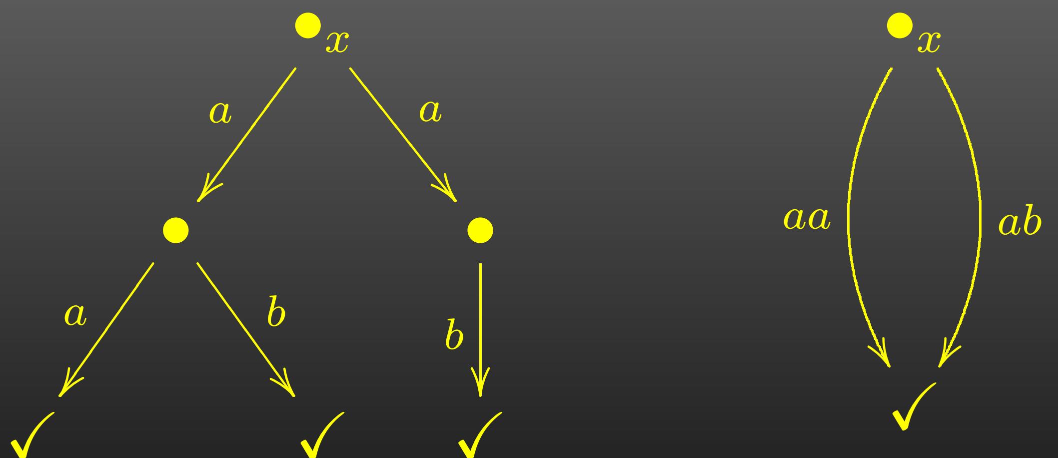

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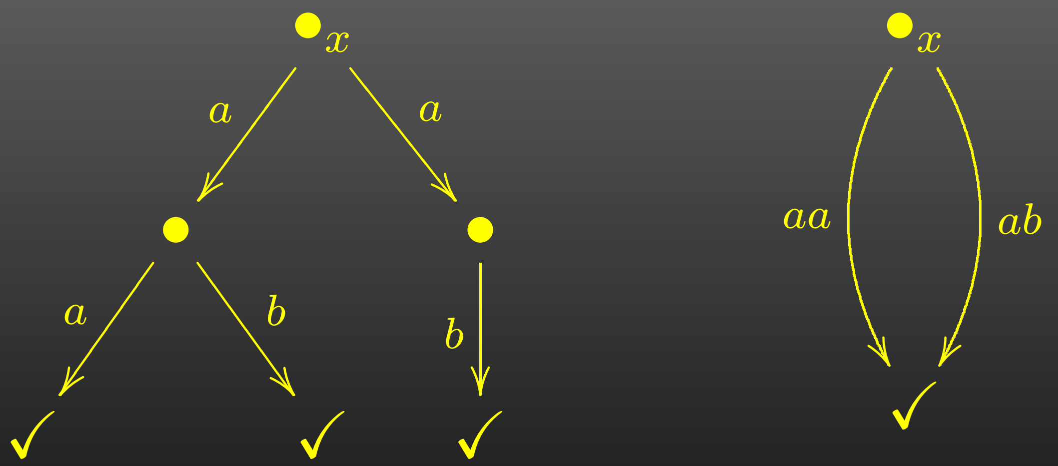

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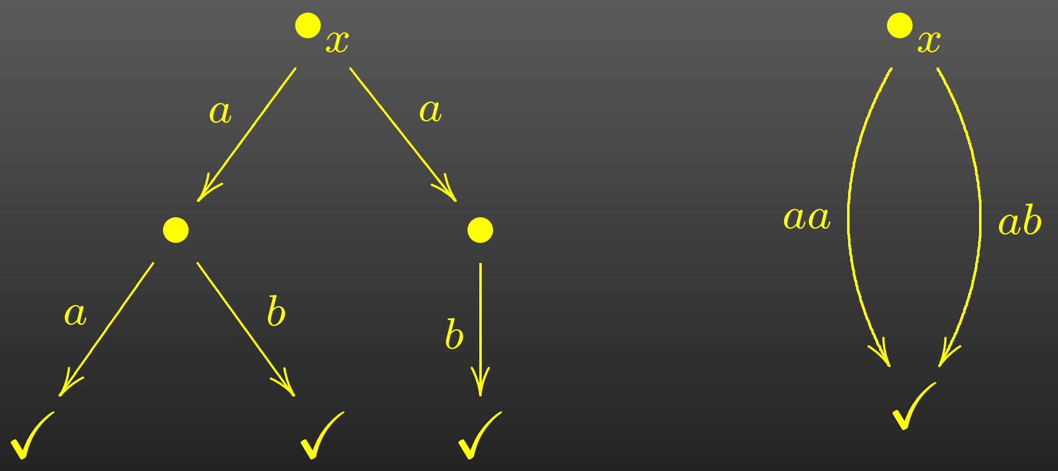

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- objects - sets
- arrows - $X \xrightarrow{f} Y$ are functions $f : X \to \mathcal{T}Y$

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Main theorem - traces

If ♣, then

$$\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})} I \xrightarrow[\cong]{\eta_I \circ \alpha} I \qquad \text{is initial} \qquad\qquad \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})} I \xleftarrow[\cong]{\eta_{\mathcal{F}I} \circ \alpha^{-1}} I \qquad \text{is final} \qquad \text{in } \mathcal{K}\ell(\mathcal{T})$$

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[$\alpha : \mathcal{F}I \stackrel{\cong}{\to} I$ denotes the initial $\mathcal{F}$-algebra in $\mathbf{Sets}$]

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proof: via limit-colimit coincidence Smyth&Plotkin ’82

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- A monad $\mathcal{T}$ s.t. $\mathcal{K}\ell(\mathcal{T})$ is $\mathbf{DCpo}_\perp$-enriched
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- $\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}$ should be locally monotone

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$\exists!$ finite trace map $\mathrm{tr}_c : X \to \mathcal{T}I$ in $\mathbf{Sets}$:

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$$\begin{array}{ccc}
\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})} X & \overset{\mathcal{F}_{\mathcal{K}\ell(\mathcal{T})}(\mathrm{tr}_c)}{- - - - - \to} & \mathcal{F}_{\mathcal{K}\ell(\mathcal{T})} I \\
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  - * for generative systems with explicit termination $\mathcal{D}(1 + A \times \_\,)$

Note: Initial $1 + A \times \_$ - algebra is

$$A^* \xrightarrow[\cong]{[\text{nil},\text{cons}]} 1 + A \times A^*$$

Finite traces - LTS with ✓

the finality diagram in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{array}{ccc}
\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}X & \xrightarrow{\mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}(\mathrm{tr}_c)} & \mathcal{F}_{\mathcal{K}\ell(\mathcal{P})}A^* \\
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amounts to

- $\langle\rangle \in \mathrm{tr}_c(x) \iff \checkmark \in c(x)$
- $a \cdot w \in \mathrm{tr}_c(x) \iff (\exists x')\langle a, x'\rangle \in c(x),\ w \in \mathrm{tr}_c(x')$

Finite traces - generative ✓

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amounts to $\mathrm{tr}_c(x)$:

- $\langle\rangle \mapsto c(x)(\checkmark)$
- $a \cdot w \mapsto \sum_{y \in X} c(x)(a, y) \cdot c(y)(w)$

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