Convexity Meets Coalgebra in Probabilistic Systems

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Coalgebra Now @ FloC 2018

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

states

object in the base category ***C***

behaviour type

functor on the base category ***C***

form a category too

$$\mathbf{CoAlg}_{\mathbf{C}}(F)$$

generic notion of behavioural equivalence

$$\approx$$

Probabilistic systems coalgebraically

a monad

Probability distribution functor on $\mathbf{Sets}$

$$\mathcal{D}X = \{\xi \colon X \to [0,1] \mid \sum_{x \in X} \xi(x) = 1, \mathrm{supp}(\xi) \text{ is finite}\}$$

for $f\colon X \to Y$ we have $\mathcal{D}f\colon \mathcal{D}X \to \mathcal{D}Y$ by

$$\mathcal{D}f(\xi)(y) = \sum_{x \in f^{-1}(y)} \xi(x) = \xi(f^{-1}(y))$$

Probabilistic systems coalgebraically

a monad !

Probability distribution monad on $\mathbf{Sets}$

$$(\mathcal{D}X, \eta, \mu)$$

unit

multiplication

$$\eta_X : X \to \mathcal{D}X$$

$$\mu_X : \mathcal{D}\mathcal{D}X \to \mathcal{D}X$$

$$\eta_X(x) = (x \mapsto 1)$$

$$\mu_X((\xi_i \mapsto p_i)) = \sum p_i \xi_i$$

Dirac distribution

convex combination

Probabilistic systems coalgebraically

a monad !

