The Power of Convex Algebra

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ENS DE LYON

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UCL

CONCUR '17

probabilistic automata

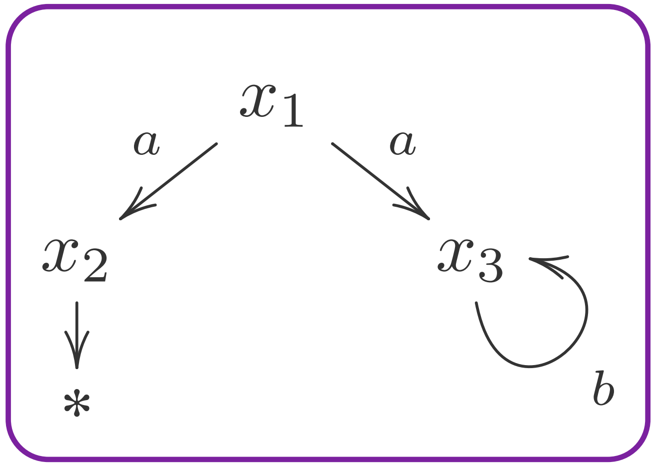
The true nature of PA as transformers of belief states

Determinisations

Determinisations

NFA

$X \to 2 \times (\mathcal{P}(X))^A$



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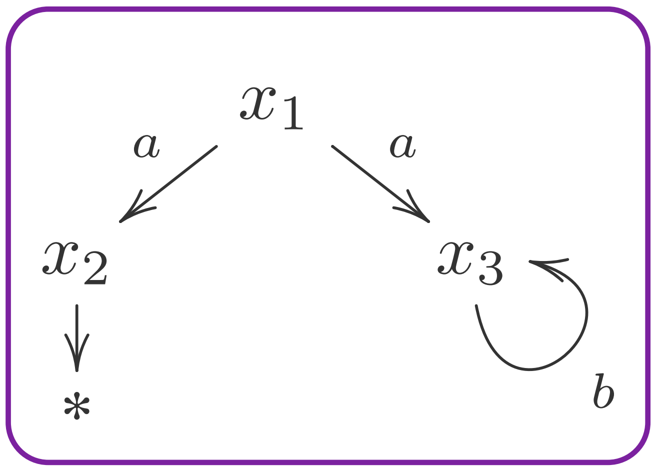

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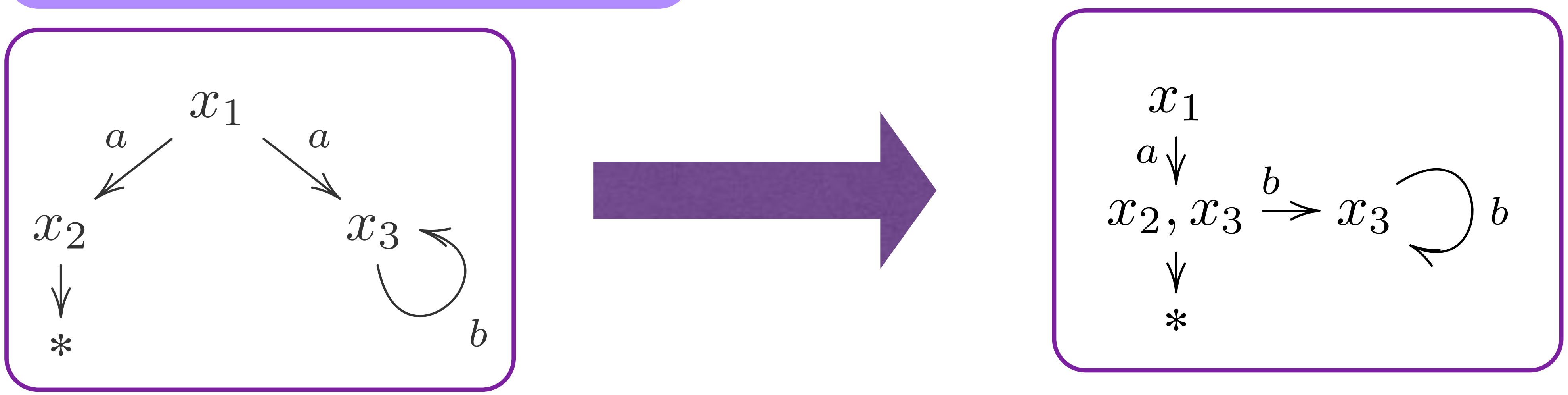

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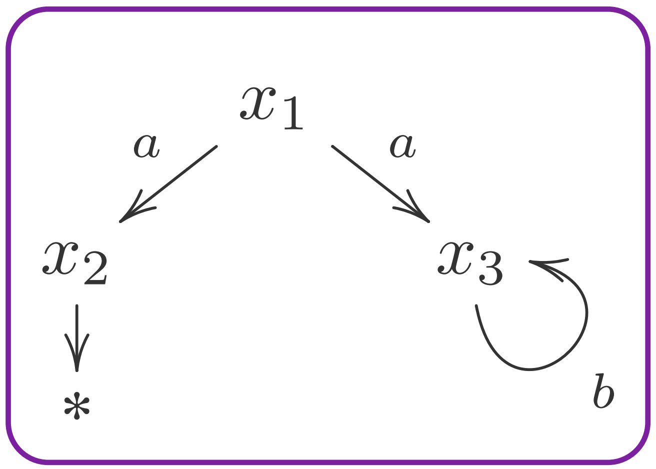
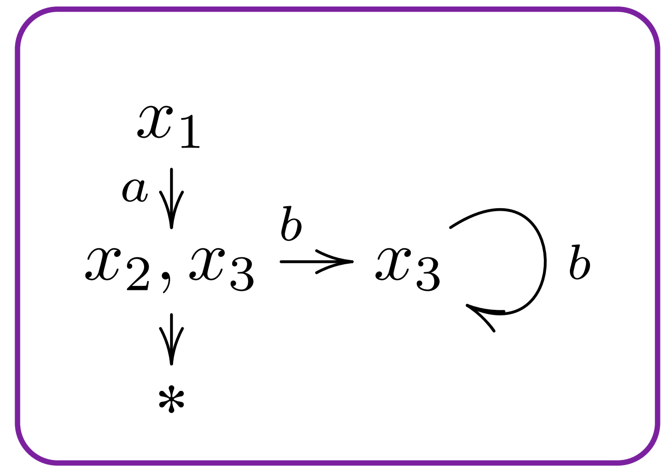




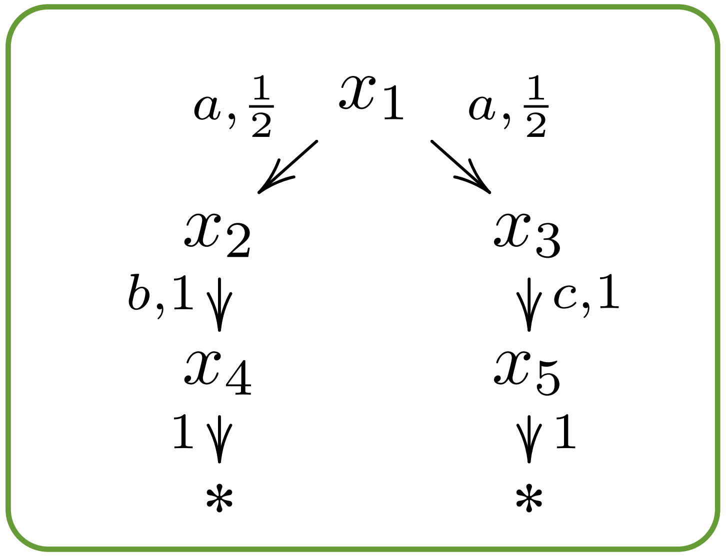
[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations

Determinisations

Generative PTS

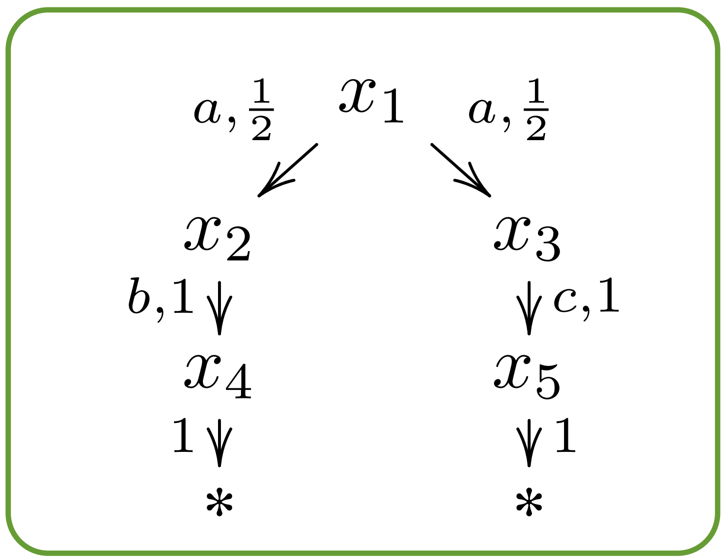
$$X \to \mathcal{D}(1 + A \times X)$$



Determinisations

Generative PTS

X → $\mathcal{D}$ (1 + A x X)



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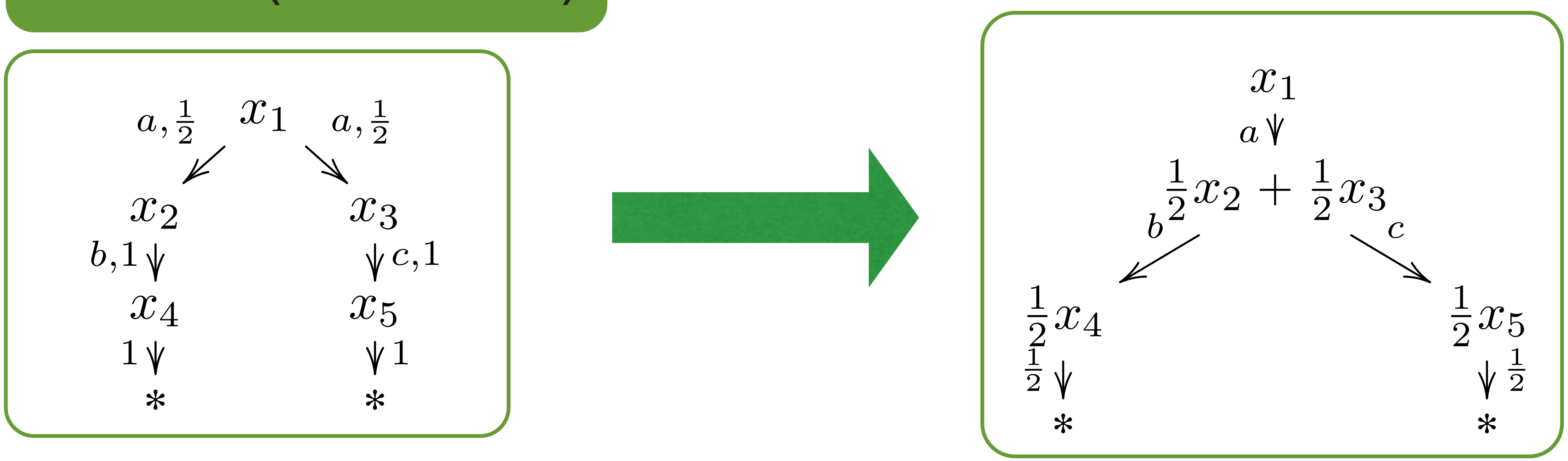

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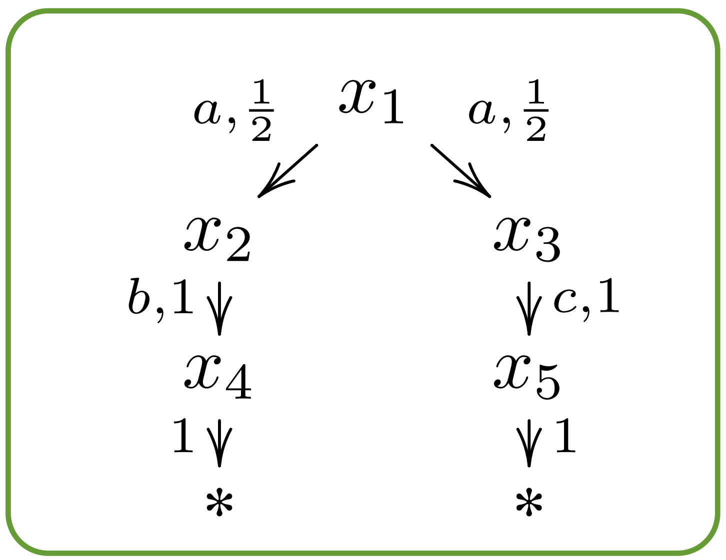
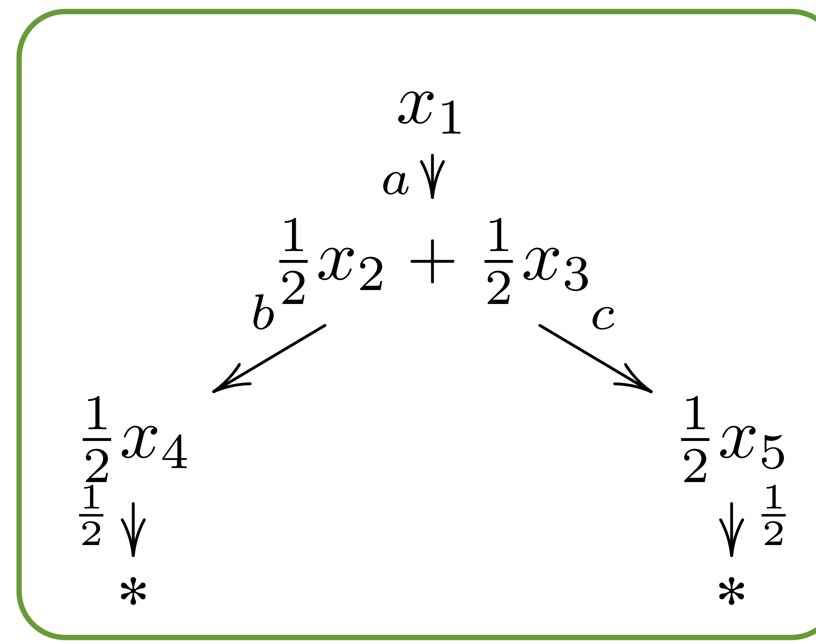




[Silva, S. MFPS'11]

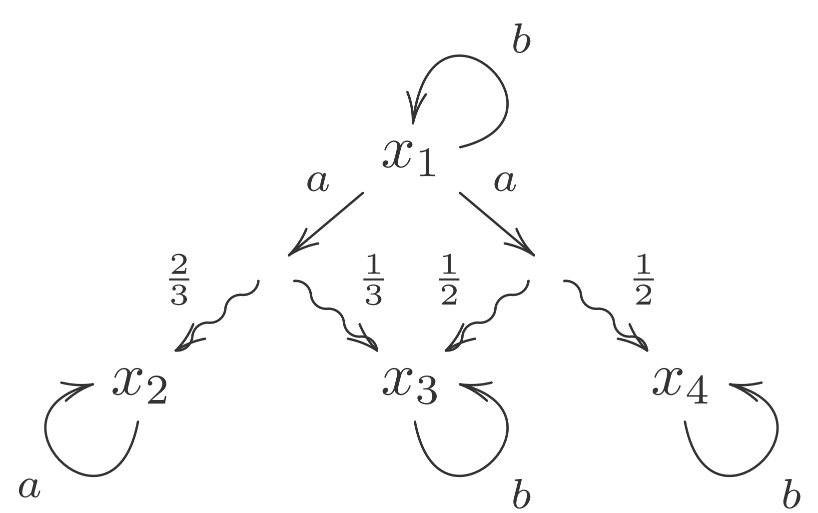
[Jacobs, Silva, S. JCSS'15]

Determinisations

Determinisations

PA

$X \to (\mathcal{PD}(X))^A$



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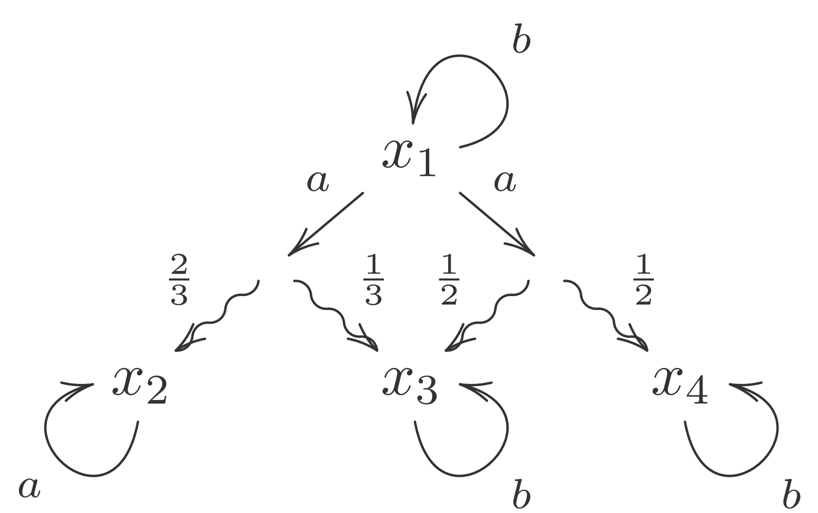

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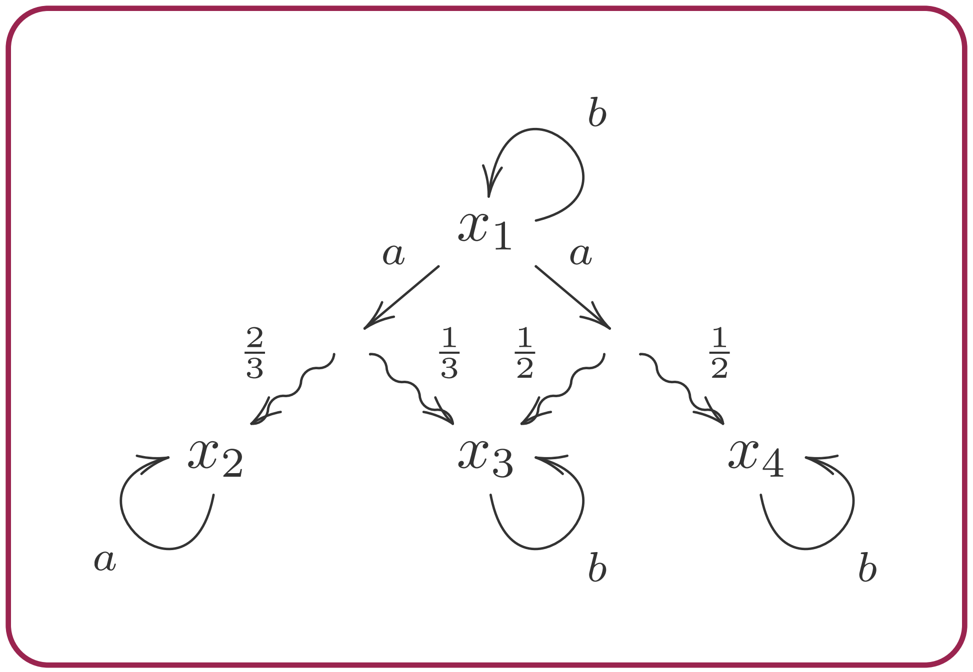

