Congruences of convex algebras
(for CA, PCA, TCA, f.p. = f.g.)

Ana Sokolova
Harald Woracek

University of Salzburg
TU Vienna

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\[ \text{CA} = \text{convex algebras} \]

\begin{itemize}
  \item Variety of algebras of type
  \[ T_{ca} := \left\{ (p_i)_{i=1}^n \in \mathbb{R}^n \mid n \in \mathbb{N}^+, p_1, \ldots, p_n \geq 0, \sum_{i=1}^n p_i = 1 \right\}. \]
  \item two axioms
    \[ f(\delta_{ij})_{i=1}^n (x_1, \ldots, x_n) = x_j, \quad n \in \mathbb{N}^+, \ j = 1, \ldots, n, \]
    \[ f(p_i)_{i=1}^n \left( f(p_{1j})_{j=1}^m (x_1, \ldots, x_m), \ldots, f(p_{nj})_{j=1}^m (x_1, \ldots, x_m) \right) = \]
    \[ = f(\sum_{i=1}^n p_i p_{ij})_{j=1}^m (x_1, \ldots, x_m) \]
\end{itemize}
CA = convex algebras

**Variety of algebras of type**

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Ana Sokolova University of Salzburg

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CA = convex algebras

Variety of algebras of type

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CA = convex algebras

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CA \supseteq PCA \supseteq TCA
CA = convex algebras

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also "⊆"

CA ⊇ PCA ⊇ TCA

PCA - ≤

TCA
CA = convex algebras

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CA \supseteq PCA \supseteq TCA

(P)CA - EM algebras for the (sub)distribution monad
CA = convex algebras

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Also \( \subseteq \)

CA \supseteq PCA \supseteq TCA

(P)CA - EM algebras for the (sub)distribution monad
Finitely generated, finitely presentable
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Finitely generated = quotients of free finitely generated ones
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Finitely presentable = quotients of free finitely generated ones under finitely generated congruences
Finitely generated, finitely presentable

Finitely generated = quotients of free finitely generated ones

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smallest congruence containing a finite set of pairs
Finitely generated, finitely presentable

Finitely generated = quotients of free finitely generated ones

Finitely presentable = quotients of free finitely generated ones under finitely generated congruences

for CA, PCA, and TCA all congruences are f.g., hence f.p. = f.g.
Free CA, PCA, TCA

CA

(n-1)-simplex in \( \mathbb{R}^n \)

PCA

n-simplex in \( \mathbb{R}^n \)

TCA

n-octahedron in \( \mathbb{R}^n \)
Our aim in this paper is to achieve full understanding of the structure of finitely generated positive convex or totally convex algebras. Consequently, knowing all congruences of the free algebras is given by the standard (4.3 and 4.4 below, where we describe the congruences on any finitely generated algebras in the monad [Do06, Do08].

Historically, work on convex algebras can be traced back (at least) to the Eilenberg-Moore category of Eilenberg-Moore algebras of the subprobability distribution monad arising from the above mentioned concrete examples, the simplex (2.1) and the regular octahedron with center at the origin. Besides its obvious intrinsic interest, our motivation to investigate finitely generated algebras started in [PR84], where they were realized to be total convex algebras in a category of Banach spaces (with linear contractions) to be used to connect with the axiomatization of trace semantics for probabilistic systems.

Sets of probability subdistributions (1) The projection axiom: Polytopes can be defined in several equivalent ways. The definition is equivalent to understand all finitely generated algebras in the category of Eilenberg-Moore algebras of the subprobability distribution monad (2.2), are of particular interest in the present paper, let us recall the definitions.

Expression (2.1) is a polytope if and only if its extremal points are finite. Note that if $\delta_{ij}$ denotes the Kronecker-delta ($\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise), then $K_n$ is a polytope and $Y_n$ is a polytope in the edge of the $n$-octahedron. The mentioned concrete examples, the simplex (2.1) and the octahedron (2.2), are of particular interest in the present context. The set of all convex, positive and totally convex algebras and their induced equational classes were extensively studied, mainly focussing on the convex, positive meet of positive convex algebras was given shortly after in [Grö03].

The connection with the axiomatization of trace semantics for probabilistic systems is presented in the next lemma. The fact that this definition is equivalent to the projection axiom will be given in the next section (Remark 2.11). The proof of the lemma is, in essence, a simple application of the Kre˘ ın-Milman theorem; we skip the details.

