Compositionality and algebraic properties of process operations

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in a coalgebraic setting

How to ...

get a process operation on coalgebras which is
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By recognizing structure on the

- base category
- functor - the type of coalgebras

Category Structure

Symmetric category:

$$\langle \mathbb{C}, \otimes, \gamma \rangle$$

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$\otimes$: bifunctor

γ: nat. iso. $\gamma : X \otimes Y \to Y \otimes X$

Example: $\langle \mathbf{Sets}, \times, \gamma \rangle, \qquad \gamma(\langle x, y \rangle) = \langle y, x \rangle$

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Example: $\langle \mathbf{Sets},\ \times,\ \alpha \rangle, \qquad \alpha(\langle x, \langle y, z\rangle\rangle) = \langle\langle x, y\rangle, z\rangle$

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symmetric functor F on a symmetric category with

$$s : F(-) \otimes F(+) \Rightarrow F(- \otimes +)$$

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\downarrow{\scriptstyle \alpha} & & & & \downarrow{\scriptstyle F\alpha} \\
(FX \otimes FY) \otimes FZ & \xrightarrow{s \otimes id} & F(X \otimes Y) \otimes FZ & \xrightarrow{s} & F((X \otimes Y) \otimes Z)
\end{array}$$

Example

$A \times \_ + 1$ on $\mathbf{Sets}$, given a partial $\cdot$ on A, with

$$s_{X,Y} : (A \times X + 1) \times (A \times Y + 1) \to A \times (X \times Y) + 1$$

defined by

$$s_{X,Y}(\langle u, v \rangle) = \begin{cases} \langle c, \langle x, y \rangle \rangle & u = \langle a, x \rangle,\ v = \langle b, y \rangle,\ c = a \cdot b \in A \\ * & \text{otherwise} \end{cases}$$

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- is a symmetric functor for $A(\cdot)$ partially commutative
- is a semigroup functor for $A(\cdot)$ a partial semigroup

Coalgebra structure

Result: If $\mathbb{C}$ and F have structure (sym./sem./mon.) then $\mathsf{Coalg}_{\mathcal{F}}$ has structure (sym./sem./mon.) with

$$\langle X, c_X \rangle \otimes \langle Y, c_Y \rangle = \langle X \otimes Y, s \circ (c_X \otimes c_Y) \rangle$$

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Hence: process operations on F-coalgebras !

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parallel composition of deterministic systems

$$x \mid y \xrightarrow{a} x' \mid y' \quad \Longleftrightarrow \quad x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c$$

Note: $x \mid y$ denotes the state $\langle x, y \rangle$

in the composite coalgebra.

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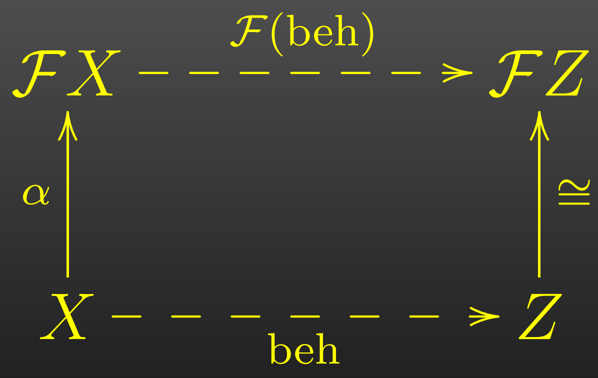

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Then: $\langle Z, \zeta \rangle$ with $\|$ is a sym./sem./mon. object in $\mathsf{Coalg}_{\mathcal{F}}$

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Then: $x \parallel y = y \parallel x$

$x \parallel (y \parallel z) = (x \parallel y) \parallel z$ ...

algebraic properties hold in $C_Z(\|)$

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- final coalgebra exists $C_Z = \langle Z, \zeta \rangle$
- $\|: C_Z \times C_Z \to C_Z$ is an operation

Also: $x \mid y \sim y \mid x$

$x \mid (y \mid z) \sim (x \mid y) \mid z$ ...

in any composite coalgebra

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Beh. of the components $\Rightarrow$ beh. of the composite

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Then:	$\mathrm{beh}_{X\otimes Y} = \| \circ (\mathrm{beh}_X \otimes \mathrm{beh}_Y)$
In Sets:	$x \sim x', y \sim y' \Rightarrow x \mid y \sim x' \mid y'$ bisimilarity (the f.c.s.) is a congruence (Plotkin,Turi)

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Then: trace semantics is compositional

Conclusion

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Future: full generality ?