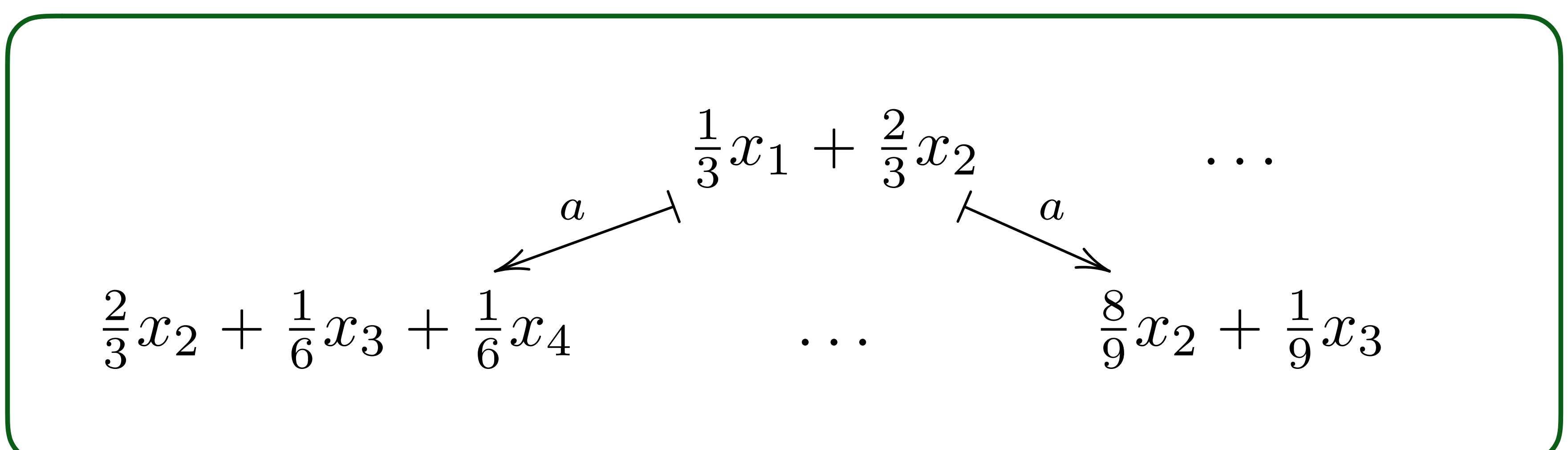

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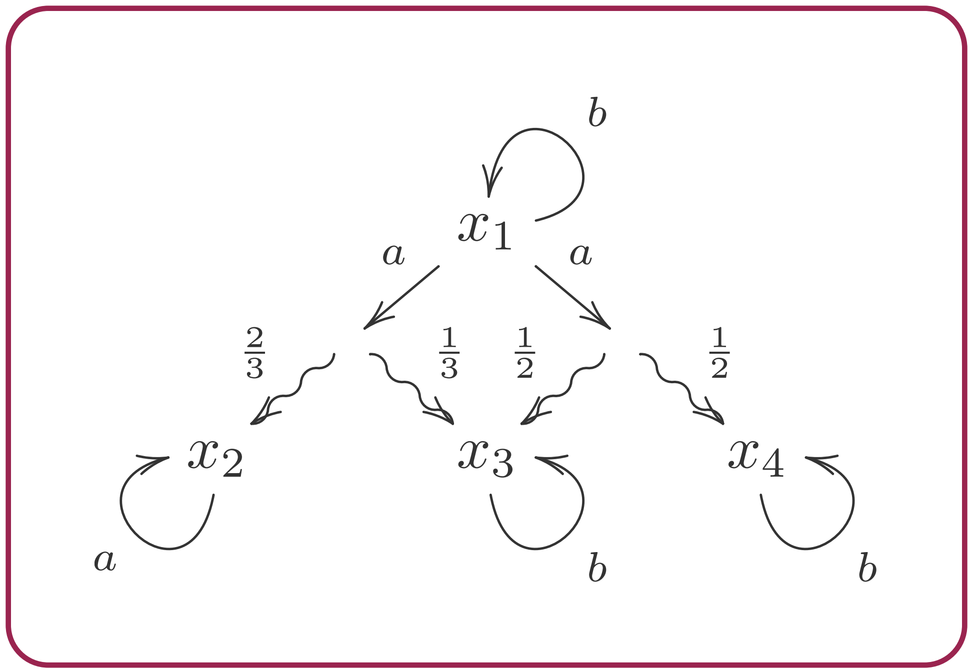
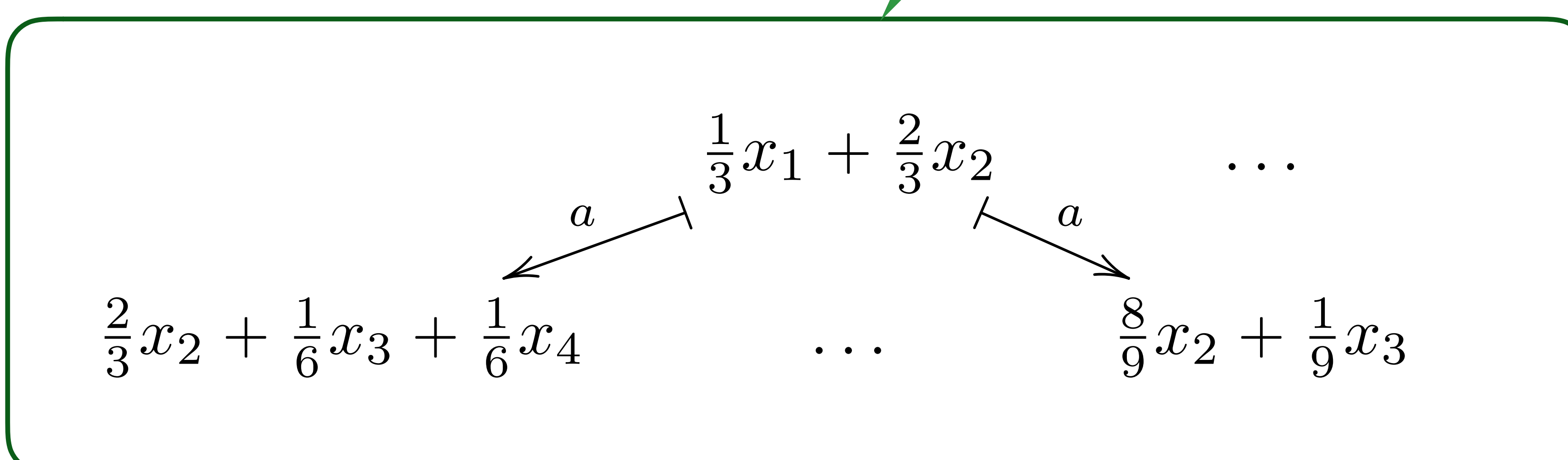


belief-state transformer



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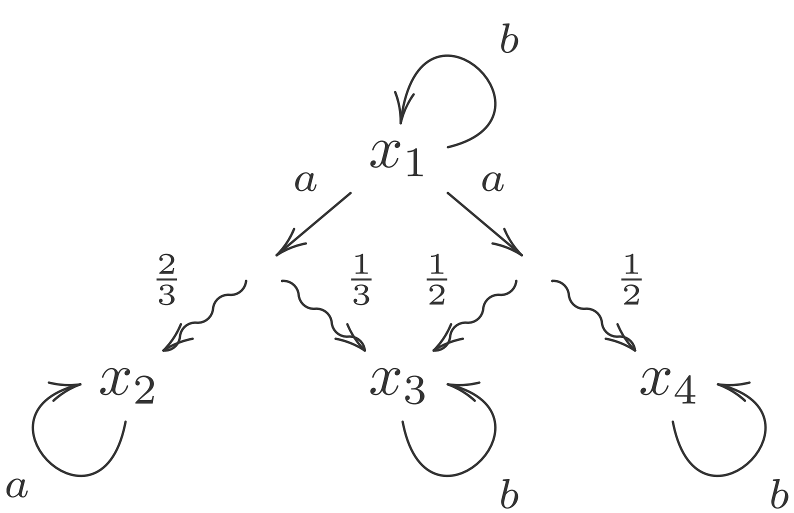


belief-state transformer

belief state

$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \qquad \dots$$

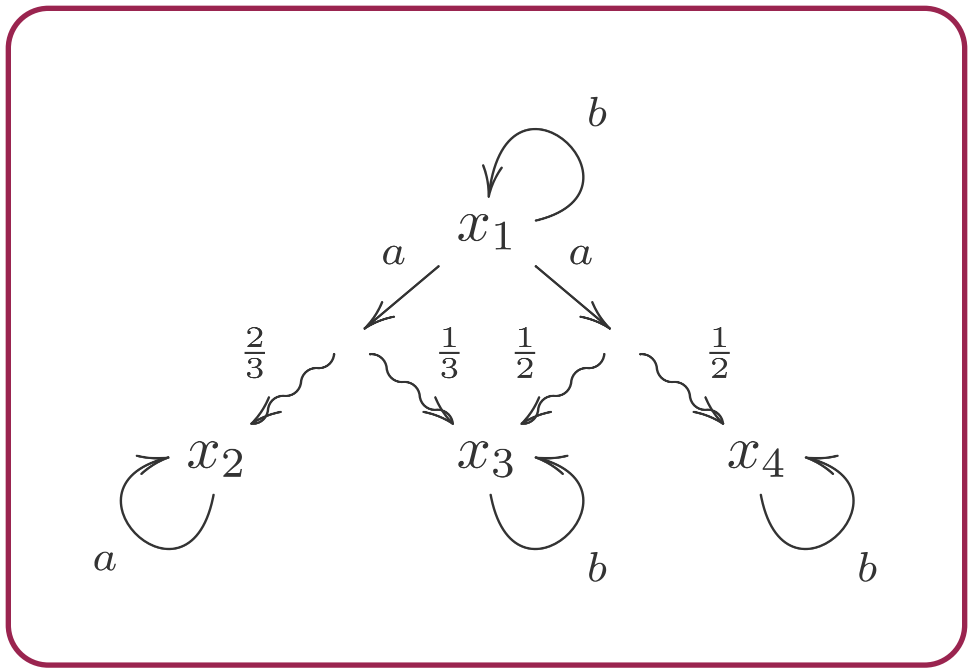
$$\frac{1}{3}x_1 + \frac{2}{3}x_2 \overset{a}{\longmapsto} \frac{2}{3}x_2 + \frac{1}{6}x_3 + \frac{1}{6}x_4 \qquad \dots \qquad \frac{1}{3}x_1 + \frac{2}{3}x_2 \overset{a}{\longmapsto} \frac{8}{9}x_2 + \frac{1}{9}x_3$$

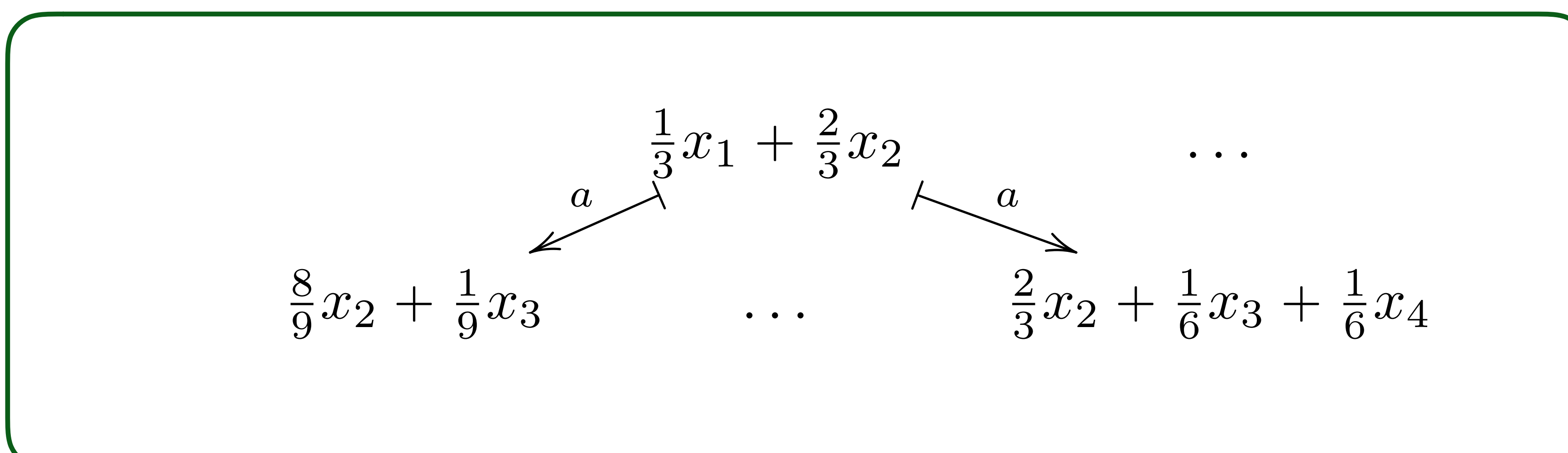
Belief-state transformer

foundation ?

PA

$X \to (\mathcal{PD}(X))^A$





Belief-state transformer

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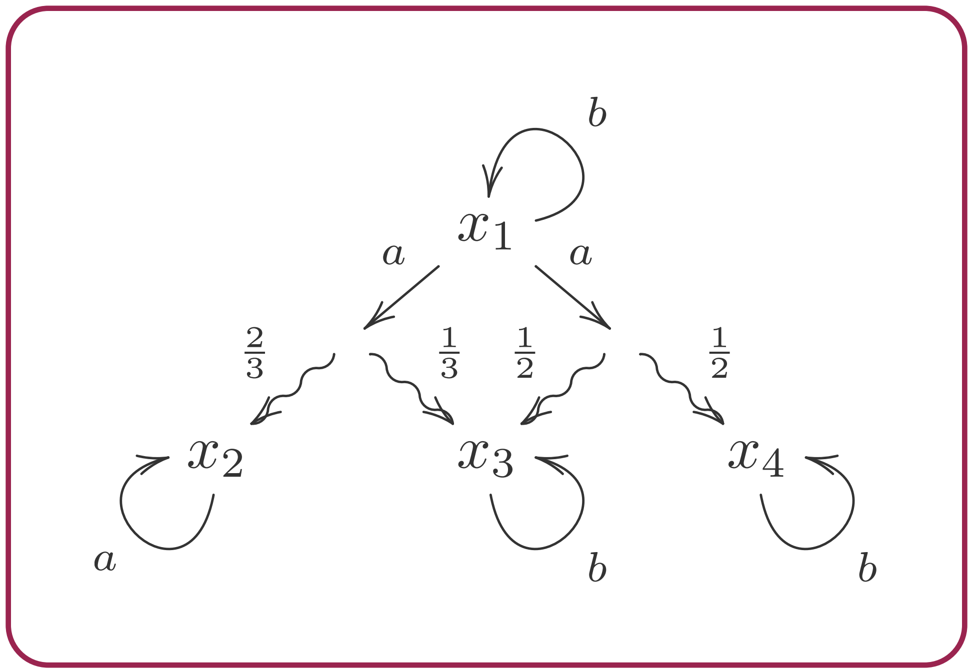
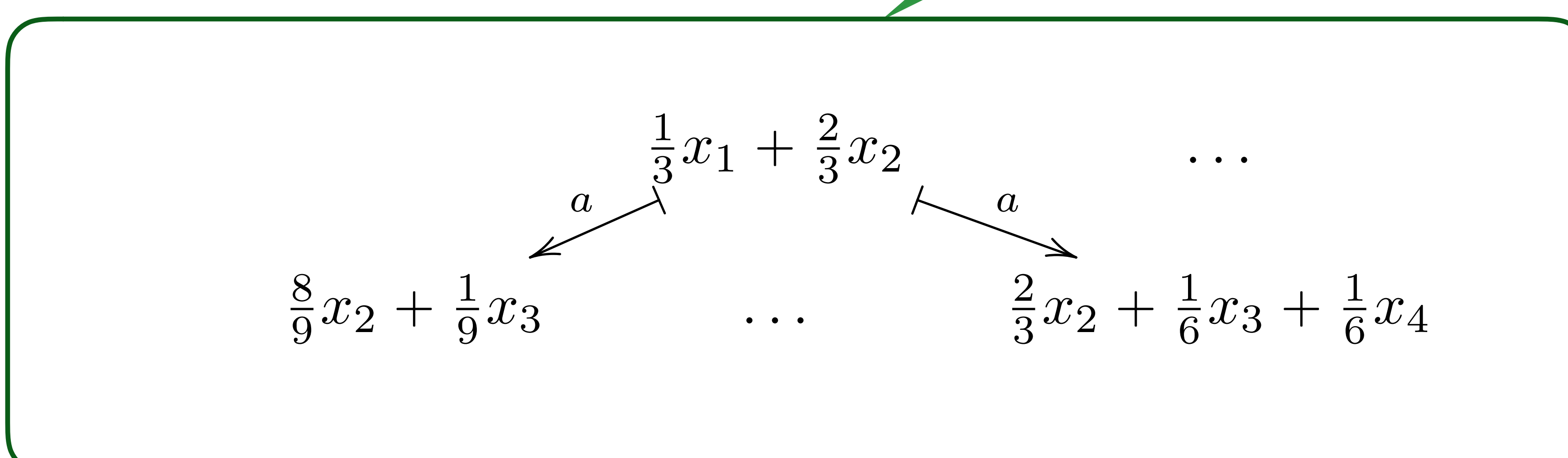


what is it?



Belief-state transformer

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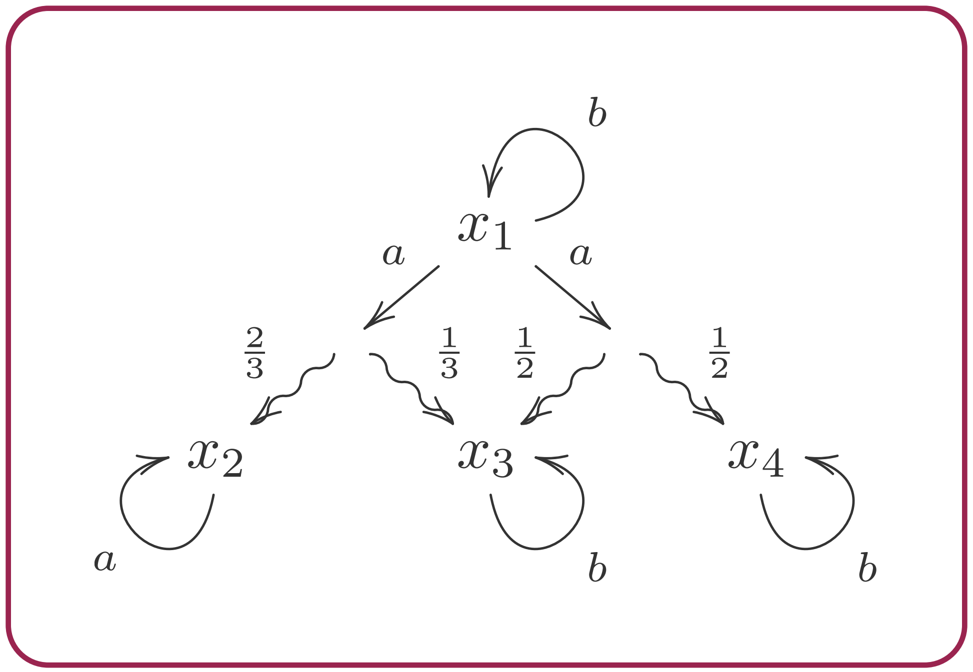
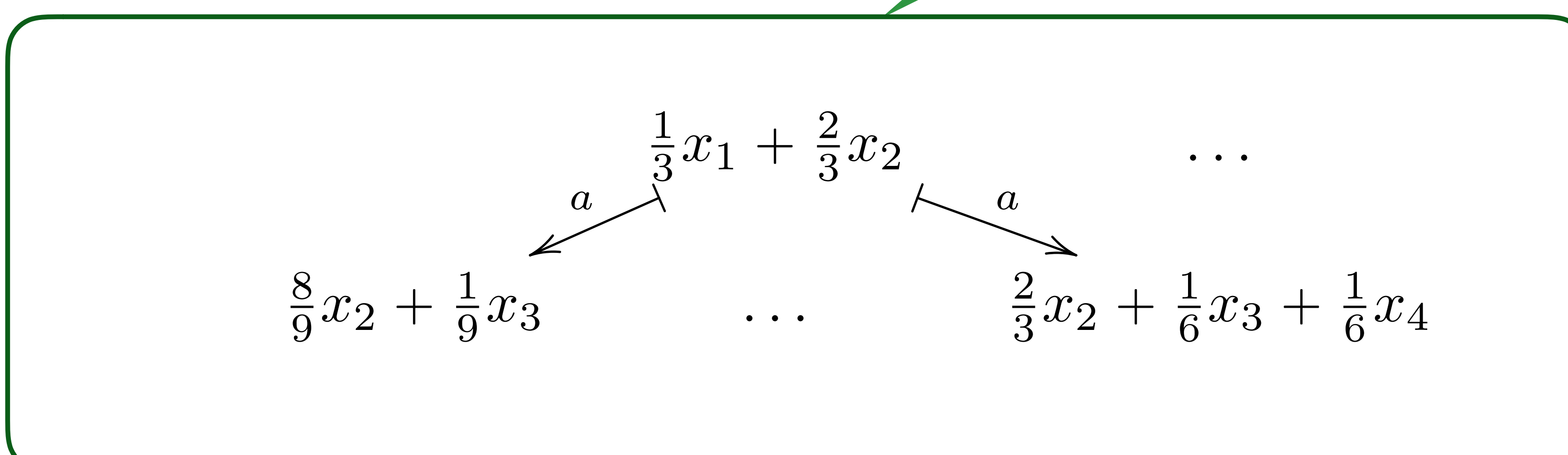
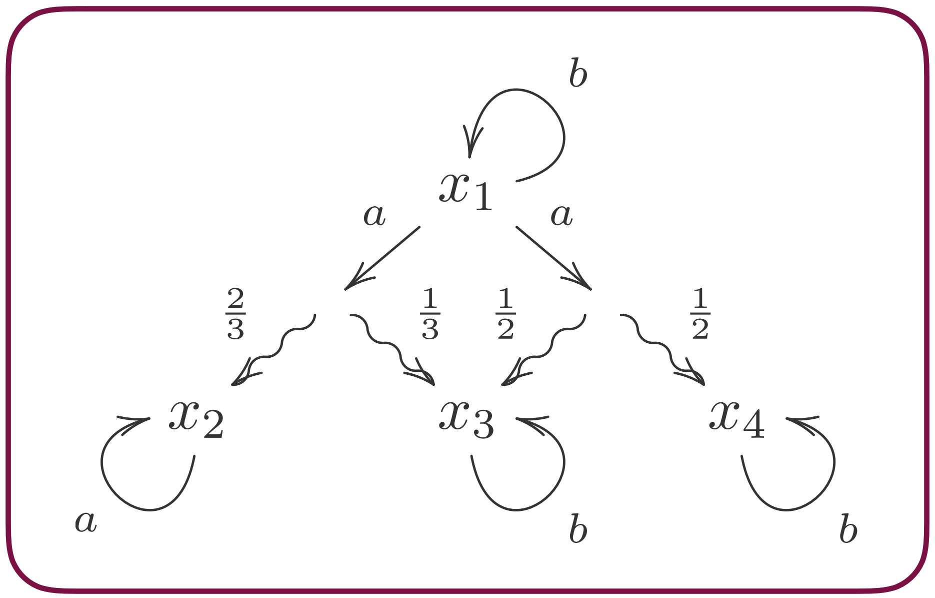