Probability distribution monad on $\mathbf{Sets}$

$$(\mathcal{D}X, \eta, \mu)$$

unit

$$\eta_X : X \to \mathcal{D}X$$

multiplication

$$\mu_X : \mathcal{D}\mathcal{D}X \to \mathcal{D}X$$

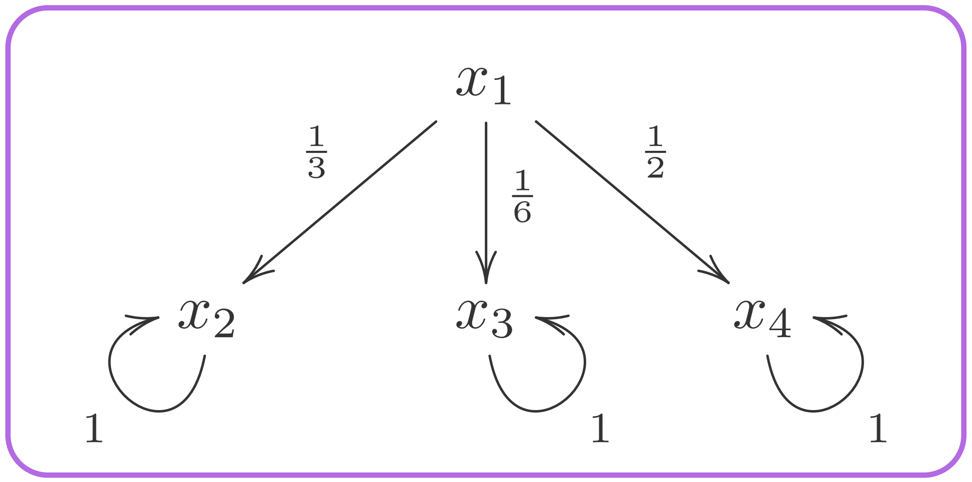
$$\eta_X(x) = 1x$$

$$\mu_X(\sum p_i \xi_i) = \sum p_i \xi_i$$

Examples

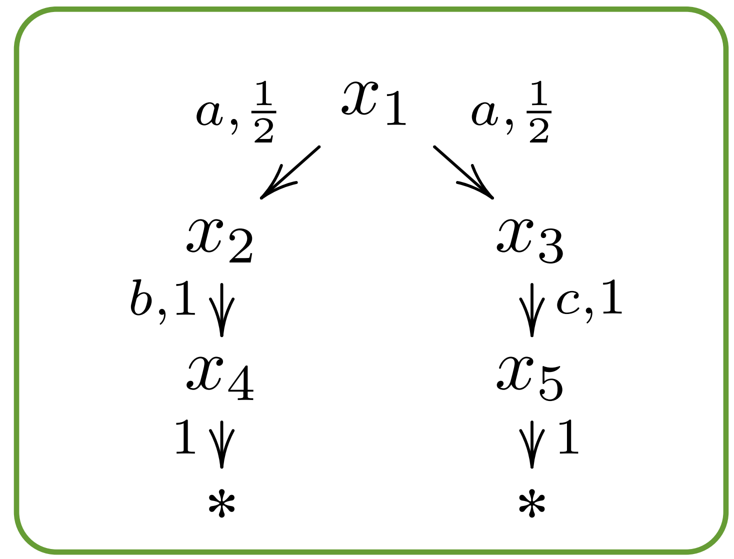
MC

$X \to \mathcal{D}(X)$



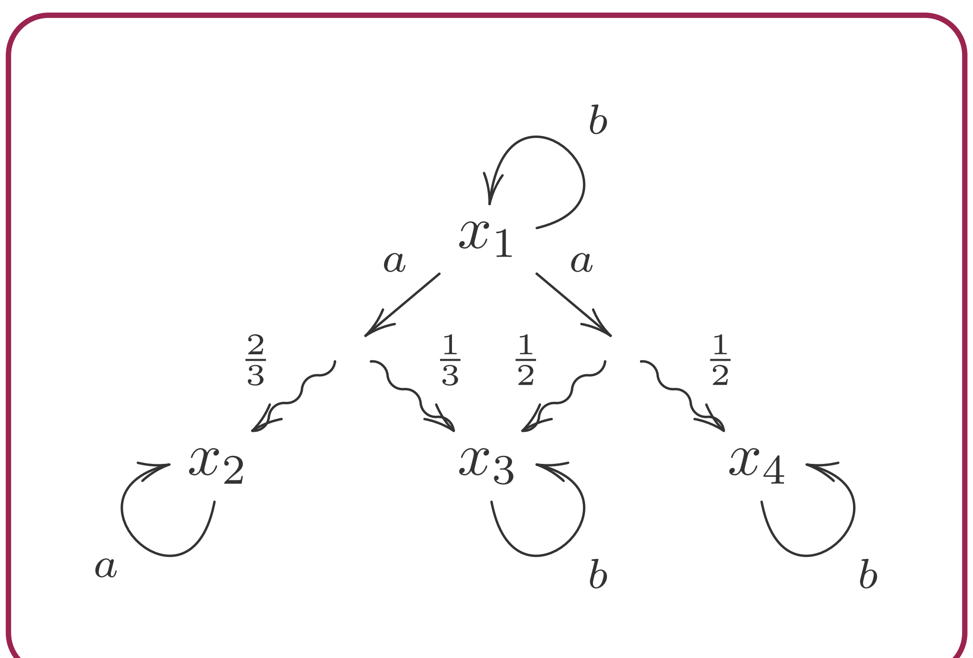
Generative PTS

$X \to \mathcal{D}\,(1 + A \times X)$



PA

$X \to \mathcal{P}\,(\mathcal{D}(X))^A$



all on
Sets

Convex Algebras

infinitely many finitary operations

convex combinations

binary ones "suffice"

- algebras

$$\boxed{\left(A, \sum_{i=1}^{n} p_i(-)_i\right)}$$

$$p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$$

- convex (affine) maps

$$\boxed{h\left(\sum_{i=1}^{n} p_i a_i\right) = \sum_{i=1}^{n} p_i h(a_i)}$$

satisfying

- Projection $\sum_{i=1}^{n} p_i a_i = a_k, \quad p_k = 1$
- Barycenter $\sum_{i=1}^{n} p_i \left(\sum_{j=1}^{m} p_{i,j} a_j\right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{n} p_i p_{i,j}\right) a_j$

Eilenberg-Moore Algebras

convex algebras abstractly

$\mathcal{EM}(\mathcal{D})$

- objects

$$\begin{tikzcd} \mathcal{D}A \arrow[d, "a"] \\ A \end{tikzcd}$$

satisfying

$$\begin{tikzcd} A \arrow[r, "\eta"] \arrow[dr, equal] & \mathcal{D}A \arrow[d, "a"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D}\mathcal{D}A \arrow[r, "\mu"] \arrow[d, "\mathcal{D}a"'] & \mathcal{D}A \arrow[d, "a"] \\ \mathcal{D}A \arrow[r, "a"] & A \end{tikzcd}$$

- morphisms

$$\begin{tikzcd} \begin{matrix} \mathcal{D}A \\ \downarrow a \\ A \end{matrix} \arrow[r, "h"] & \begin{matrix} \mathcal{D}B \\ \downarrow b \\ B \end{matrix} \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{D}A \arrow[r, "\mathcal{D}h"] \arrow[d, "a"'] & \mathcal{D}B \arrow[d, "b"] \\ A \arrow[r, "h"] & B \end{tikzcd}$$

Free Convex Algebras

carried by distributions

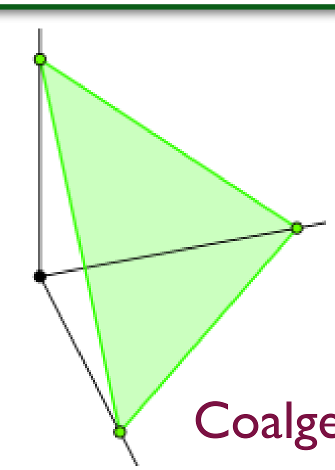
convex combinations as expected

wherever there are distributions, there is convexity

$$\mathbb{D}_X = (\mathcal{D}X, \sum_{i=1}^{n} p_i(-)_i) \qquad p_i \in [0,1], \sum_{i=1}^{n} p_i = 1$$

$$\sum p_i \xi_i = \xi \quad \Leftrightarrow \quad \forall x \in X.\ \xi(x) = \sum p_i \xi_i(x)$$

finitely generated
free convex algebras are
simplexes



Convexity in Probabilistic Systems Semantics

Traces

Generative PTS

$\mathcal{D}$ (1 + A x (-))

$$
\begin{tikzcd}
 & x_1 \arrow[dl, "{a, \frac{1}{2}}"'] \arrow[dr, "{a, \frac{1}{4}}"] & \\
x_2 \arrow[d, "{b, \frac{1}{3}}"'] & & x_3 \arrow[d, "{c, \frac{1}{2}}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
$$