Remark 2.11. Clearly, knowing all congruences of the free algebras gives all finitely generated algebras in the category of Eilenberg-Moore algebras of the subprobability distribution monad. Moreover, knowing all congruences of the free algebras gives all finitely generated algebras in the category of Eilenberg-Moore algebras of the subprobability distribution monad.
Congruences on polytopes

- $K \subseteq \mathbb{R}^n$ is a polytope if $K = \text{co } Y$ for a finite set $Y$
- k is convex, compact, with finitely many extremal points $K = \text{co } (\text{ext } K)$
- Every polytope is a CA/PCA
- An equivalence on $K$ is a CA/PCA congruence iff it is a convex set (in $\mathbb{R}^n \times \mathbb{R}^n$)

$\text{co } Y := \{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in [0, 1], \sum_{y \in Y} \lambda_y = 1 \}$

Ana Sokolova  University of Salzburg
**Congruences on polytopes**

- \( K \subseteq \mathbb{R}^n \) is a polytope if \( K = \text{co} \ Y \) for a finite set \( Y \)
equivalently
- \( K \) is convex, compact, with finitely many extremal points
- \( K = \text{co} \ (\text{ext} \ K) \)

Every polytope is a CA/PCA

An equivalence on \( K \) is a CA/PCA congruence iff it is a convex set (in \( \mathbb{R}^n \times \mathbb{R}^n \))

How do they look like?

\[
\text{co} \ Y := \left\{ \sum_{y \in Y} \lambda_y y \mid \lambda_y \in [0, 1], \sum_{y \in Y} \lambda_y = 1 \right\}
\]
Example

$Y = \{0,1\}, K = \text{co} Y$ in $\mathbb{R}$

Let $K = [0,1]$.

$K$ has exactly 5 congruences:

1. $\Theta_1 = \Delta$
2. $\Theta_2 = \{(0,0), (1,1)\} \cup (0,1) \times (0,1)$
3. $\Theta_3 = \{(0,0)\} \cup (0,1] \times (0,1]$  (all finitely generated)
4. $\Theta_4 = [0,1) \times [0,1) \cup \{(1,1)\}$, and
5. $\Theta_5 = [0,1] \times [0,1]$. 
Four important `guys`

$K$ - polytope

$\theta$ - CA/PCA congruence on $K$
Four important `guys'

K - polytope

\( \Theta \) - CA/PCA congruence on K

\[ V_K := \mathcal{P}(\text{ext } K) \setminus \{\emptyset\} \]

join-semilattice
of vertices
**Four important `guys`**

- **$K$ - polytope**
- **$\Theta$ - CA/PCA congruence on $K$**

\[
V_K := \mathcal{P}(\text{ext } K) \setminus \{\emptyset\}
\]

join-semilattice of vertices

\[
\varphi_\Theta(Y) = \text{span} \{x_2 - x_1 \mid x_1, x_2 \in \text{co } Y, x_1 \Theta x_2\}, \quad Y \in V_K.
\]

interior of the convex hull

$\varphi_\Theta : V_K \to \text{Sub } \mathbb{R}^n$
Four important `guys` 

$K$ - polytope

$\Theta$ - CA/PCA congruence on $K$

$V_K := \mathcal{P}(\text{ext } K) \setminus \{\emptyset\}$

$\varphi_\Theta(Y) = \text{span}\{x_2 - x_1 \mid x_1, x_2 \in \text{co } Y, x_1 \Theta x_2\}$, $Y \in V_K$

$\{Y_1, Y_2\} \in E_\Theta \iff \Theta \cap (\text{co } Y_1 \times \text{co } Y_2) \neq \emptyset$, $Y_1, Y_2 \in V_K$

$\varphi_\Theta : V_K \rightarrow \text{Sub } \mathbb{R}^n$

graph $\mathcal{G}_\Theta$

interior of the convex hull
Four important `guys`

$K$ - polytope

$\Theta$ - CA/PCA congruence on $K$

$V_K := \mathcal{P}(\text{ext } K) \setminus \{\emptyset\}$

$\varphi_\Theta(Y) = \text{span} \left\{ x_2 - x_1 \mid x_1, x_2 \in \partial Y, x_1 \Theta x_2 \right\}, \quad Y \in V_K.$

$\{Y_1, Y_2\} \in E_\Theta \iff \Theta \cap (\partial Y_1 \times \partial Y_2) \neq \emptyset, \quad Y_1, Y_2 \in V_K.$

$\approx_\Theta$

connectivity equivalence

$\varphi_\Theta : V_K \rightarrow \text{Sub } \mathbb{R}^n$

join-semilattice of vertices

graph $G_\Theta$

interior of the convex hull
(i) The map $\varphi_\Theta$ is monotone.