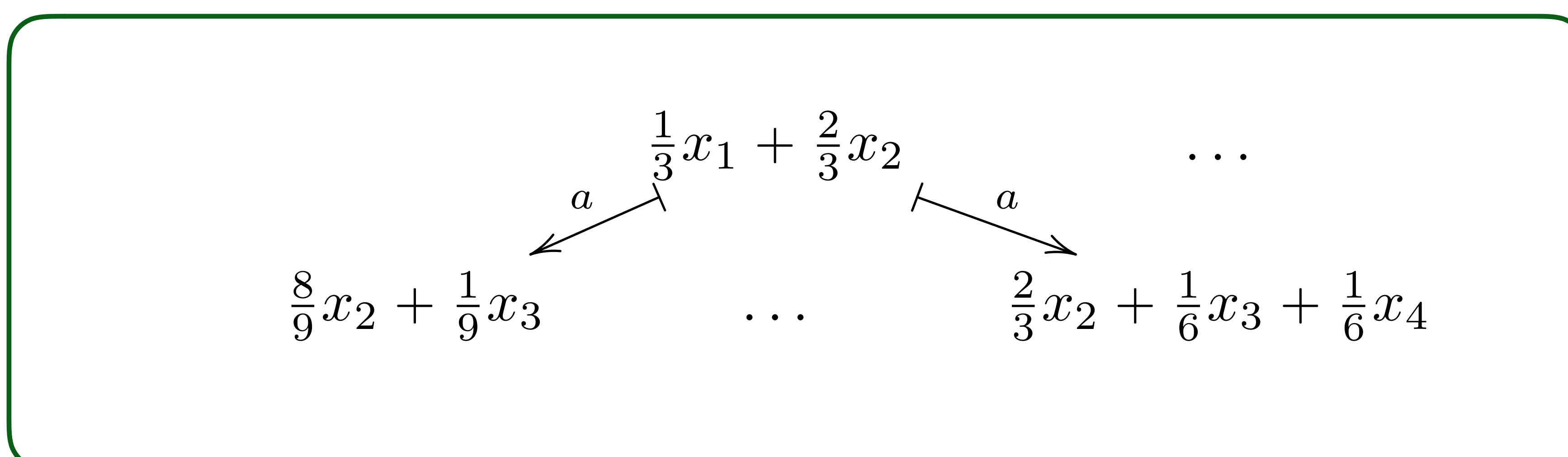
how does it emerge?

what is it?

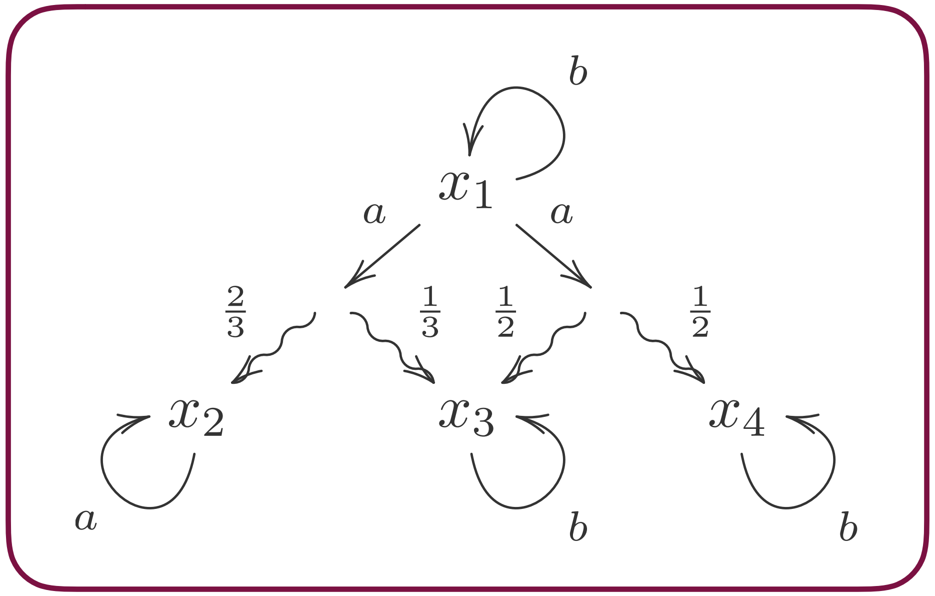


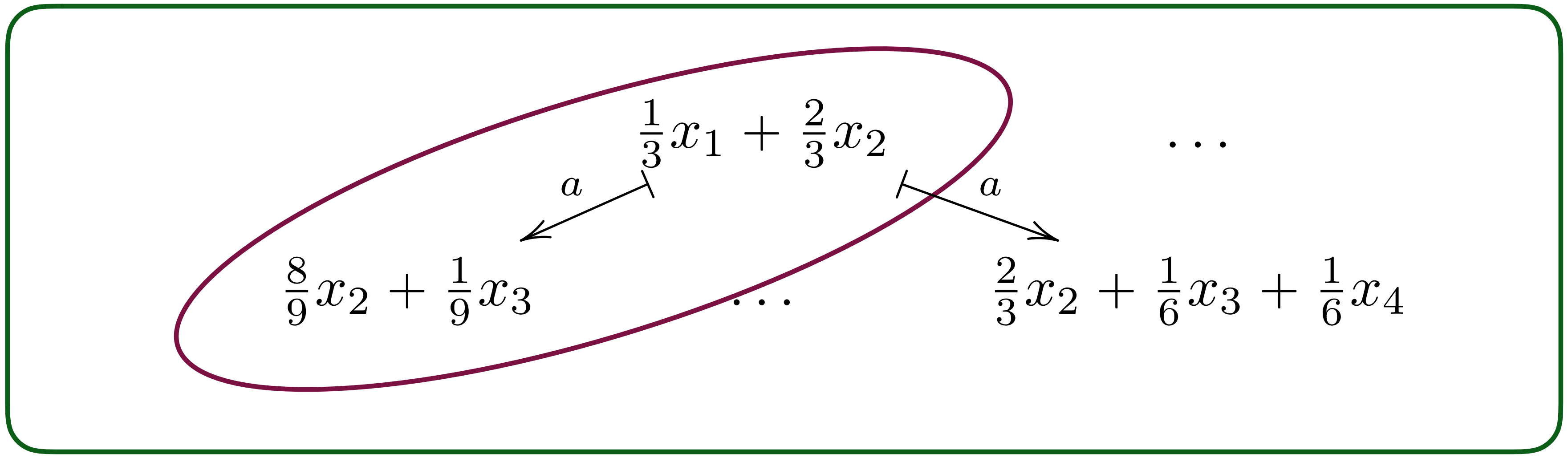
Belief-state transformer





Belief-state transformer





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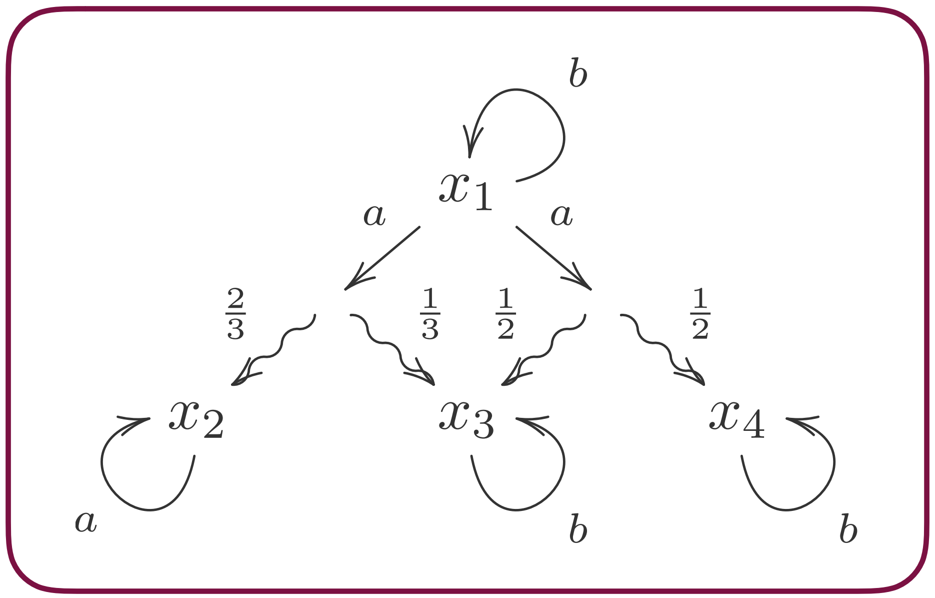
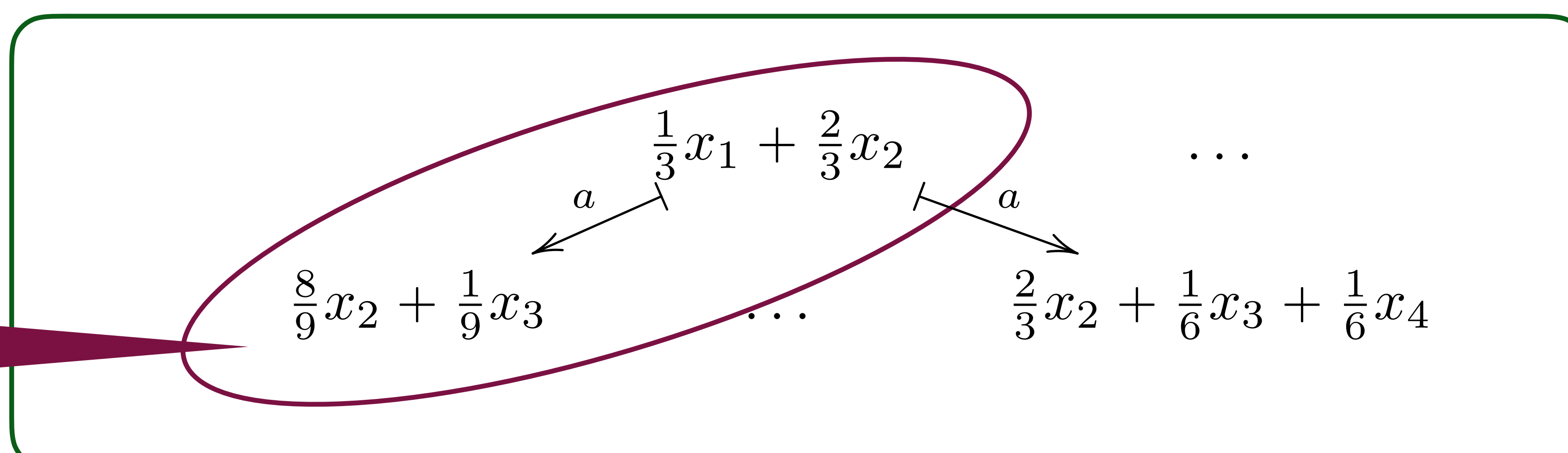
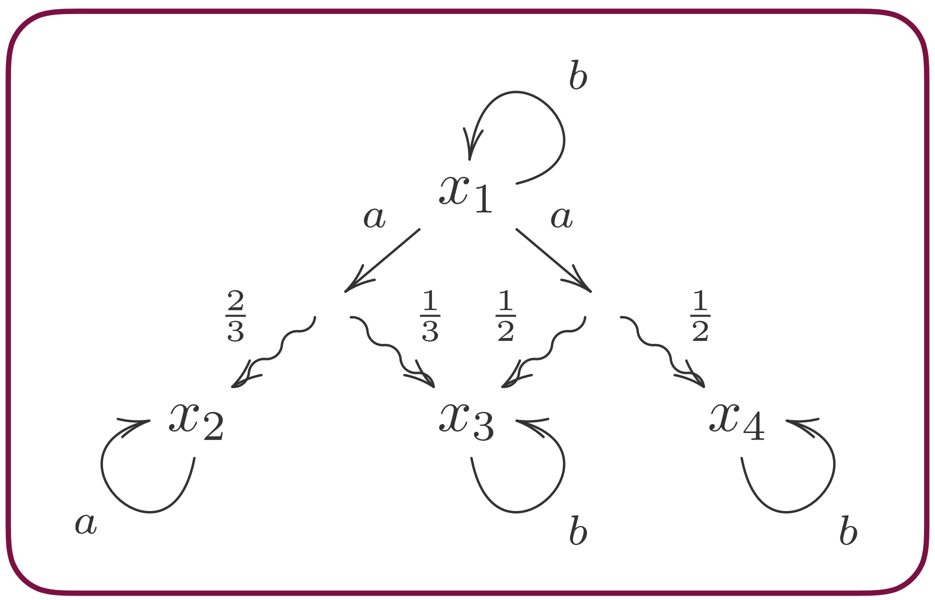
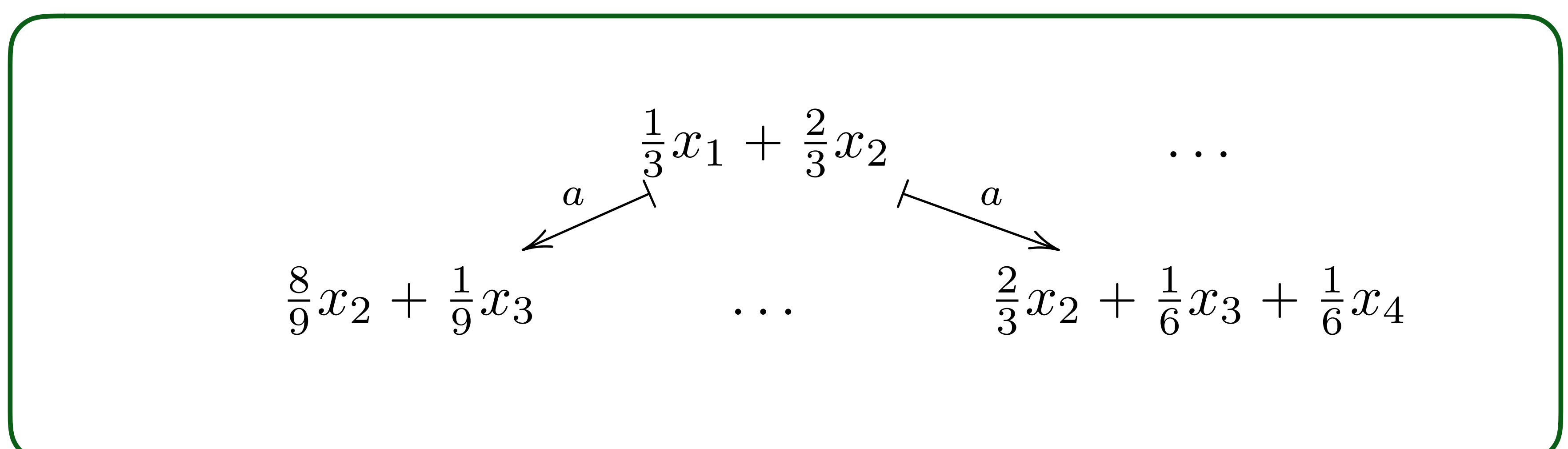




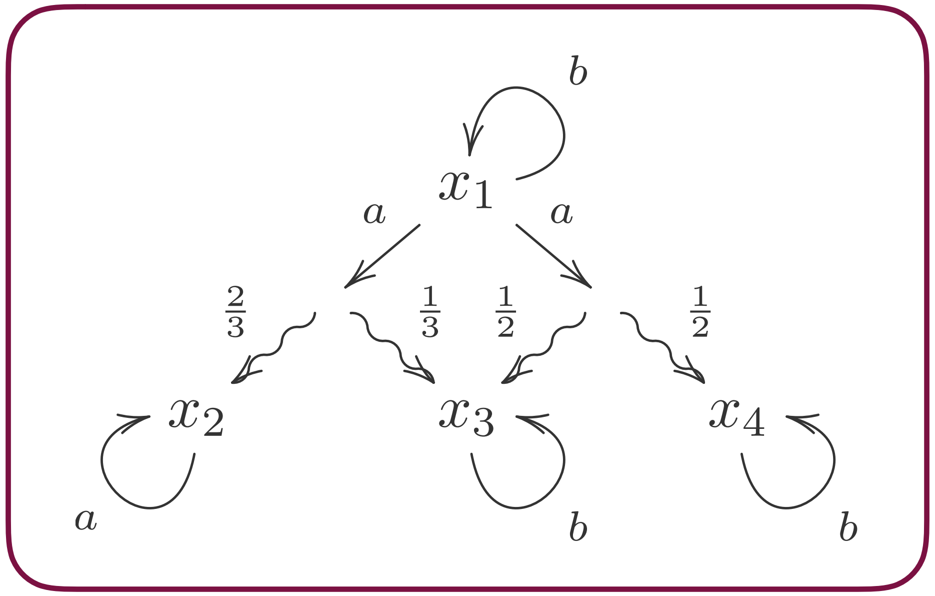
$$\frac{1}{3}\left(\frac{2}{3}x_2 + \frac{1}{3}x_3\right) + \frac{2}{3}(1x_2)$$