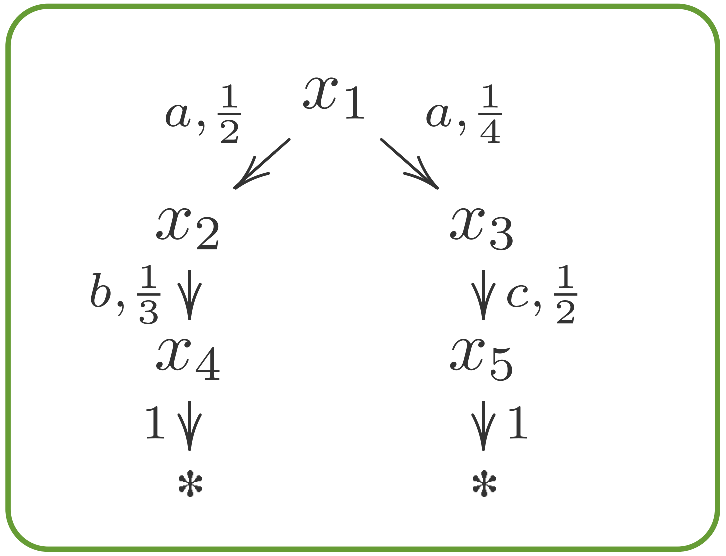
$$\mathrm{tr}(x_1)(ab) = \frac{1}{6} \qquad \mathrm{tr}(x_1)(ac) = \frac{1}{8}$$

$$\mathrm{tr} \colon X \to \mathcal{D}A^*$$

Traces via determinisation

Generative PTS

$\mathcal{D}$ (1 + A x (-))



$$\mathrm{tr}(x_1)(ab) = \frac{1}{6} \qquad \mathrm{tr}(x_1)(ac) = \frac{1}{8}$$

$$\mathrm{tr}\colon X \to \mathcal{D}A^*$$

trace = bisimilarity after determinisation

Happens in convex algebra

Trace axioms for generative PTS

Axioms for bisimilarity

+

$$p \cdot a \cdot (p_1 \mathsf{E}_1 \oplus p_2 \mathsf{E}_2) \quad \equiv \quad p_1 \cdot a \cdot p\mathsf{E}_1 \oplus p_2 \cdot a \cdot p\mathsf{E}_2 \qquad (D)$$

soundness and completeness

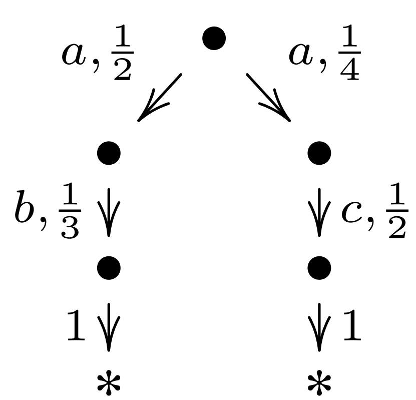
Happens in convex algebra

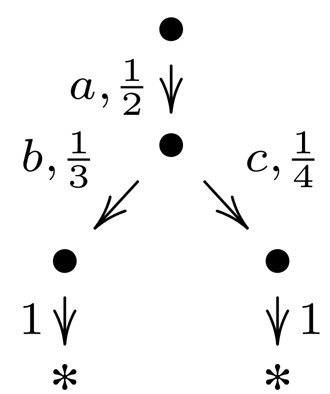
[Silva, S. MFPS'11]

Trace axioms for generative PTS

Generative PTS

$\mathcal{D}(1 + A \times (-))$





$$\frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot * \overset{(D)}{\equiv} \frac{1}{2} \cdot a \cdot \frac{1}{4} \cdot c \cdot 1 \cdot *$$

$$\left(\frac{1}{2} \cdot a \cdot \frac{1}{3} \cdot b \cdot 1 \cdot *\right) \oplus \left(\frac{1}{4} \cdot a \cdot \frac{1}{2} \cdot c \cdot 1 \cdot *\right) \overset{(Cong)}{\equiv} \left(\frac{1}{2} \cdot a \cdot \frac{1}{3} \cdot b \cdot 1 \cdot *\right) \oplus \left(\frac{1}{2} \cdot a \cdot \frac{1}{4} \cdot c \cdot 1 \cdot *\right)$$

$$\overset{(D)}{\equiv} \frac{1}{2} \cdot a \cdot \left(\frac{1}{3} \cdot b \cdot 1 \cdot * \oplus \frac{1}{4} \cdot c \cdot 1 \cdot *\right)$$

The quest for completeness

Inspired lots of new research:

- A. S., H. Woracek Congruences of convex algebras JPAA'15
- S. Milius Proper functors CALCO'17

if f.p. = f.g. and ★ then completeness

does not hold

our axiomatisation would be proven complete if only one particular convex functor were proper

it works !

[S., Woracek FoSSaCS'18]

Finitely generated, finitely presentable

finitely generated (f.g.) = quotients of free finitely generated ones

finitely presentable (f.p.) = quotients of free finitely generated ones under finitely generated congruences

smallest congruence containing a finite set of pairs

Theorem

Every congruence of convex algebras is f.g.
Hence f.p. = f.g.

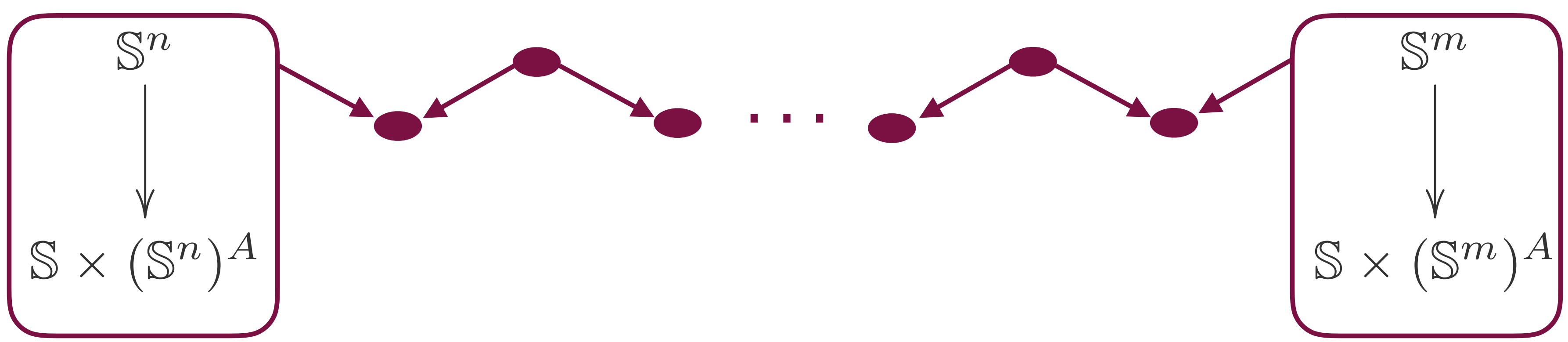
[S., Woracek JPAA'15]