(ii) Let $C$ be a component of the graph $G_\Theta$. Then $C$ contains a largest element with respect to inclusion. Denoting this largest element by $Y(C)$, we have $\{Y, Y(C)\} \in E_\Theta$, $Y \in C$.

(iii) The relation $\approx_\Theta$ is a congruence of the join-semilattice $V_K$.

(iv) Let $C$ be a component of $G_\Theta$ and $Y(C)$ its largest element. Then

$$\varphi_\Theta(Y) = \varphi_\Theta(Y(C)) \cap \text{dir } Y \quad \text{for } Y \in C,$$

$$[\check{\circ} Y + \varphi_\Theta(Y(C))] \cap \check{\circ} Y(C) \neq \emptyset \quad \text{for } Y \in C.$$ 

Set

$$Z(C) := \bigcup_{Y \in C} \check{\circ} Y,$$

then the congruence $\Theta$ can be recovered from $\varphi_\Theta$ and $G_\Theta$ as

$$\Theta = \bigcup_{C \text{ component of } G_\Theta} \{(x_1, x_2) \in Z(C) \times Z(C) : x_2 - x_1 \in \varphi_\Theta(Y(C))\}.$$
Let $K$ be a polytope in $\mathbb{R}^n$. Let $\sim$ be a congruence relation of the join-semilattice $V_K$ with the property that each congruence class $C$ of $\sim$ contains a largest element, say $Y(C)$. Moreover, let

$$
\varphi: \{Y(C) \mid C \text{ class of } \sim\} \to \text{Sub} \mathbb{R}^n
$$

be a monotone map such that, for each class $C$ of $\sim$,

$$
\varphi(Y(C)) \subseteq \text{dir} Y(C), \quad \left[\text{co } Y + \varphi(Y(C))\right] \cap \text{co} Y(C) \neq \emptyset \quad \text{for } Y \in C.
$$

Then there exists a unique congruence $\Theta \in \text{Con}_{CA} K$ such that

$$
\approx_\Theta = \sim, \quad \varphi_\Theta(Y(C)) = \varphi(Y(C)) \quad \text{for } C \text{ a class of } \sim.
$$

This congruence $\Theta$ can be computed from $\sim$ and $\varphi$ by means of the formula

$$
\Theta = \bigcup_{C \text{ class of } \sim} \left\{(x_1, x_2) \in Z(C) \times Z(C) : x_2 - x_1 \in \varphi(Y(C))\right\},
$$

where again $Z(C) := \bigcup_{Y \in C} \text{co} Y$. Its associated function $\varphi_\Theta$ is given as

$$
\varphi_\Theta(Y) = \varphi(Y(C)) \cap \text{dir} Y \quad \text{for } Y \in C,
$$

and the set of edges $E_\Theta$ of its associated graph $G_\Theta$ is given as

$$
\{Y_1, Y_2\} \in E_\Theta \iff \left(Y_1 \sim Y_2 \land \left[\text{co } Y_1 + \varphi([Y_1]_\sim)\right] \cap \text{co} Y_2 \neq \emptyset\right)
$$

where $[Y_1]_\sim$ denotes the equivalence class of $Y_1$. 

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Order theorem

\[ \Theta_1 \subseteq \Theta_2 \iff (E_{\Theta_1} \subseteq E_{\Theta_2} \land \varphi_{\Theta_1} \leq \varphi_{\Theta_2}) \]

And so we fully know the congruence lattice on \( K \)
And all are finitely generated

as congruences

But there are always some that are not finitely generated as subalgebras of $K \times K$

Hence in CA,PCA,TCA f.g. algebras are not closed under kernel pairs!
Conclusions

We saw:

* For CA, PCA, TCA, f.p. = f.g.
* We actually know all congruences on CA, PCA, TCA
* And many other things...
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Conclusions

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* For CA, PCA, TCA, \text{f.p.} = \text{f.g.}

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* And many other things...

\text{THANK YOU}