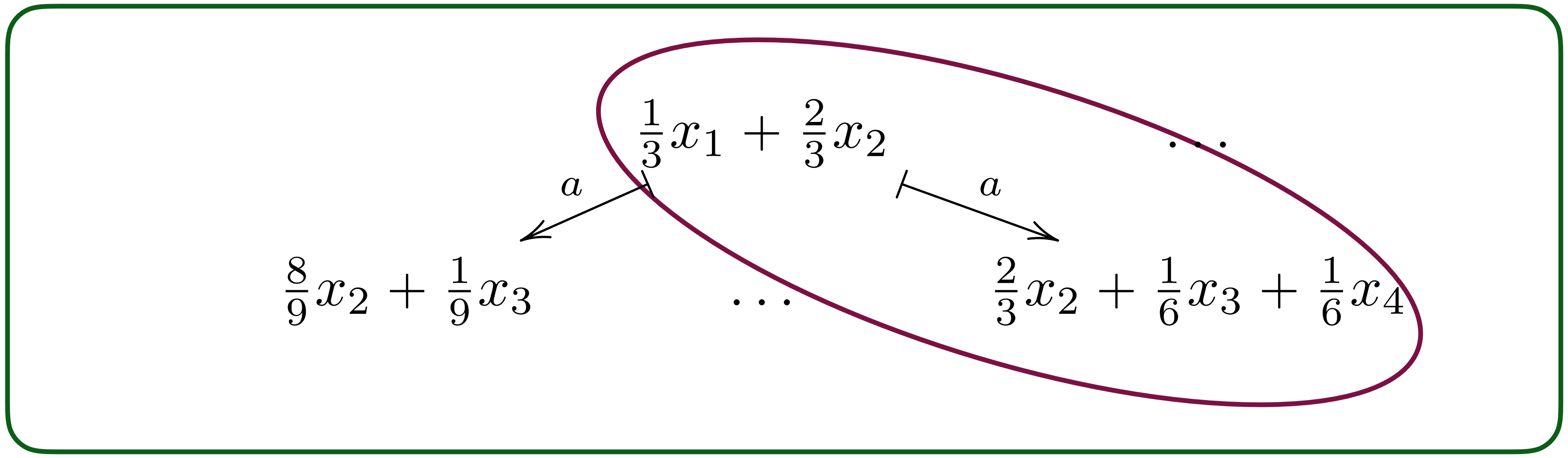
Belief-state transformer





Belief-state transformer





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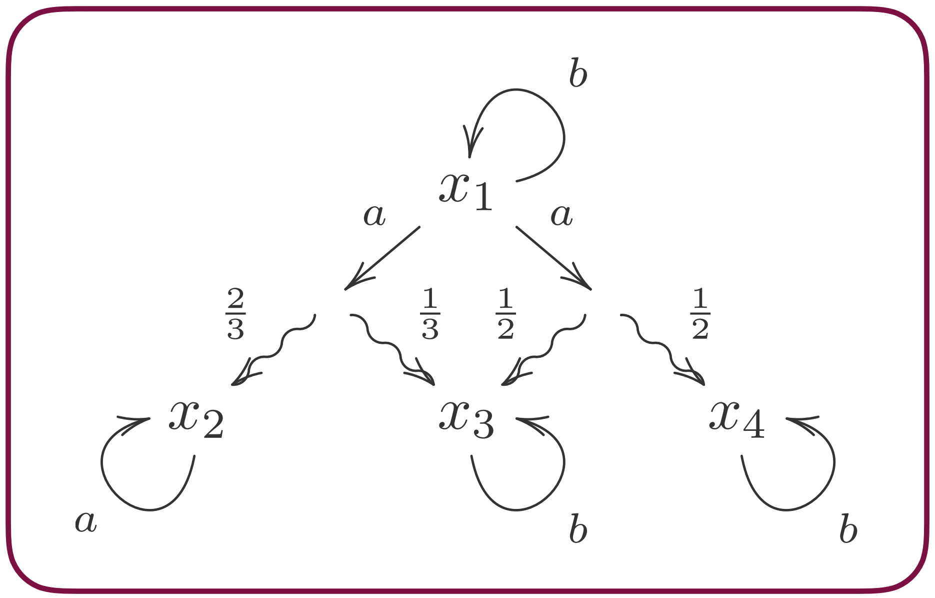
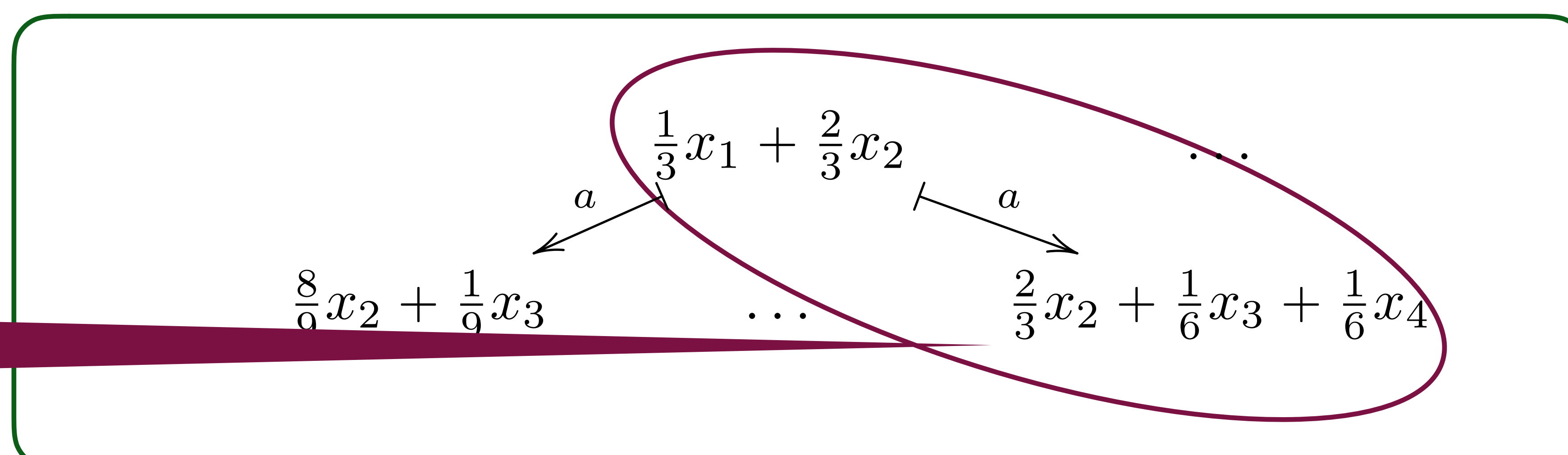


$$\frac{1}{3}\left(\frac{1}{2}x_3 + \frac{1}{2}x_4\right) + \frac{2}{3}(1x_2)$$



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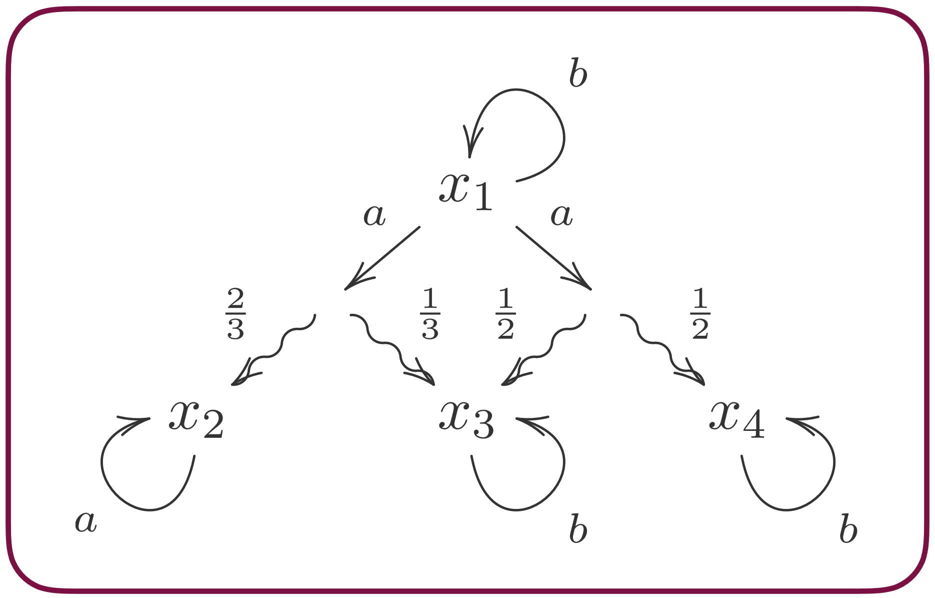

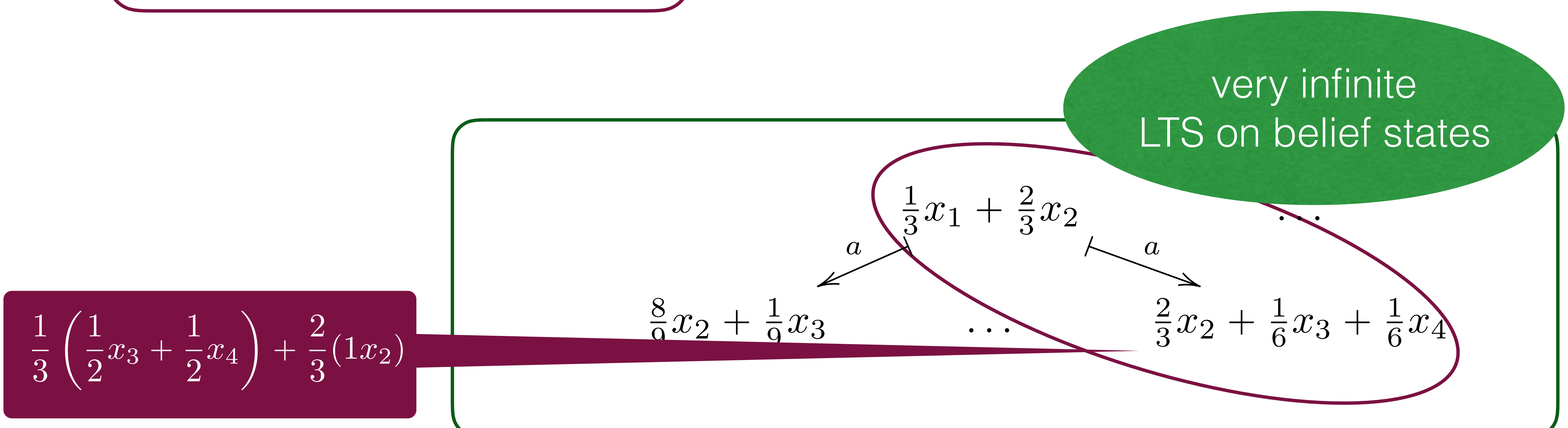

Probabilistic Automata

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity

Probabilistic Automata

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strong bisimilarity

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity
2. Convex bisimilarity
3. Distribution bisimilarity

strong bisimilarity

probabilistic / combined bisimilarity

Probabilistic Automata

Can be given different semantics:

1. Bisimilarity — strong bisimilarity
2. Convex bisimilarity — probabilistic / combined bisimilarity
3. Distribution bisimilarity — belief-state bisimilarity