Proper semirings

Ésik&Maletti 2010

A semiring is proper iff for every two equivalent states x ≡ y in WA with f.f.g. carriers, there is a zigzag of WA whose all nodes have f.f.g. carriers that relates them

free finitely generated



Milius 2017

Proper functors

functor F on an algebraic category

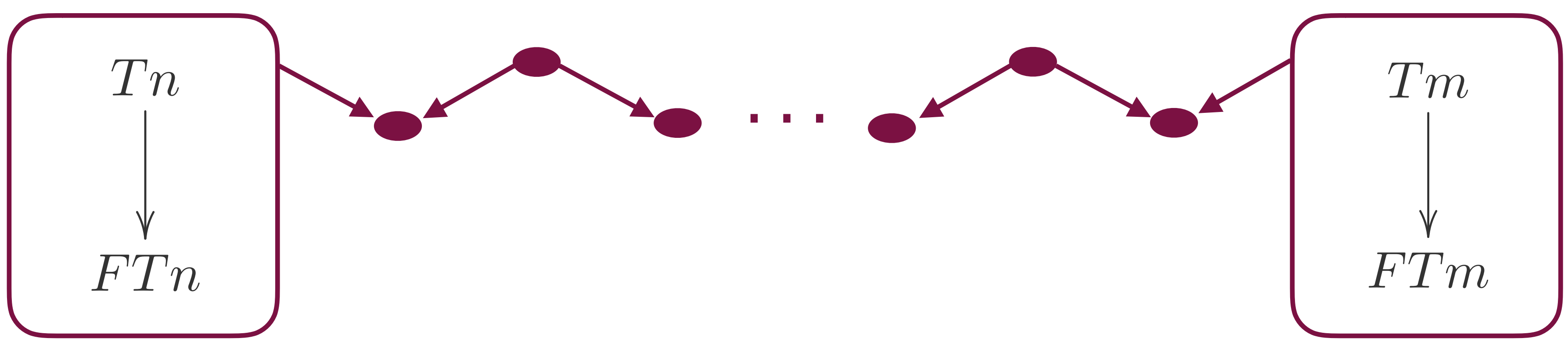
$\mathtt{Set}^{T}$

behaviour equivalence

F-coalgebras

A ~~semiring~~ is proper iff for every two equivalent states x ≡ y in ~~WA~~ with f.f.g. carriers, there is a zigzag of ~~WA~~ whose all nodes have f.f.g. carriers that relates them

free finitely generated

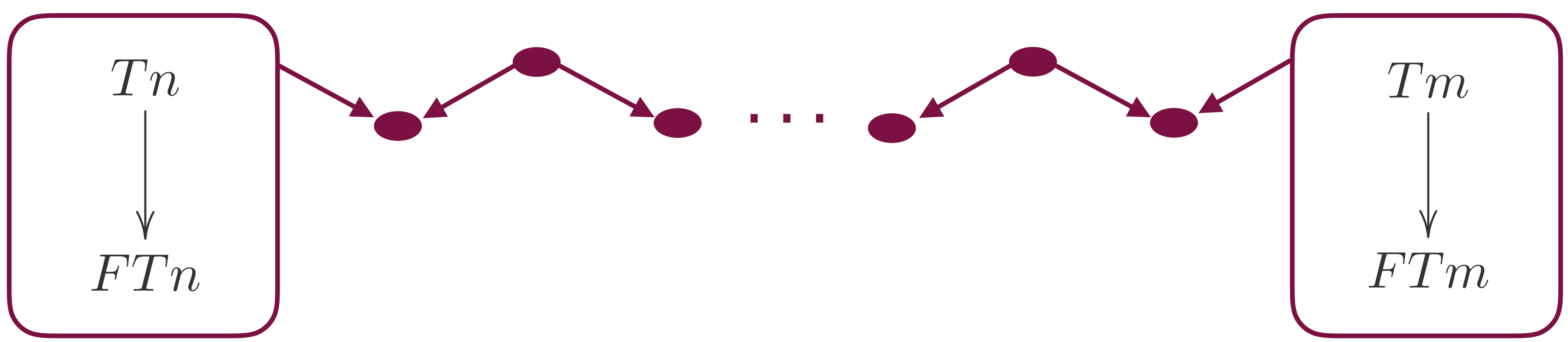


Proper functors

Milius 2017

A functor F on an algebraic category $\mathtt{Set}^T$, for a finitary monad T, is proper iff for every two behaviourally equivalent states x ≡ y in F-coalgebras with f.f.g. carriers, there is a zigzag of F-coalgebras whose all nodes have f.f.g. carriers that relates them.

free finitely generated



Proper functors enable "easy" completeness proofs of axiomatizations of expression languages…

proving properness is difficult

Previous results

Proper:

- Boolean semiring — Bloom & Ésik ‘93
- Finite commutative ordered semirings — Ésik & Kuich ‘01 — 7
- Eucledian domains, skew fields — Béal & Lombardy & Sakarovich ‘05 — 2
- $\mathbb{N}, \mathbb{B}, \mathbb{Z}$, skew fields — Béal & Lombardy & Sakarovich ‘05 — 1
- Noetherian semirings, commutative rings, finite semirings — Ésik & Maletti ‘10 — 1

these are all known (im)proper semirings

Improper:

- Tropical semiring — Ésik & Maletti ‘10

We have

[S., Woracek FoSSaCS'18]

Framework for proving properness

Instantiate it on known semirings

- Noetherian 1
- Naturals $\mathbb{N}$ 1

Prove new semirings proper

- Non-negative rationals $\mathbb{Q}_+$ 1
- Non-negative reals $\mathbb{R}_+$ 1

Prove new convex functors proper

- $[0,1] \times (-)^A$ 1
- F*, a subfunctor of the above 3

on positive convex algebras

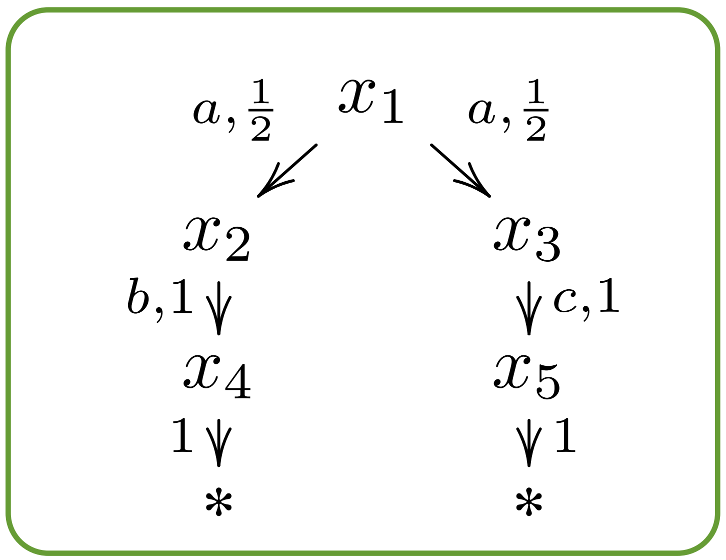
$\mathtt{Set}^{\mathcal{D}}$

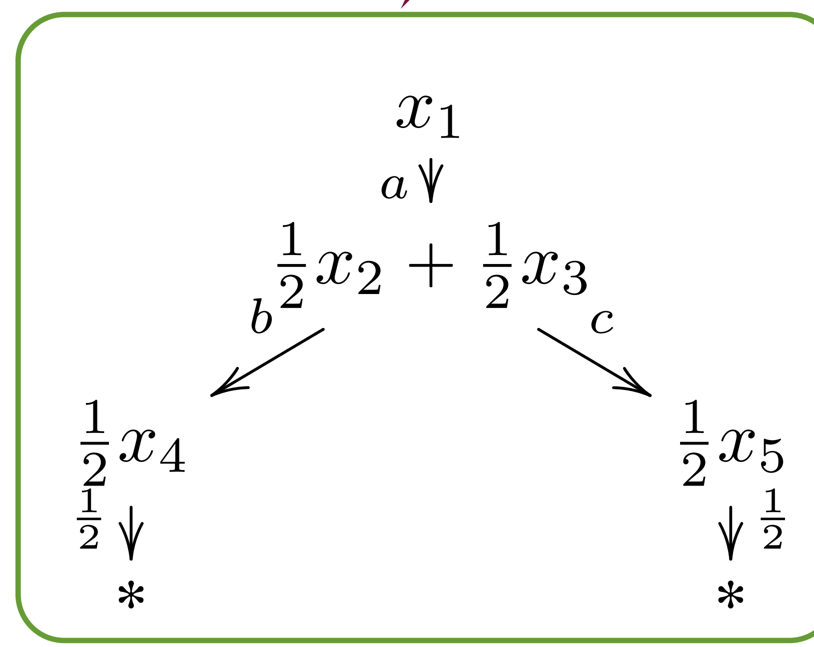
Determinisations

belief-state transformer

Generative PTS

$X \to \mathcal{D}(1 + A \times X)$





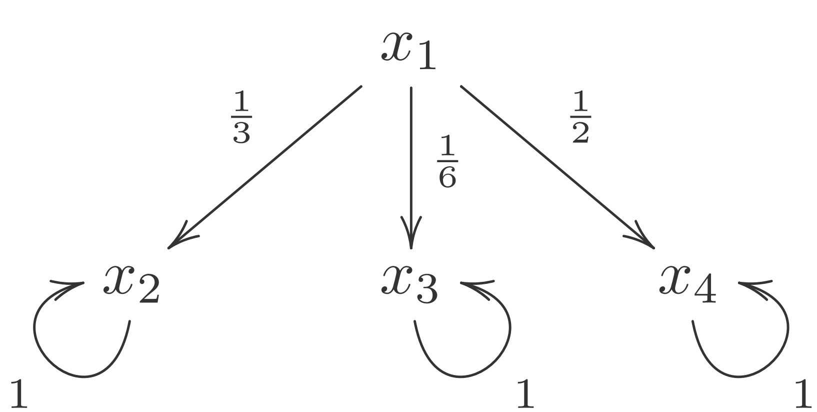
[Silva, S. MFPS'11]

[Jacobs, Silva, S. JCSS'15]

Belief-state transformers

MC

$X \to \mathcal{D}(X)$



belief-state transformer

belief state

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \ldots$$

$$\downarrow$$

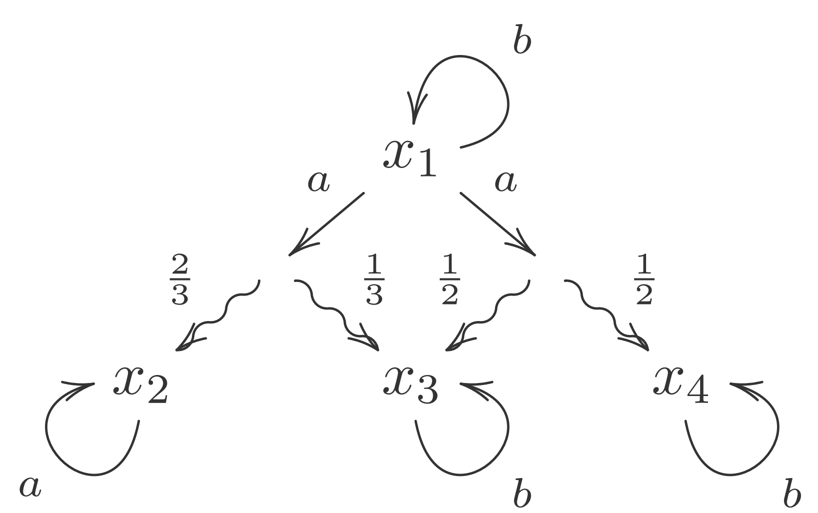
$$\frac{7}{9}x_2 + \frac{1}{18}x_3 + \frac{1}{6}x_4$$

$$\frac{1}{3}\left(\frac{1}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$

Belief-state transformers

PA

$X \to (\mathcal{PD}(X))^A$



belief-state transformer

belief state

$$
\begin{tikzcd}
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & \cdots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \cdots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4
\end{tikzcd}
$$

$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity

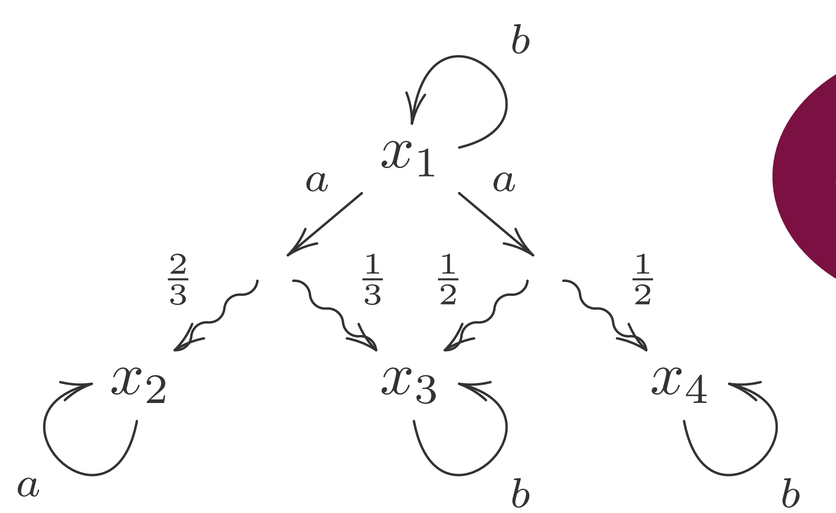
strong bisimilarity

probabilistic / combined bisimilarity

belief-state bisimilarity

PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$



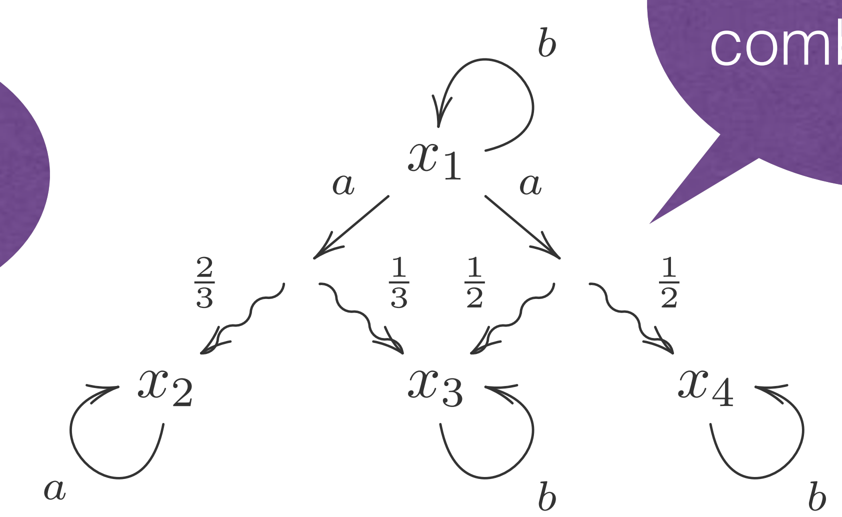
on **Sets**

$\sim \; = \; \approx$

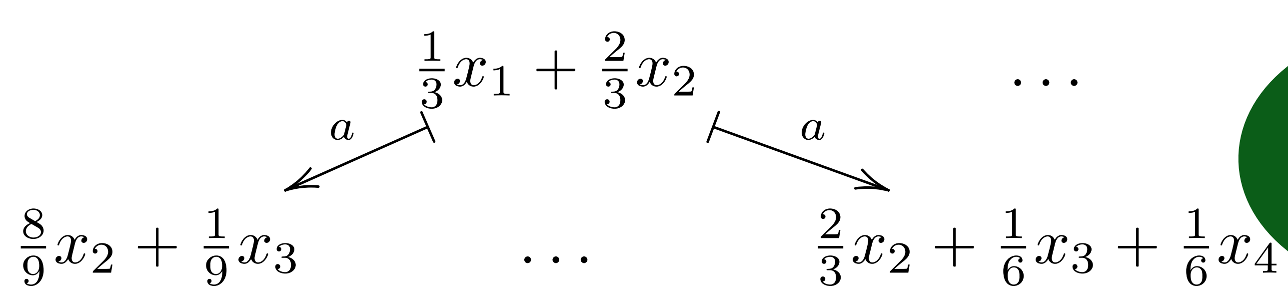
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}(X))^A$