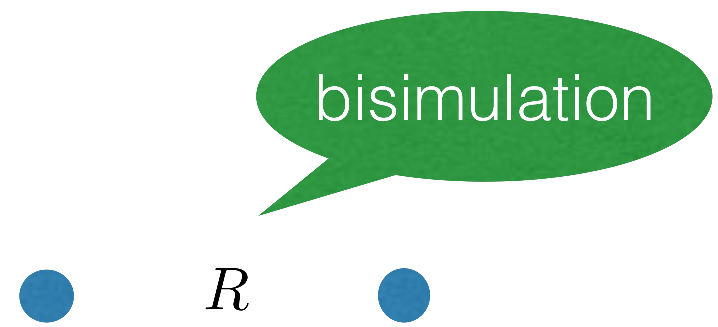
Bisimilarity

Bisimilarity

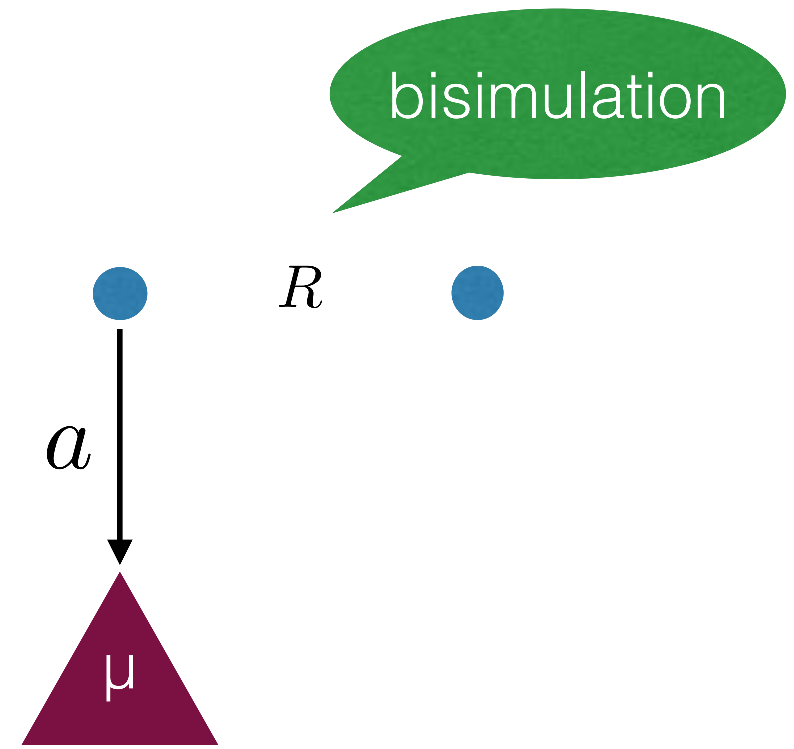
bisimulation

R

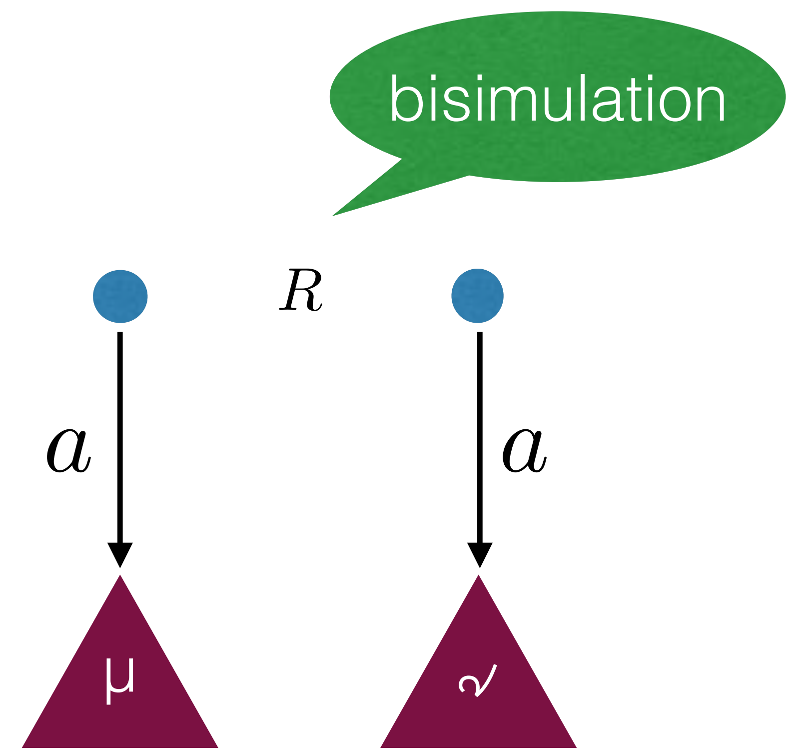
Bisimilarity



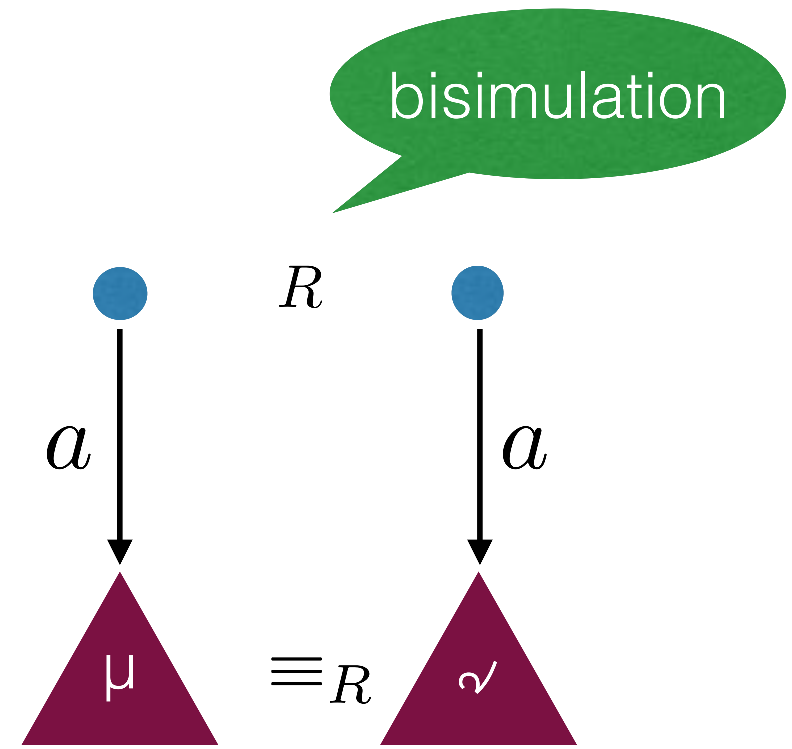
Bisimilarity



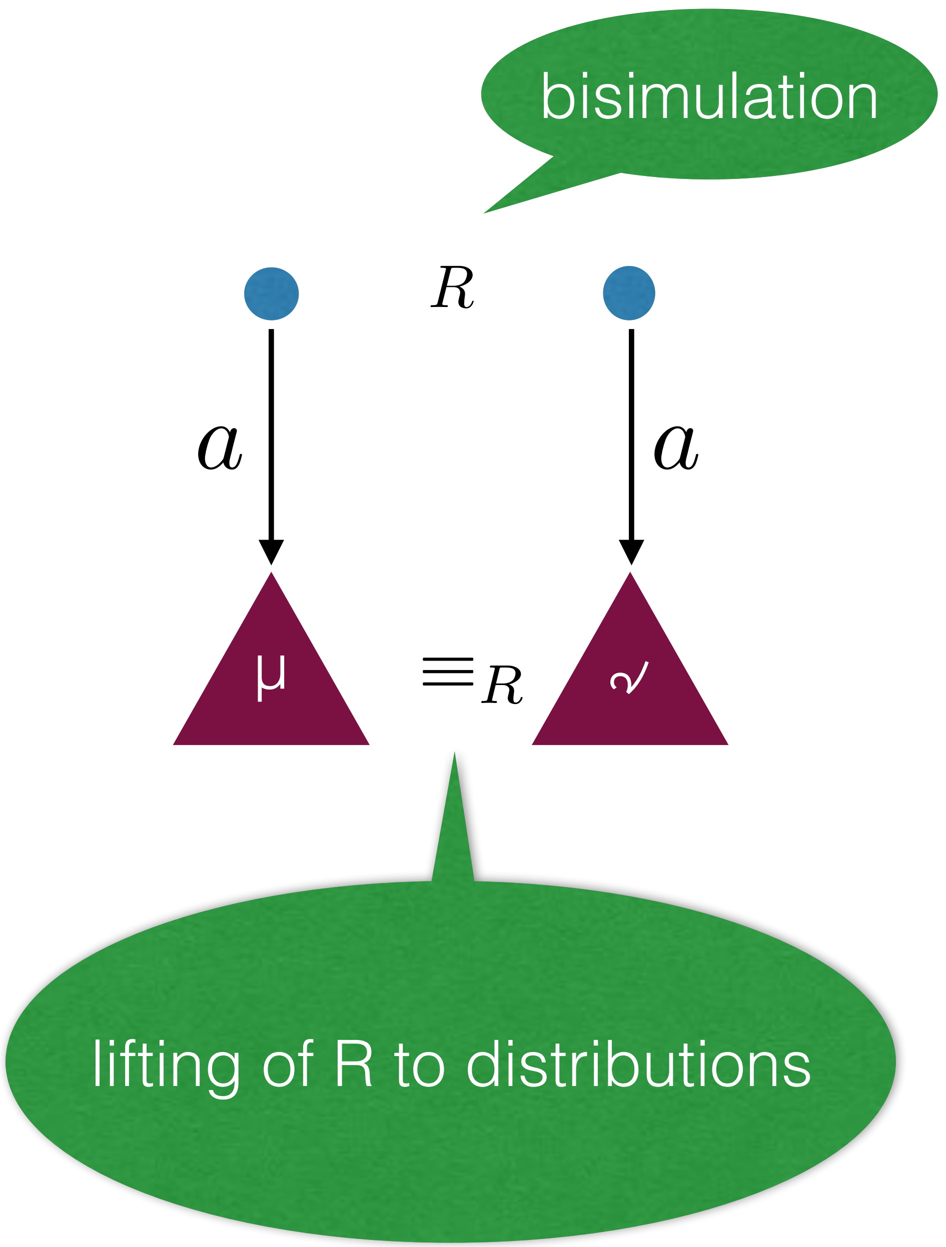
Bisimilarity



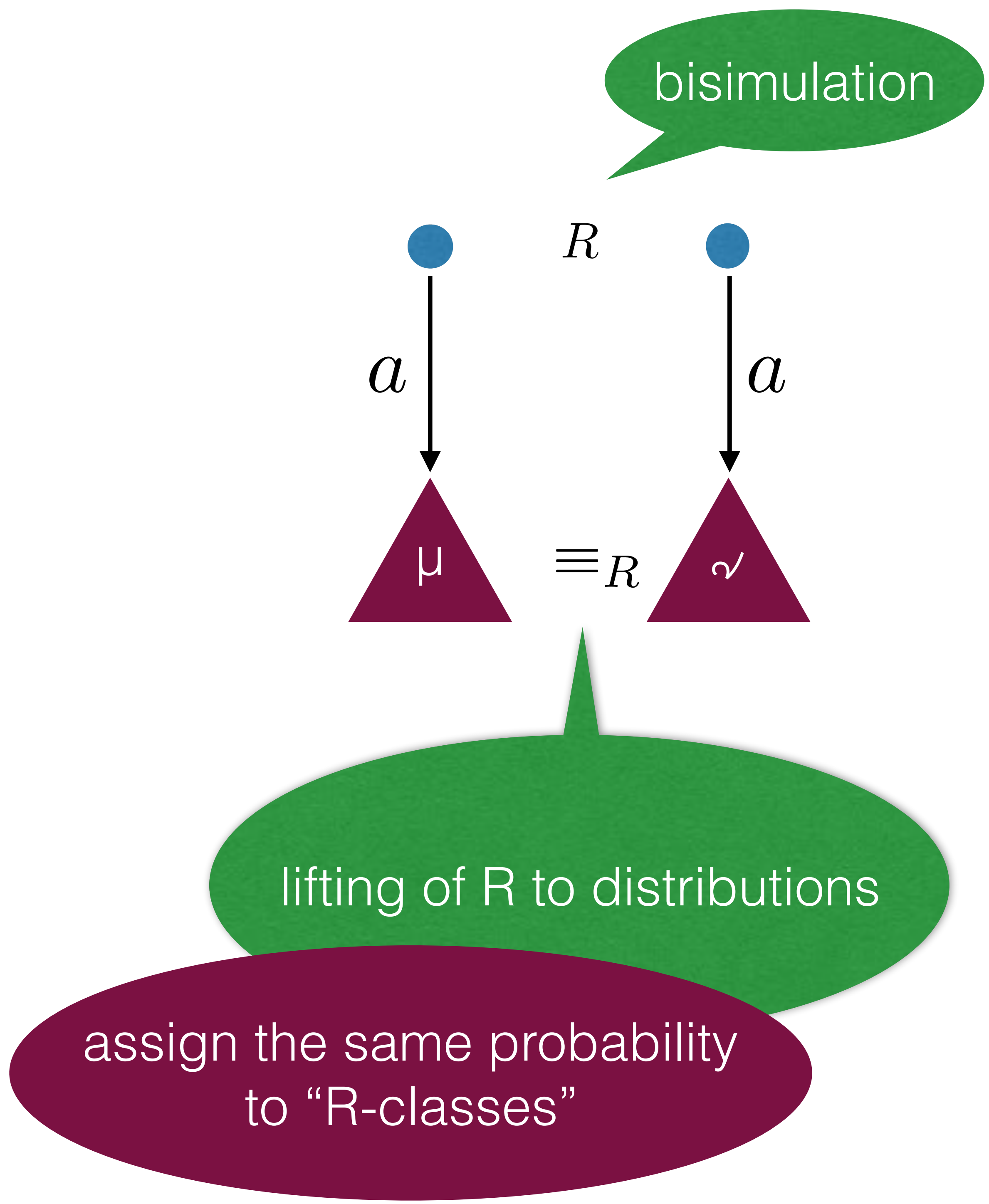
Bisimilarity



Bisimilarity



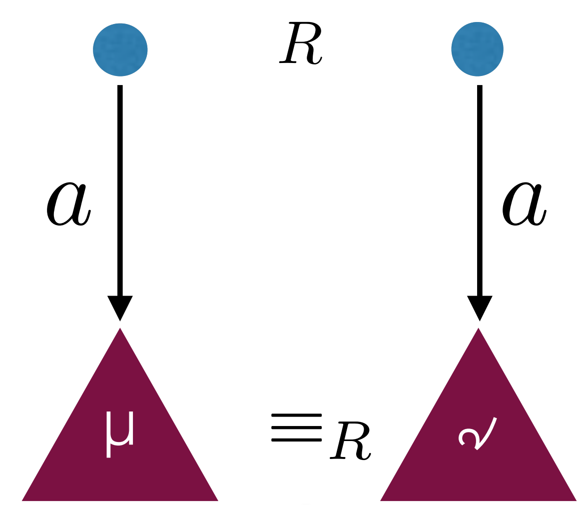
Bisimilarity



Bisimilarity

$\sim$ largest bisimulation

bisimulation



lifting of R to distributions

assign the same probability to "R-classes"

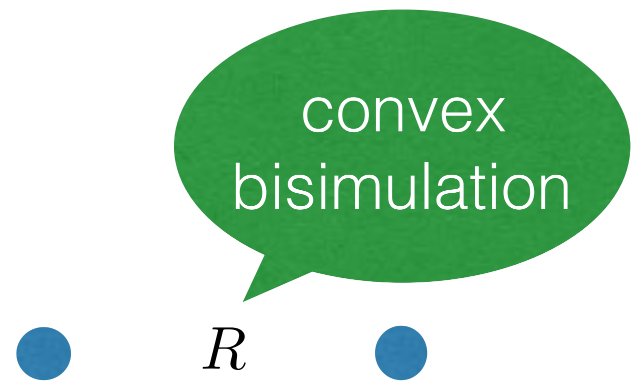
Convex bisimilarity

Convex bisimilarity

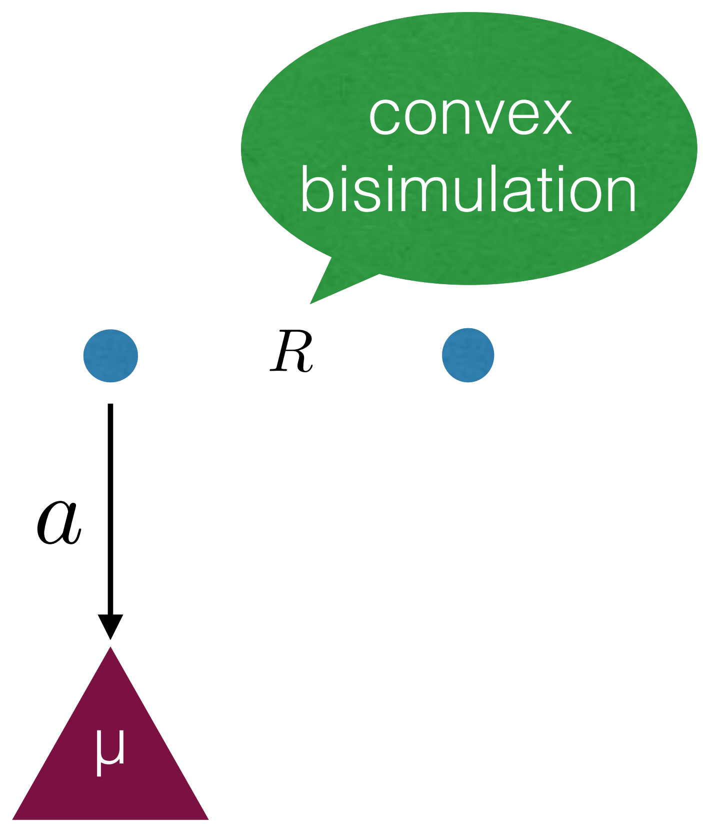
convex bisimulation

R

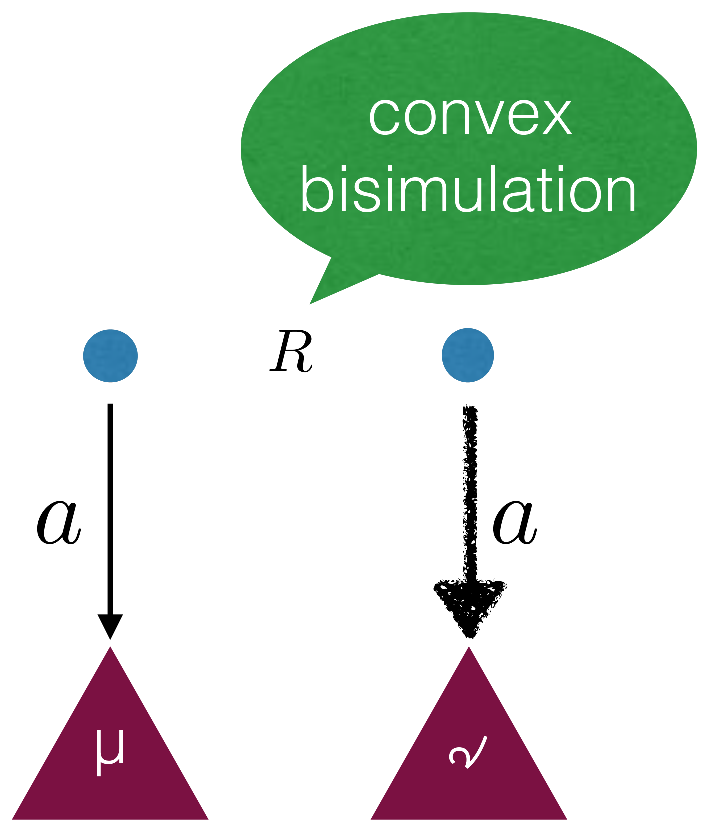
Convex bisimilarity



Convex bisimilarity



Convex bisimilarity



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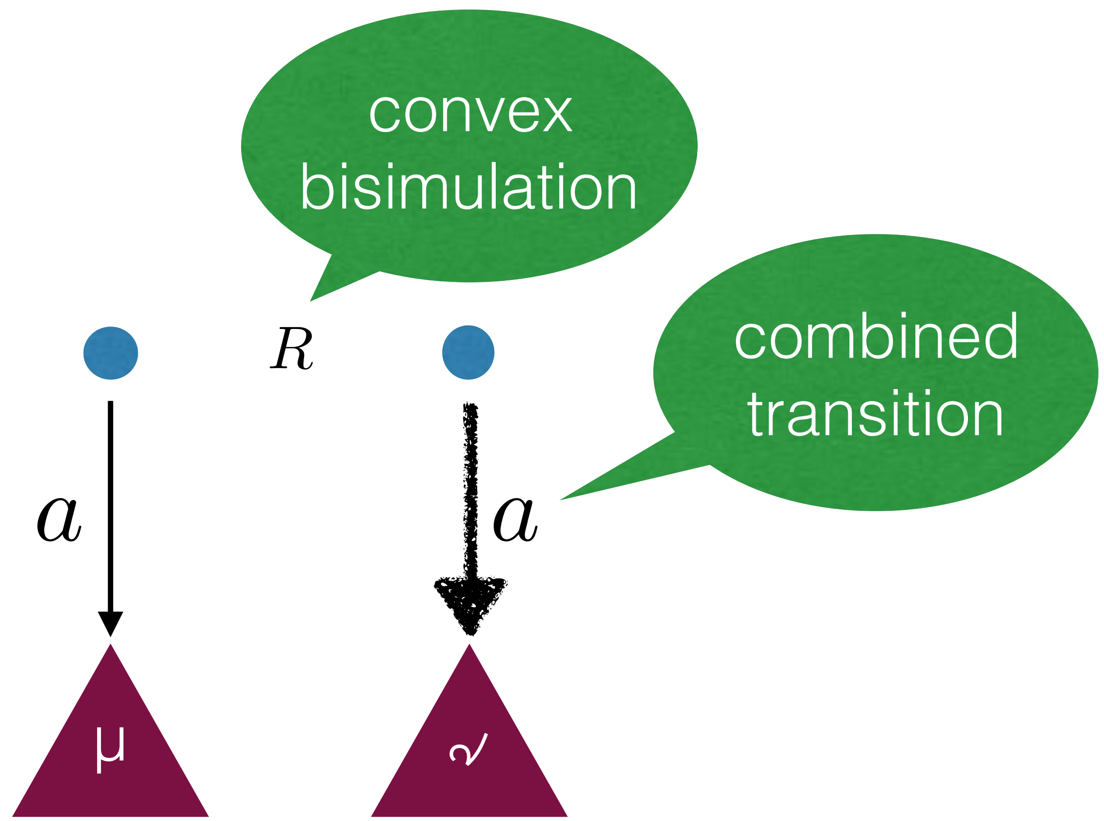
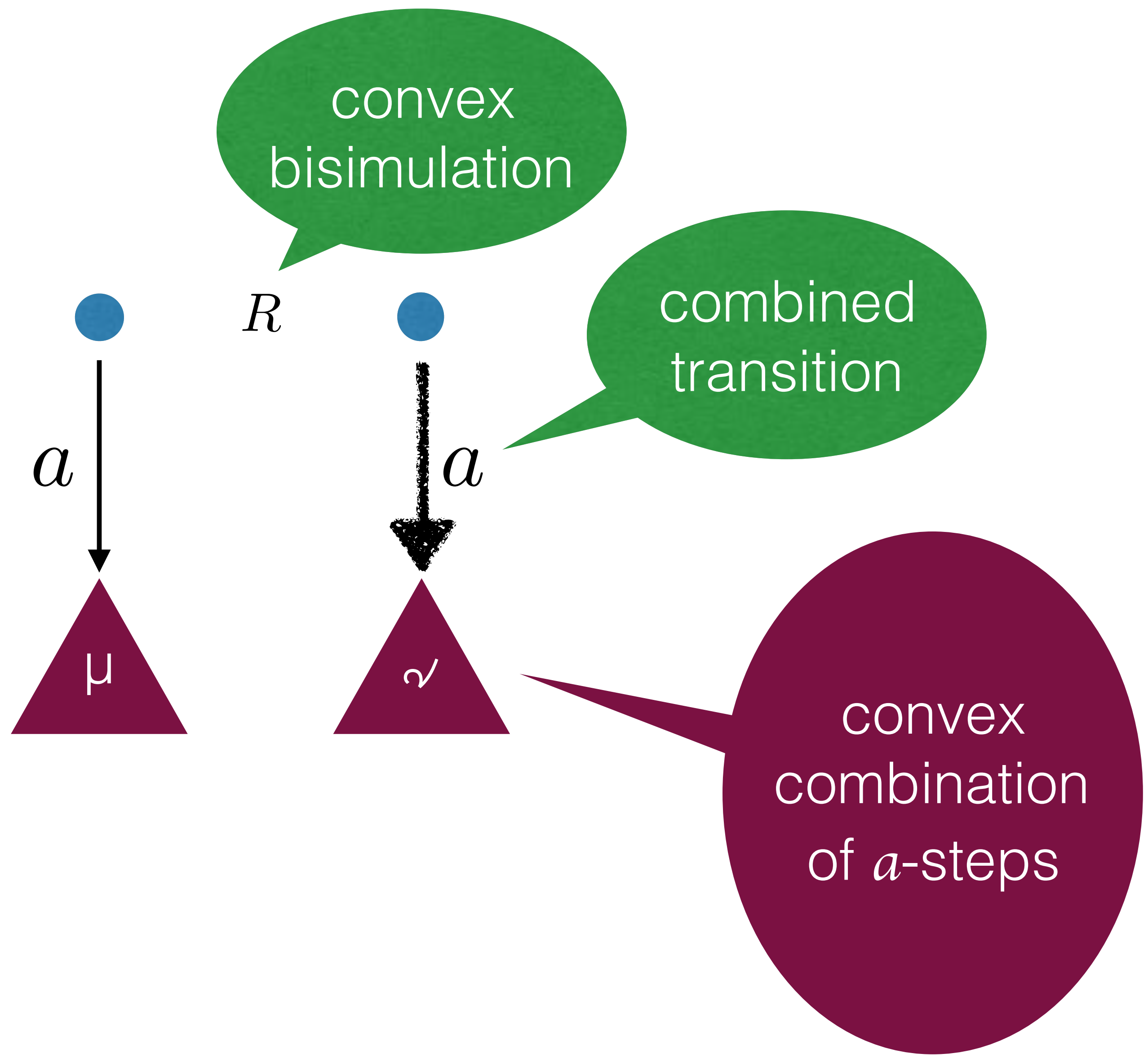
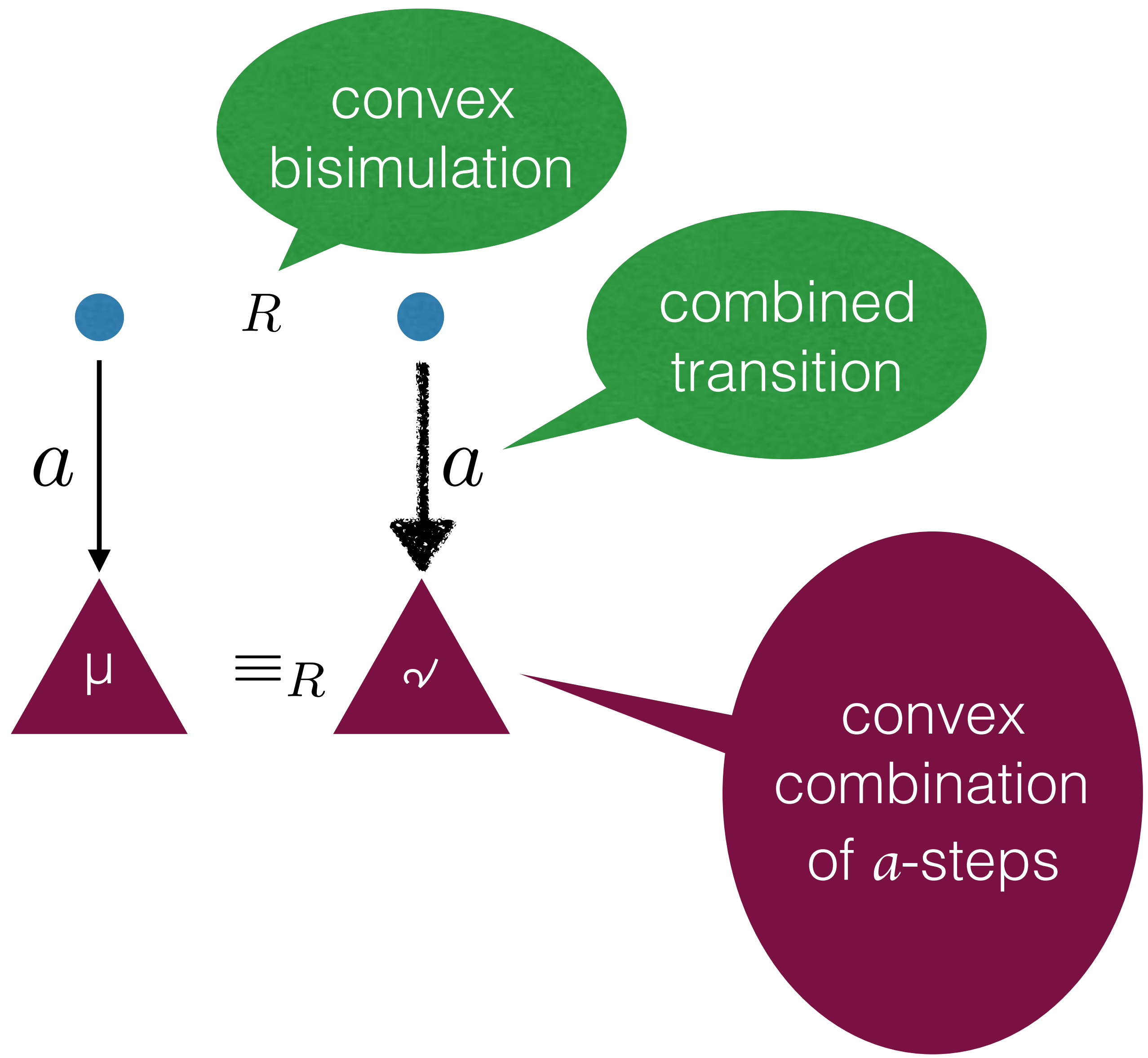