and all convex combinations



$X \to (\mathcal{P}_c(X)+1)^A$



on convex algebras

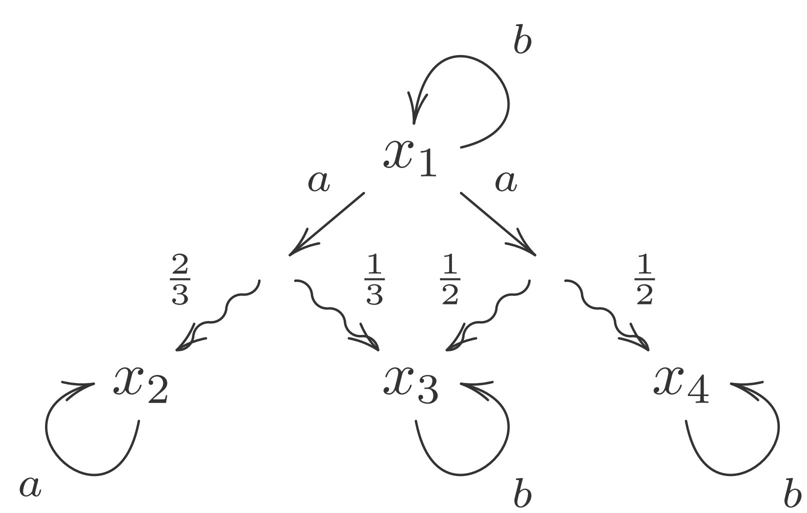
$\sim_d \; = \; \approx$

Belief-state transformer

foundation ?

PA

$X \to \mathcal{P}(\mathcal{D}(X))^A$



how does it emerge?

what is it?

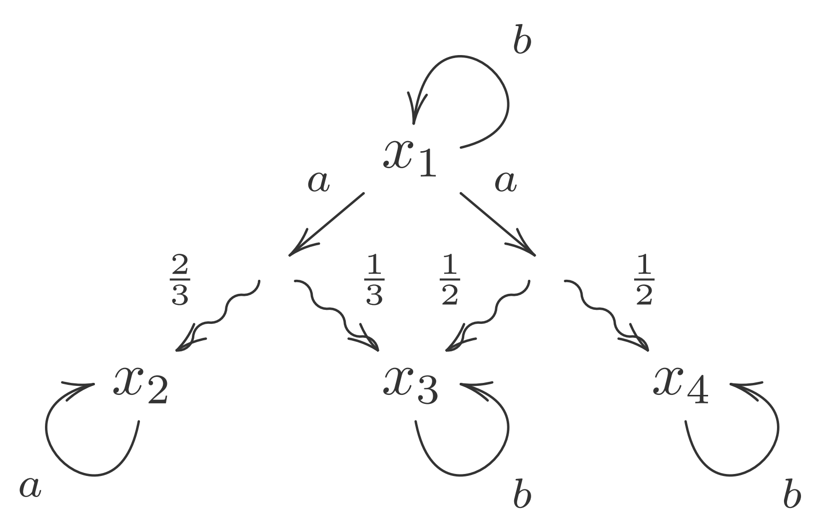
$$
\begin{tikzcd}
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \ldots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \ldots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}
$$

Belief-state transformer

foundation ?

PA

$X \to \mathcal{P}(\mathcal{D}(X))^A$



how does it emerge?

coalgebra over free convex algebra

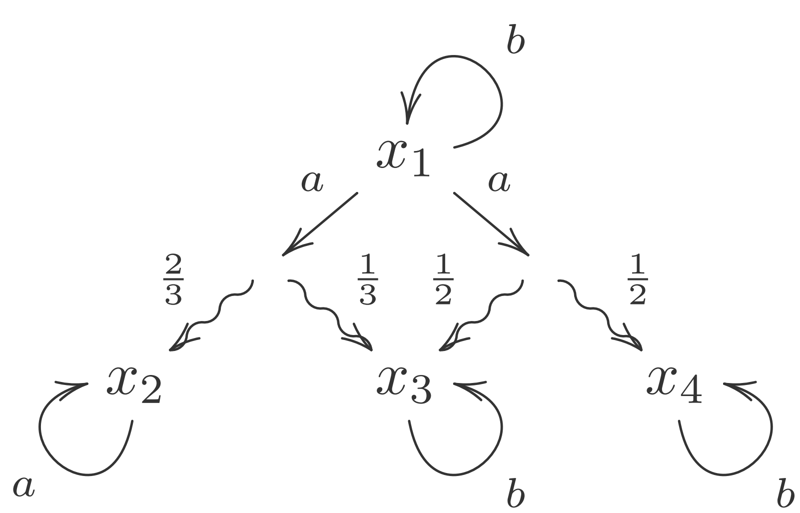
$$
\begin{tikzcd}
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \dots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \dots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}
$$

Belief-state transformer

foundation ?