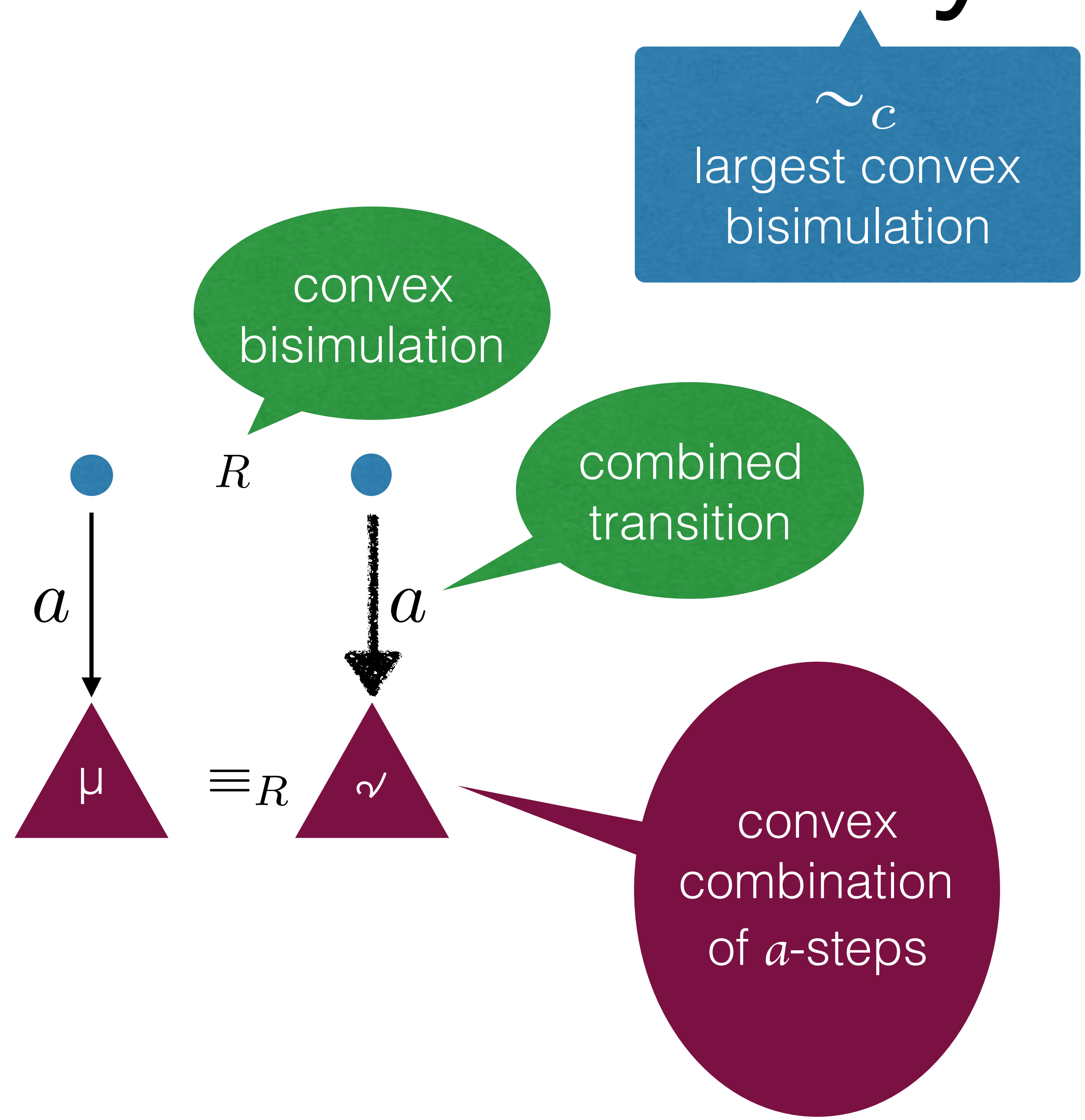
Convex bisimilarity



Convex bisimilarity



Convex bisimilarity



Distribution bisimilarity

[Hermanns, Krcal, Kretinsky CONCUR'13]

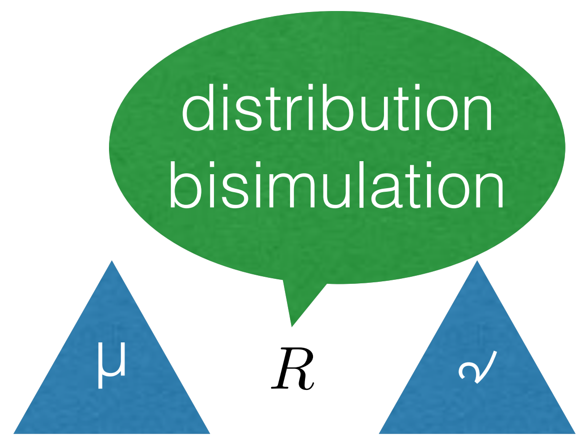
Distribution bisimilarity

distribution bisimulation

R

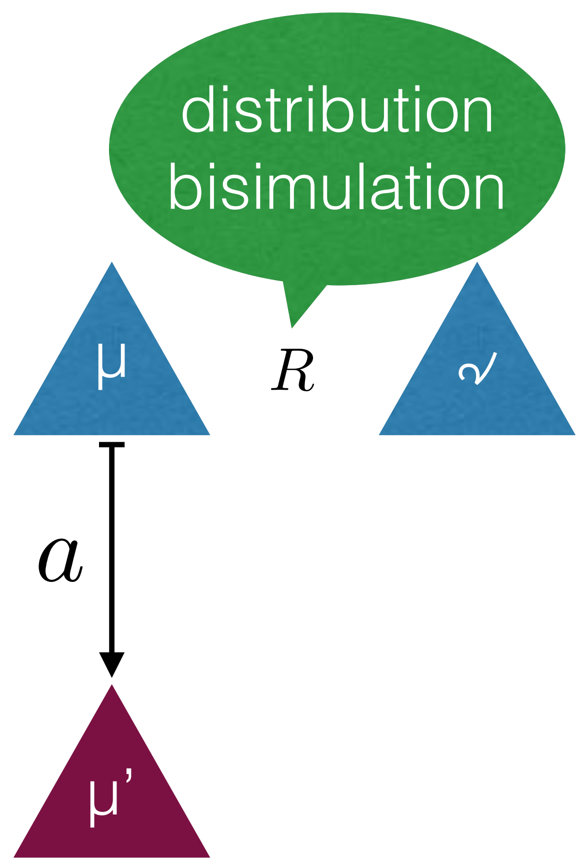
[Hermanns, Krcal, Kretinsky CONCUR'13]

Distribution bisimilarity



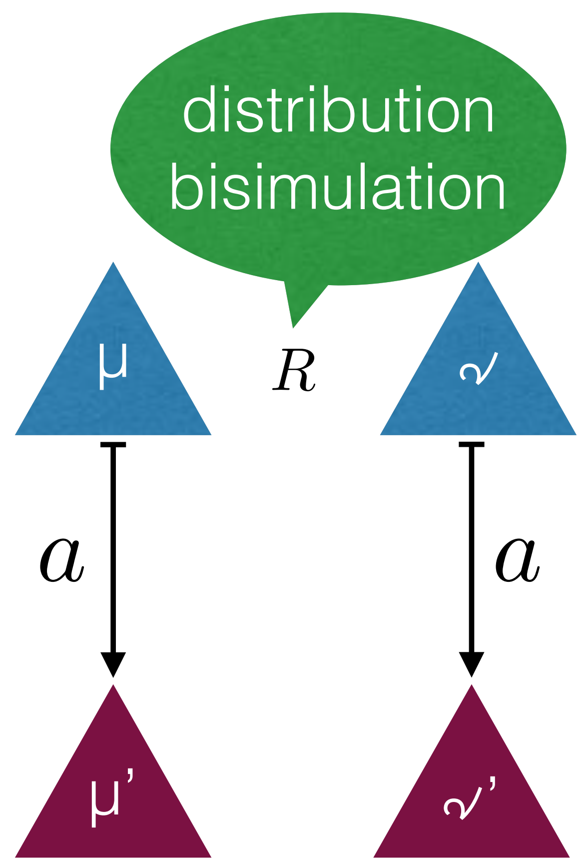
[Hermanns, Krcal, Kretinsky CONCUR'13]

Distribution bisimilarity



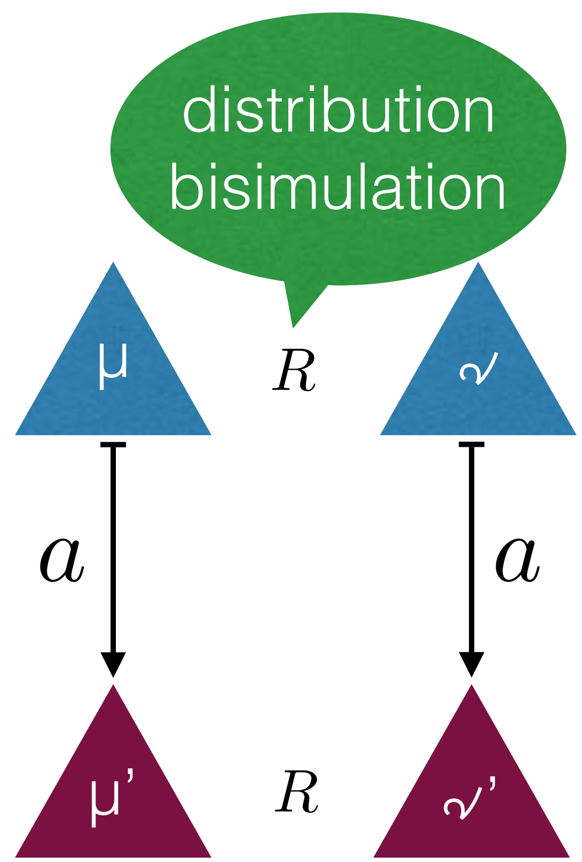
[Hermanns, Krcal, Kretinsky CONCUR'13]

Distribution bisimilarity



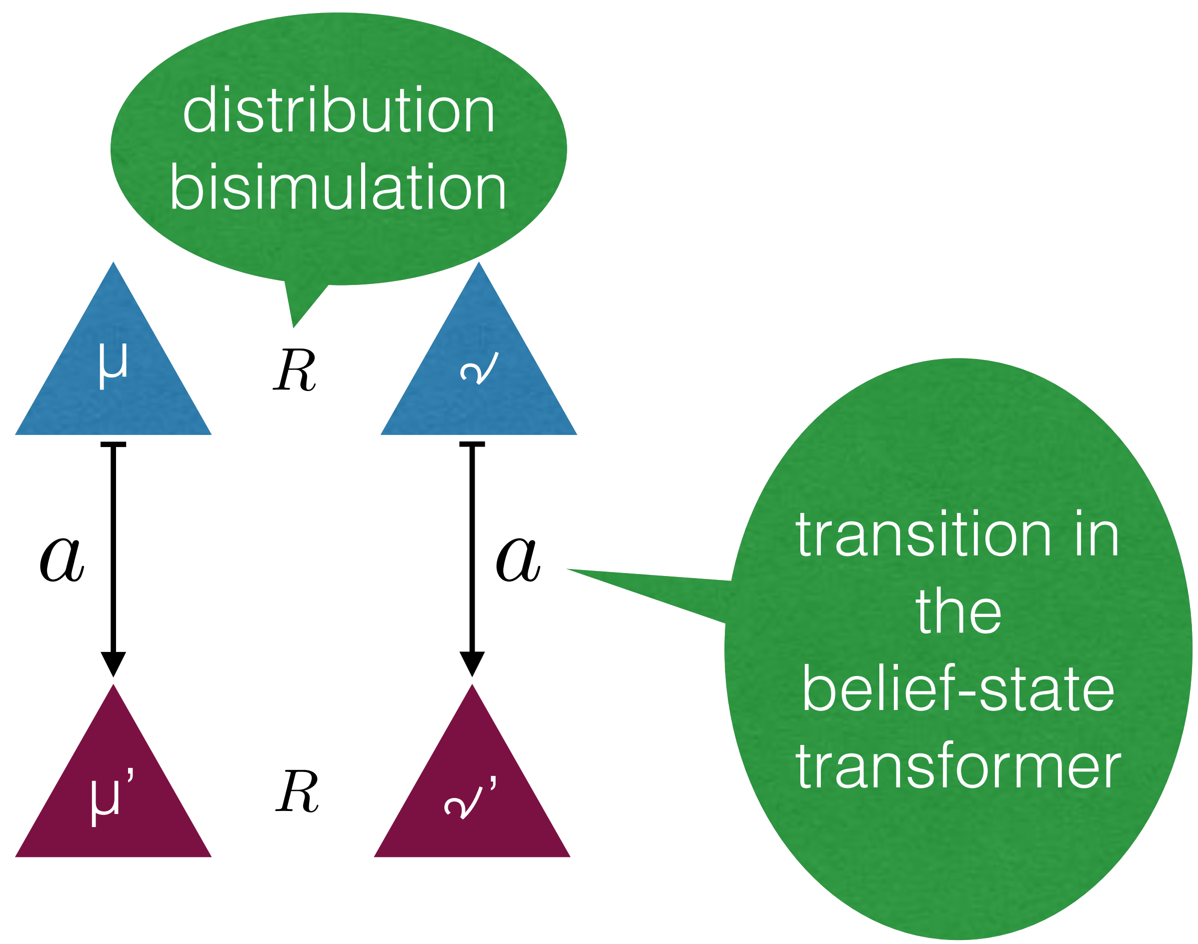
[Hermanns, Krcal, Kretinsky CONCUR'13]

Distribution bisimilarity



[Hermanns, Krcal, Kretinsky CONCUR’13]

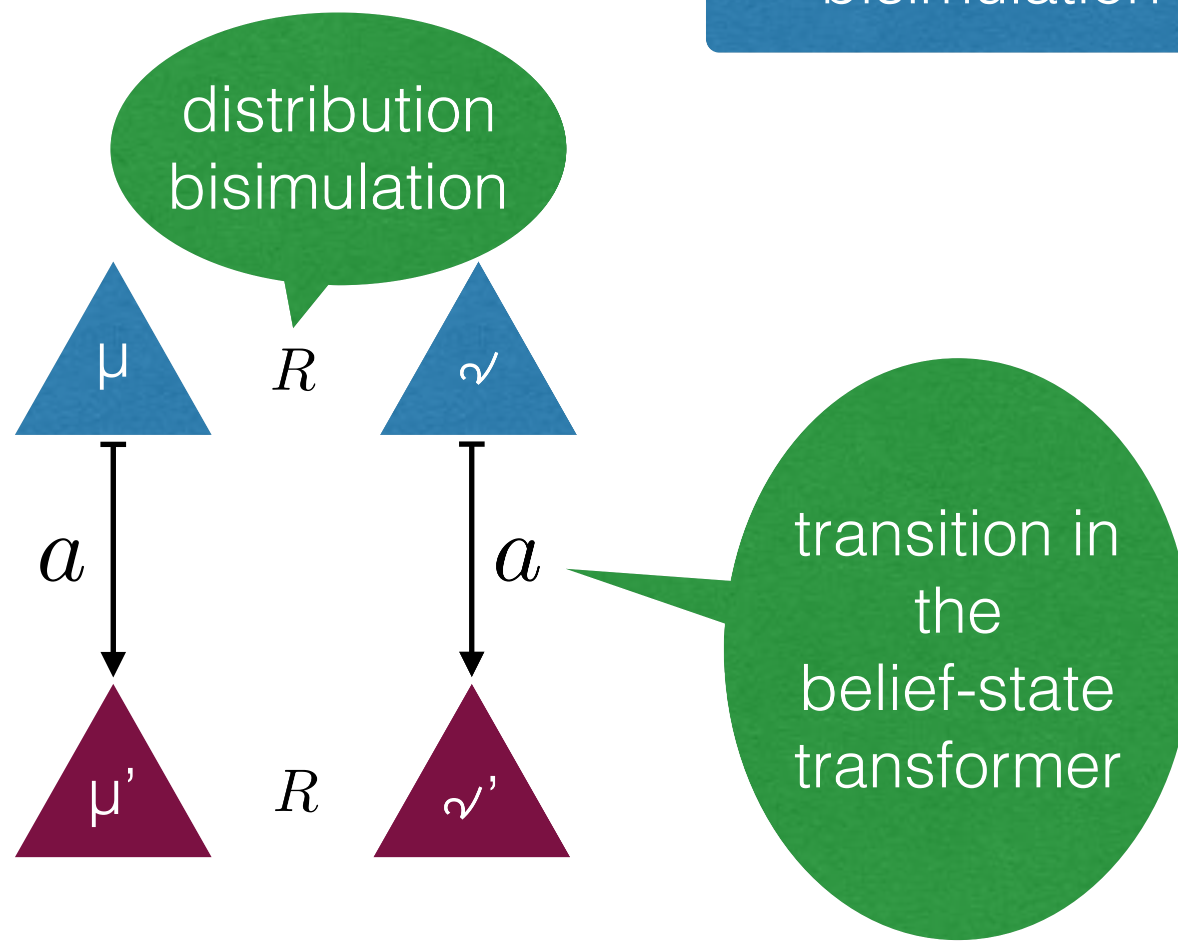
Distribution bisimilarity



[Hermanns, Krcal, Kretinsky CONCUR'13]

Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

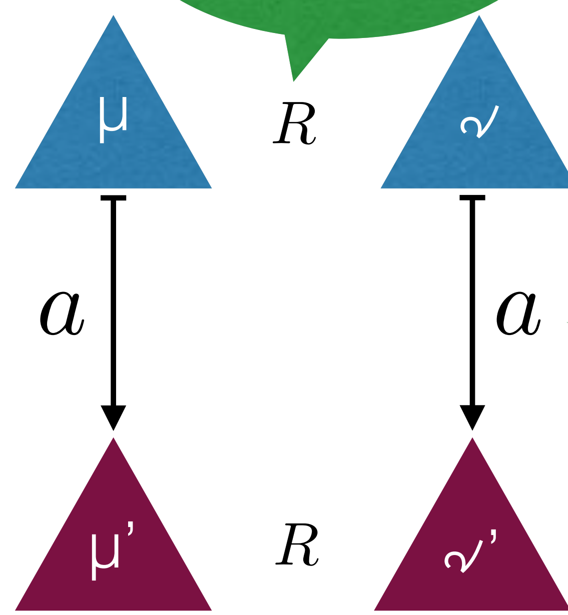


[Hermanns, Krcal, Kretinsky CONCUR’13]

Distribution bisimilarity

$\sim_d$
largest distribution bisimulation

distribution bisimulation



transition in the belief-state transformer

$\sim_d$
is LTS bisimilarity on the belief-state transformer

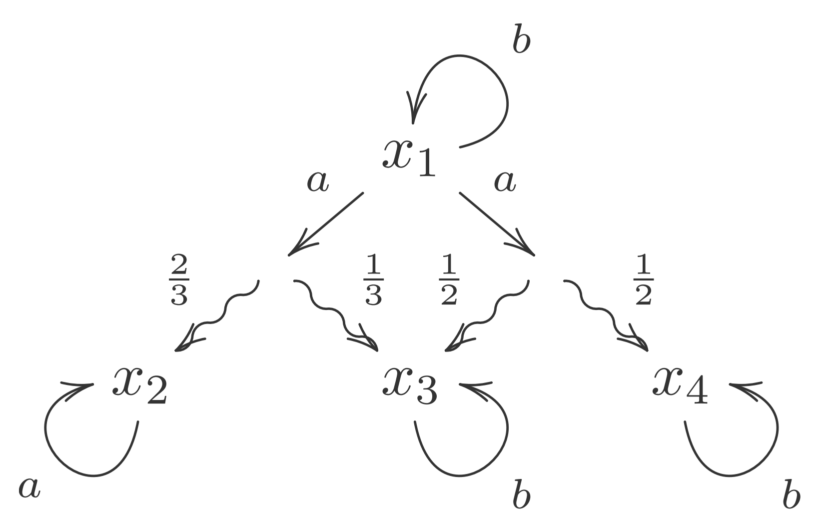
[Hermanns, Krcal, Kretinsky CONCUR'13]

Belief-state transformer

foundation ?

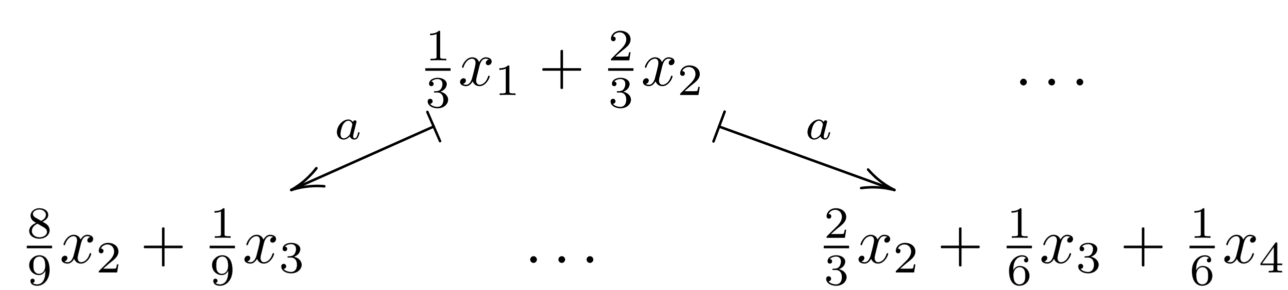
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

what is it?



Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

states

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

states

object in the base category ***C***

Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$$X \xrightarrow{c} FX$$

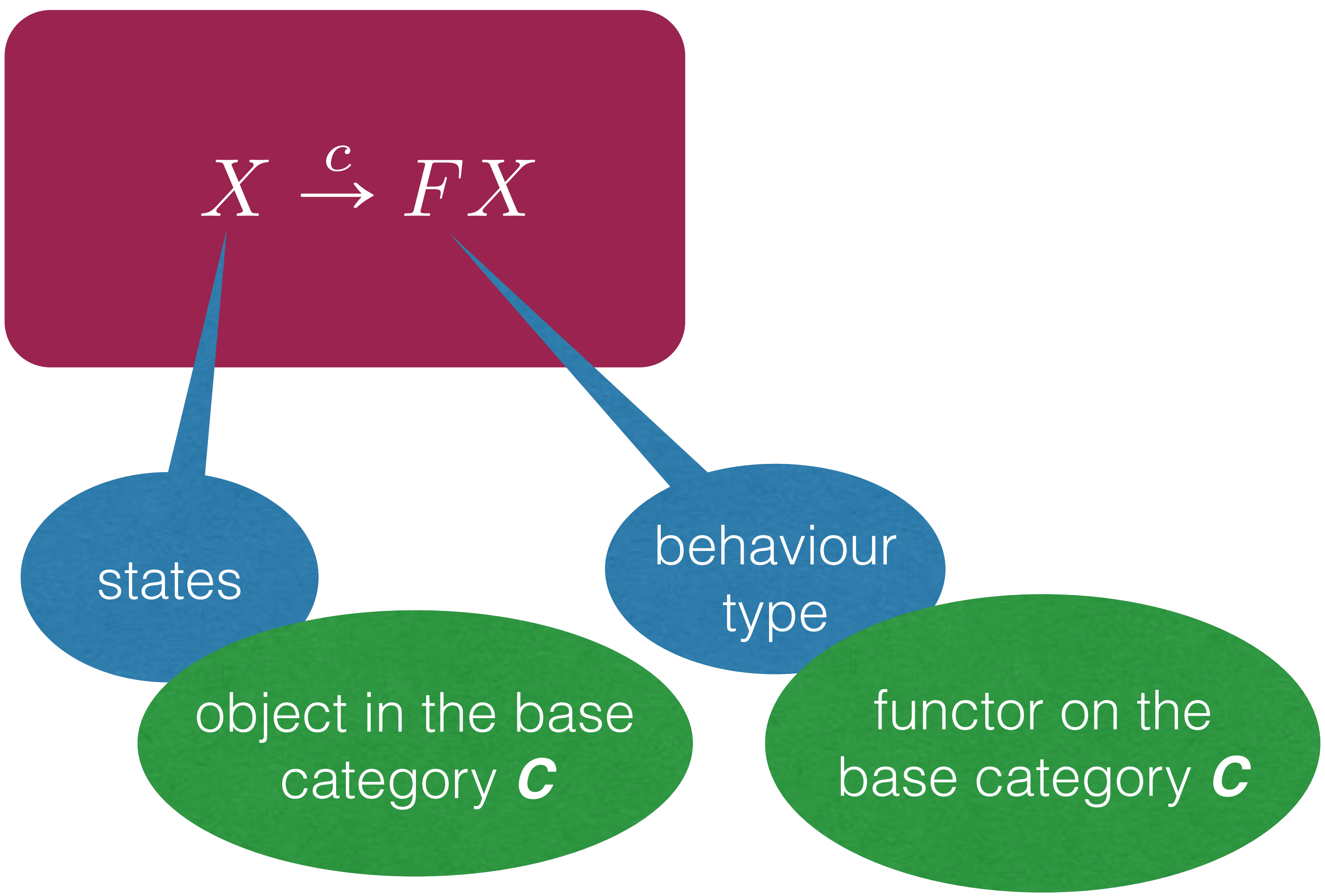
states

object in the base category ***C***

behaviour type

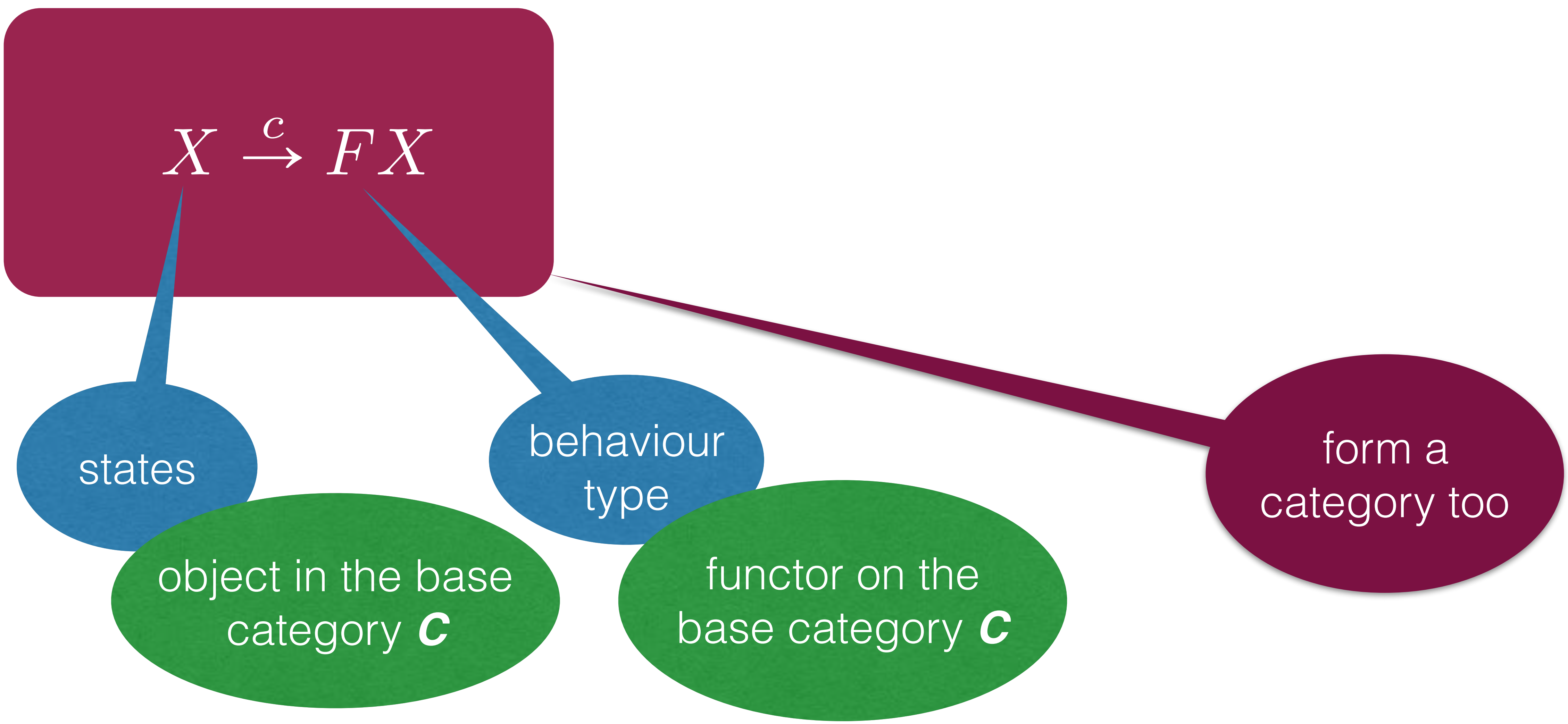
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.



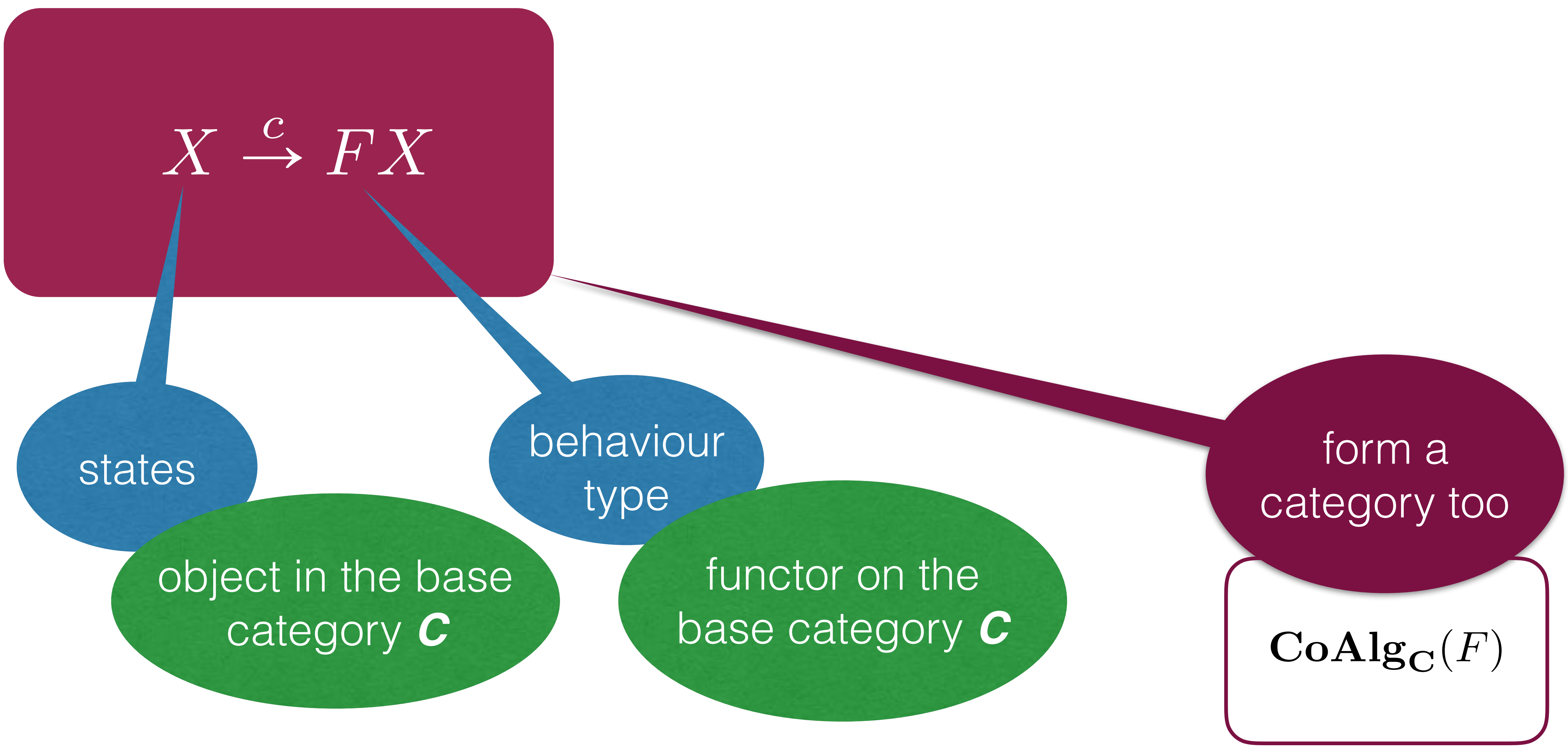
Coalgebras

Uniform framework for dynamic transition systems, based on category theory.



Coalgebras

Uniform framework for dynamic transition systems, based on category theory.



Coalgebras

Uniform framework for dynamic transition systems, based on category theory.

$X \xrightarrow{c} FX$

generic notion of behavioural equivalence

$\approx$

states

object in the base category ***C***

behaviour type

functor on the base category ***C***

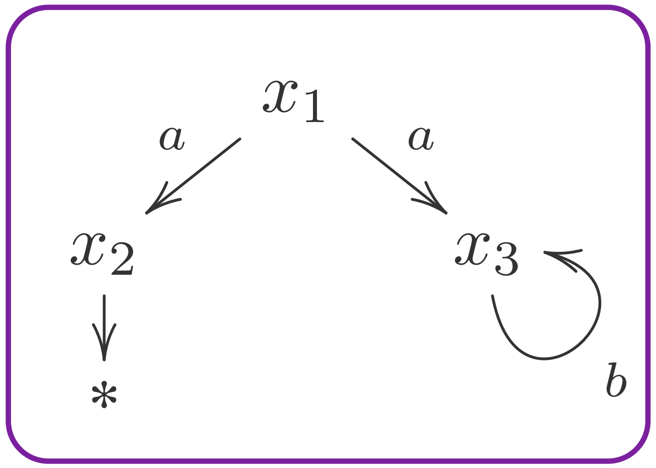
form a category too

$\mathbf{CoAlg_C}(F)$

Examples

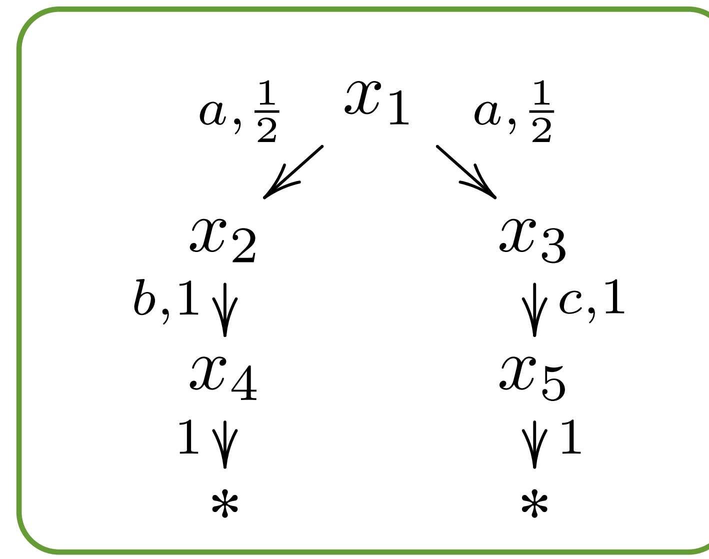
NFA

$X \to 2 \times (\mathcal{P}(X))^A$



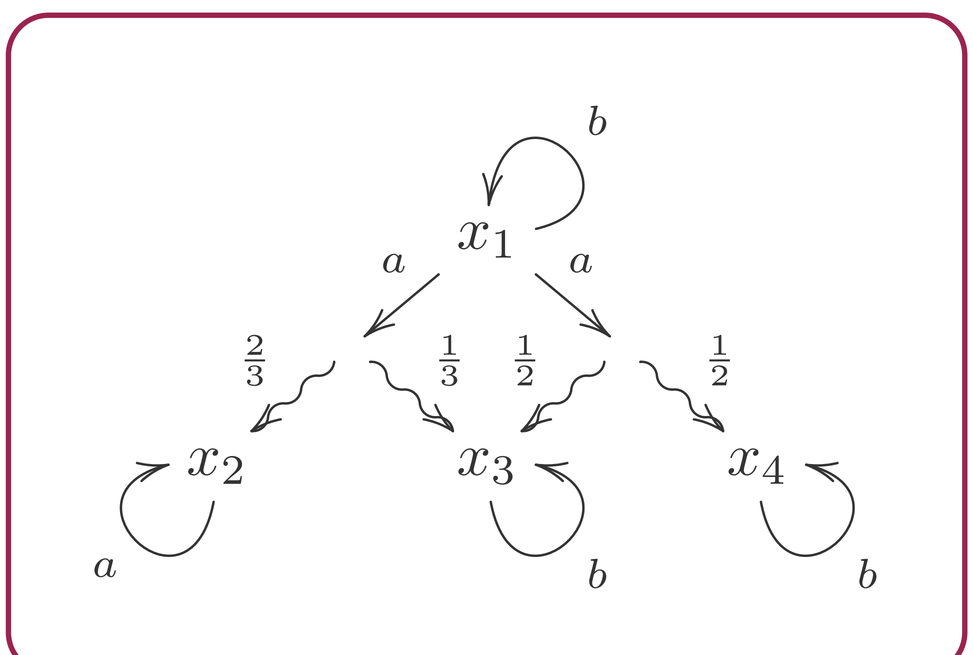
Generative PTS

$X \to \mathcal{D}(1 + A \times X)$



PA

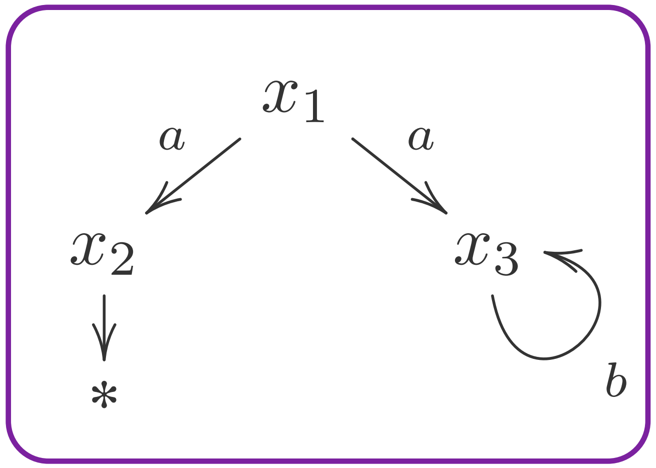
$X \to (\mathcal{P}\mathcal{D}(X))^A$



Examples

NFA

$X \to 2 \times (\mathcal{P}(X))^A$



Generative PTS

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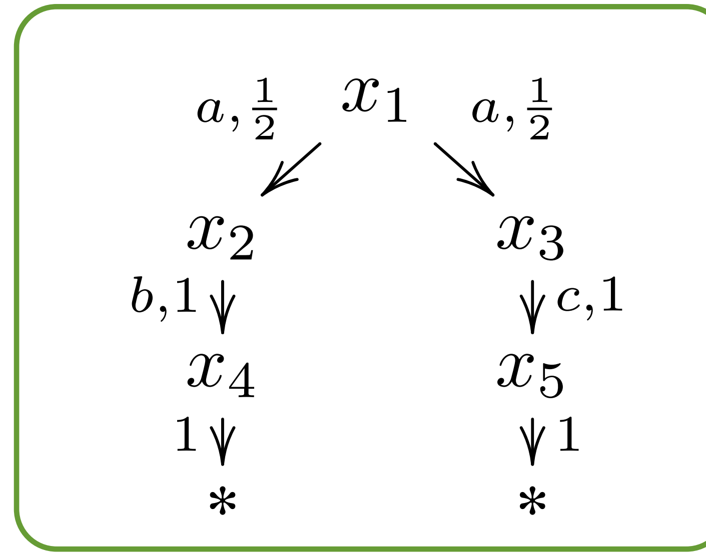