PA

$X \to \mathcal{P}(\mathcal{D}(X))^A$



via a generalised[3] determinisation

coalgebra over free convex algebra

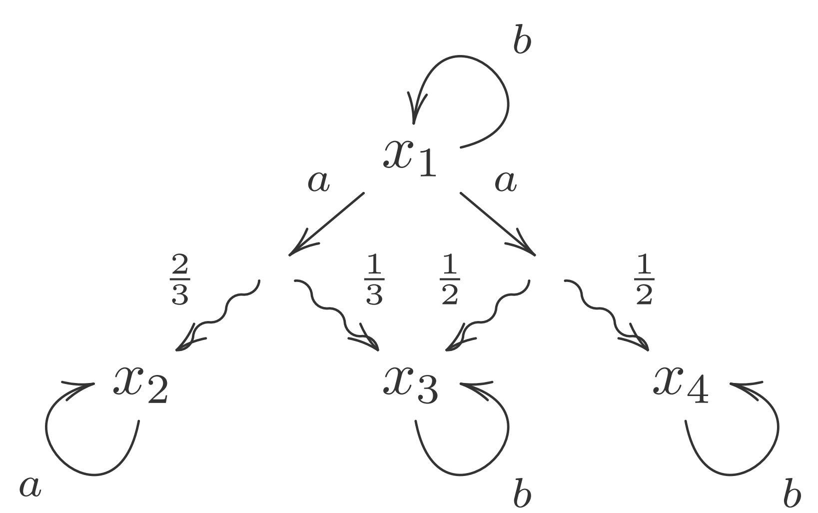
$$
\begin{tikzcd}[column sep=small]
 & \frac{1}{3}x_1 + \frac{2}{3}x_2 \arrow[dl, mapsto, "a"'] \arrow[dr, mapsto, "a"] & & \dots \\
\frac{8}{9}x_2 + \frac{1}{9}x_3 & \dots & \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 &
\end{tikzcd}
$$

Belief-state transformer

foundation !

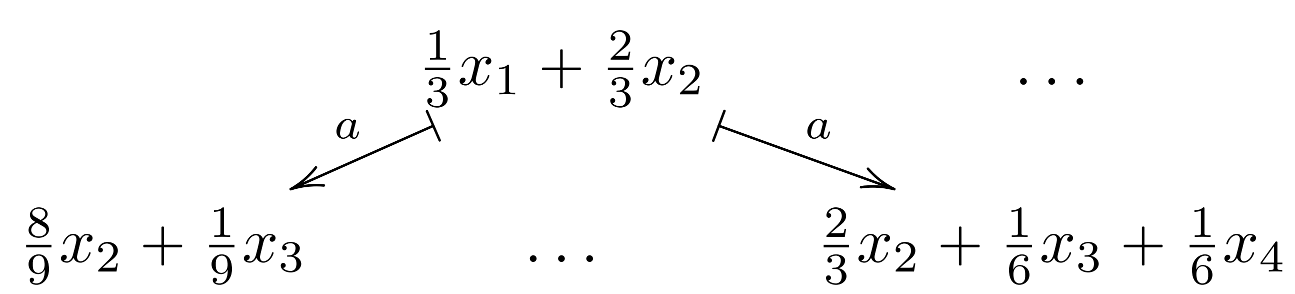
PA

$X \to \mathcal{P}(\mathcal{D}(X))^A$



via a generalised³ determinisation

coalgebra over free convex algebra

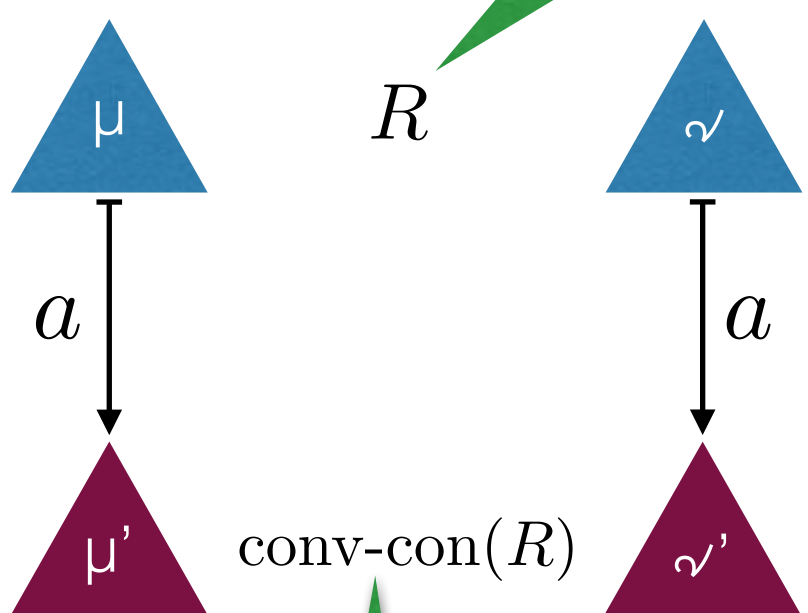


[Bonchi, SIlva, S. CONCUR'17]

Coinductive proof method for distribution bisimilarity

[Bonchi, SIlva, S. CONCUR’17]

bisimulation up-to convex hull



convex and equivalence closure of *R*

to prove μ $\sim_d$ ν it suffices to find a bisimulation up-to convex hull *R* with μ *R* ν

there always exists a finite one!

by S., Woracek JPAA’15

Termination?

- We looked at one-point extensions of convex algebras, for termination.

Every convex algebra can be extended by a single point

- What are all the possible ways?

there are many possible ways

we can give full description for…

single naturally functorial way

MC and PA belief-state transformers

[S., Woracek CALCO'17]

It’s time to terminate this talk..

convexity appears
at many places
in probabilistic systems
sematics

next: algorithms ?

$\mathcal{EM}(\mathcal{G})$?

Thank You!