PA

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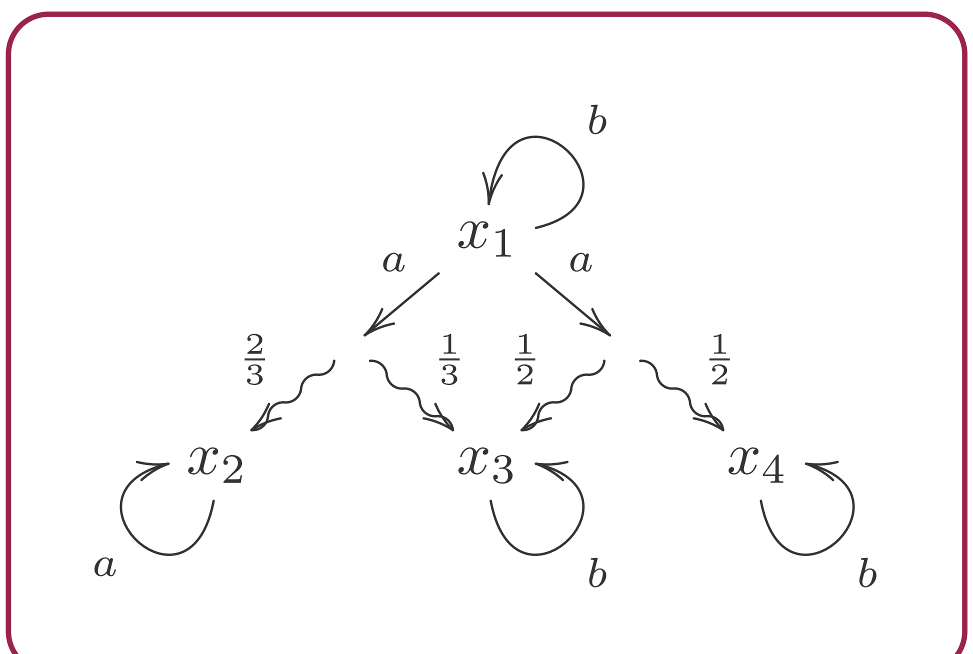
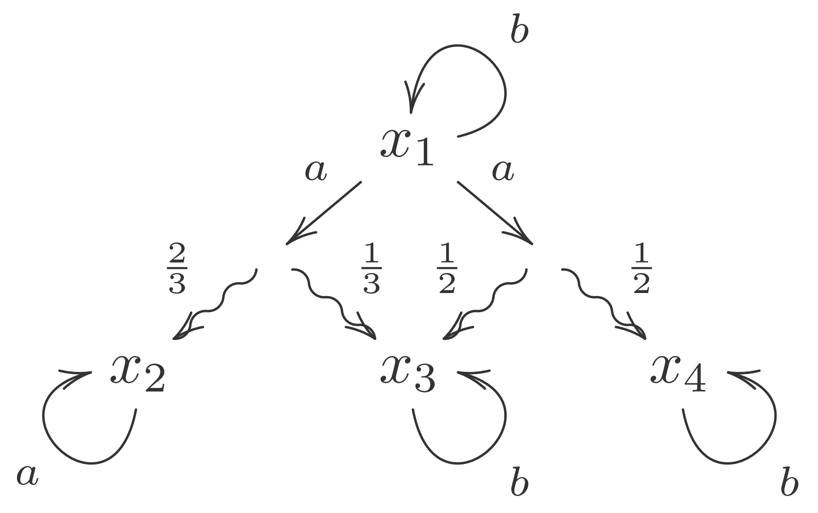


all on **Sets**

PA coalgebraically

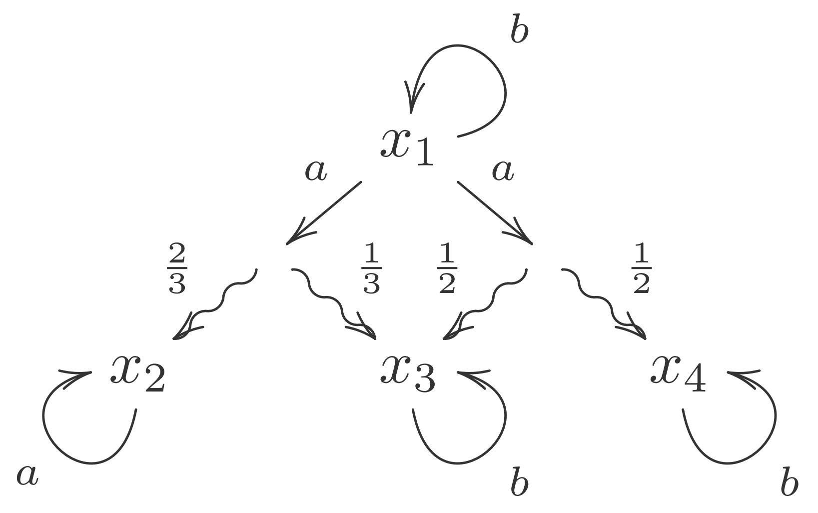
PA coalgebraically

$X \to (\mathcal{P}\,\mathcal{D}(X))^A$



PA coalgebraically

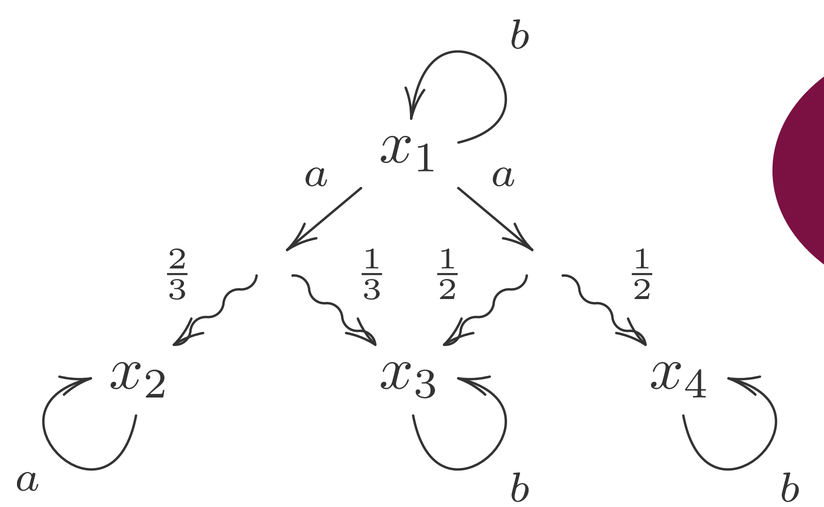
$X \to (\mathcal{P}\mathcal{D}(X))^A$



on **Sets**

PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$

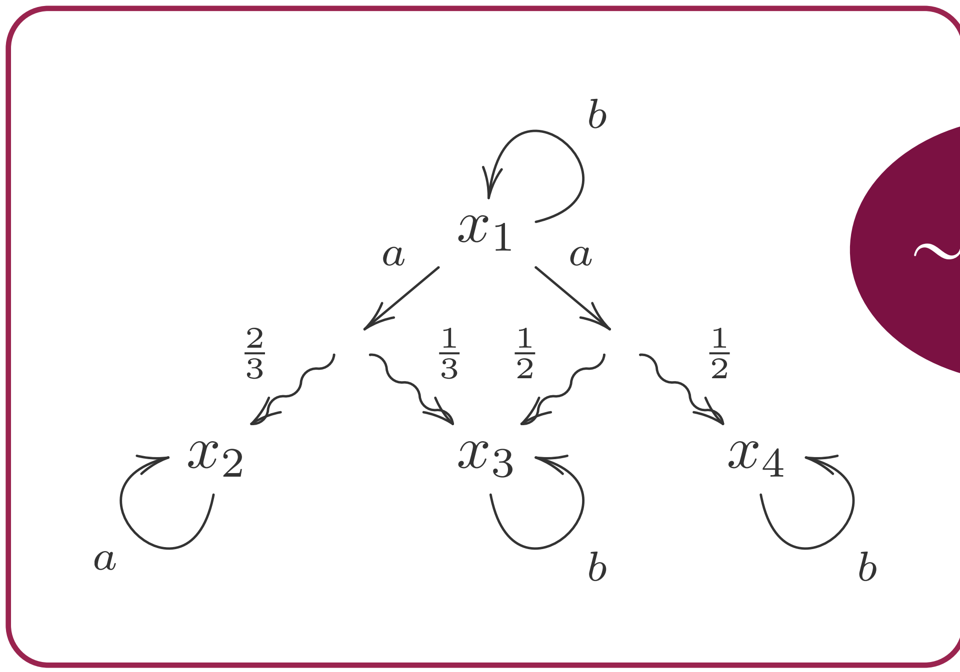


on **Sets**

$\sim \; = \; \approx$

PA coalgebraically

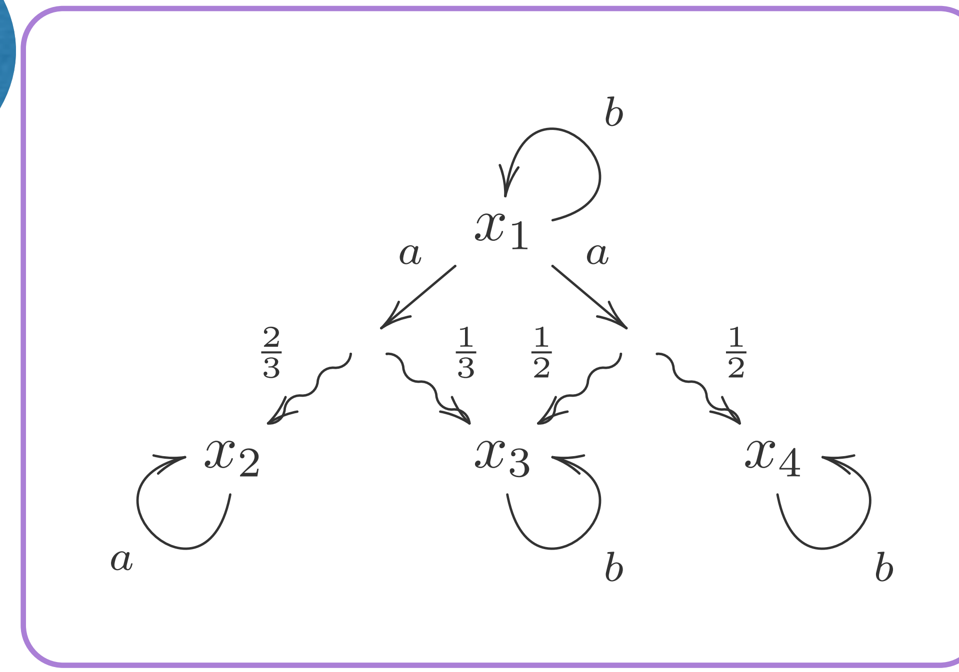
$X \to (\mathcal{P}\mathcal{D}(X))^A$



on **Sets**

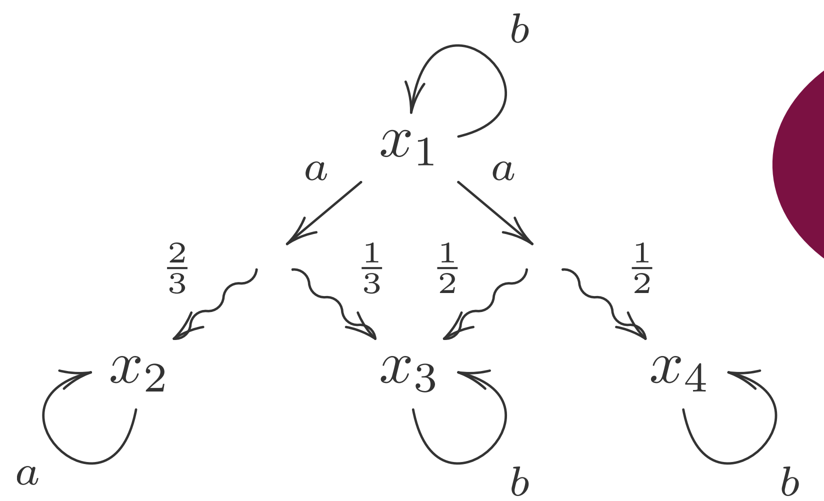
$\sim \, = \, \approx$

$X \to (\mathcal{C}(X))^A$



PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$

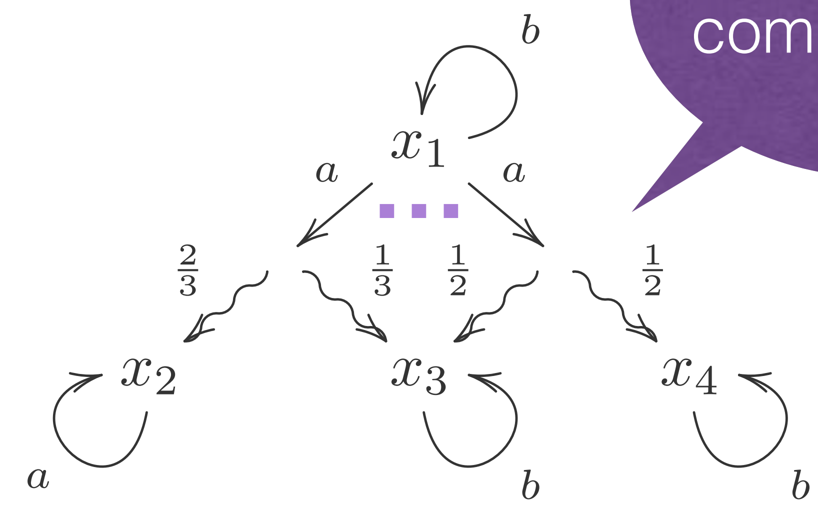


on **Sets**

$\sim \; = \; \approx$

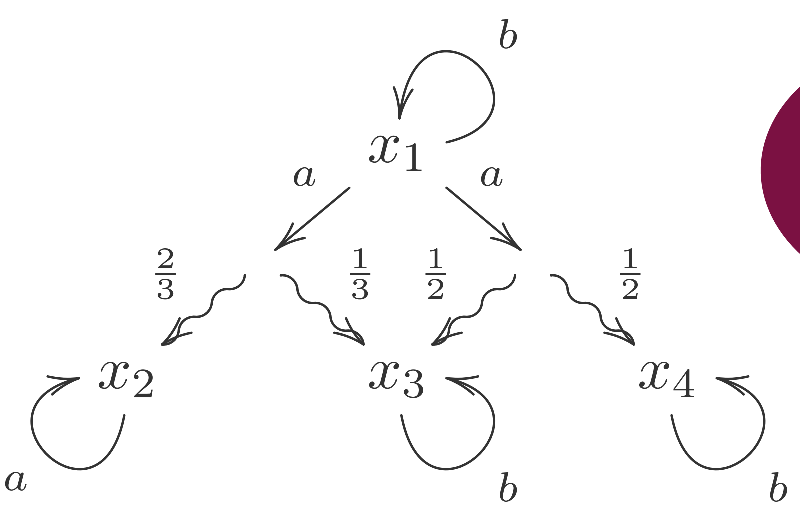
$X \to (\mathcal{C}(X))^A$

and all convex combinations



PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$



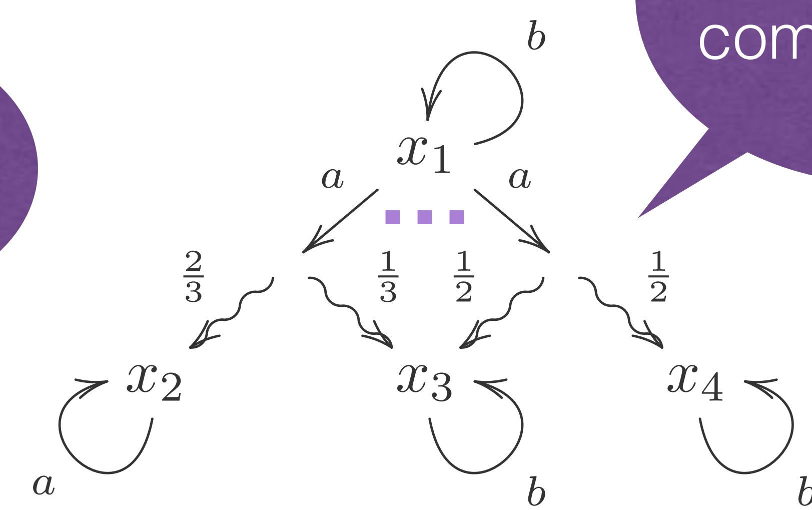
on **Sets**

$\sim \; = \; \approx$

$\sim_c \; = \; \approx$

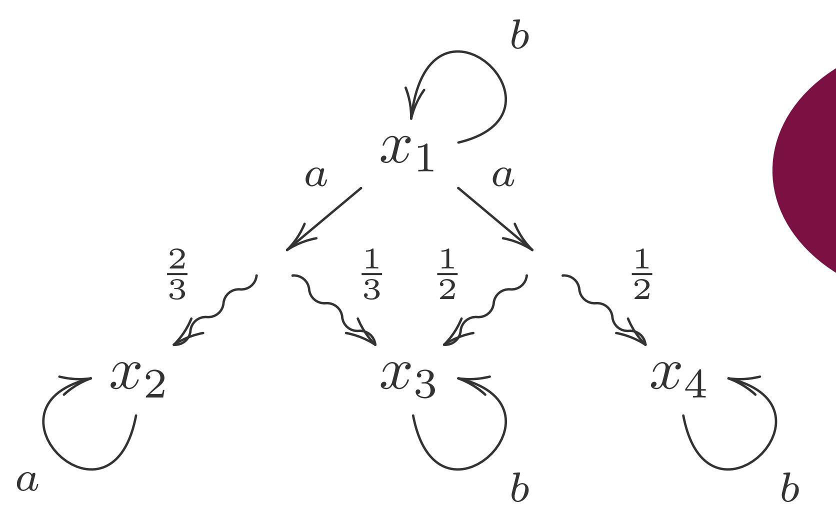
$X \to (\mathcal{C}(X))^A$

and all convex combinations



PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$



on **Sets**

$\sim \; = \; \approx$

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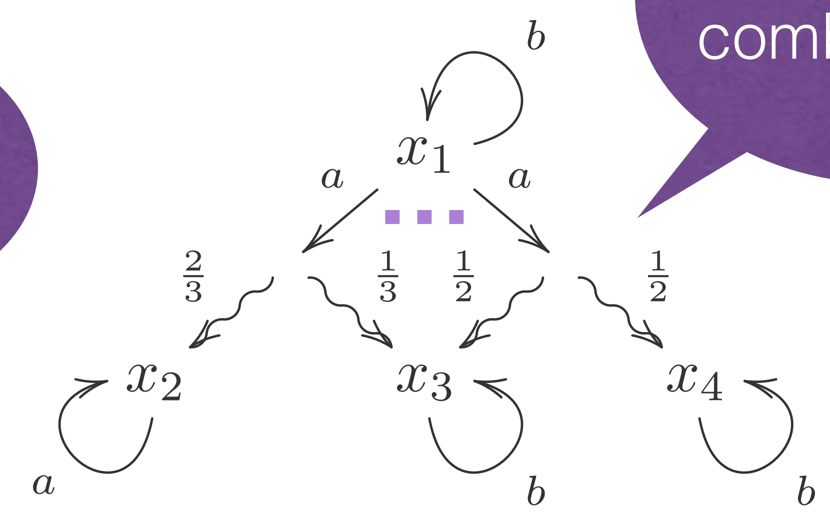
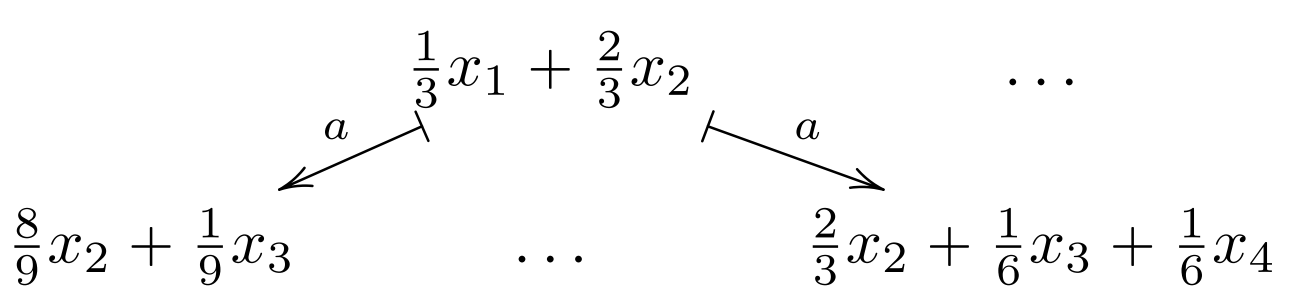
$X \to (\mathcal{C}(X))^A$

and all convex combinations

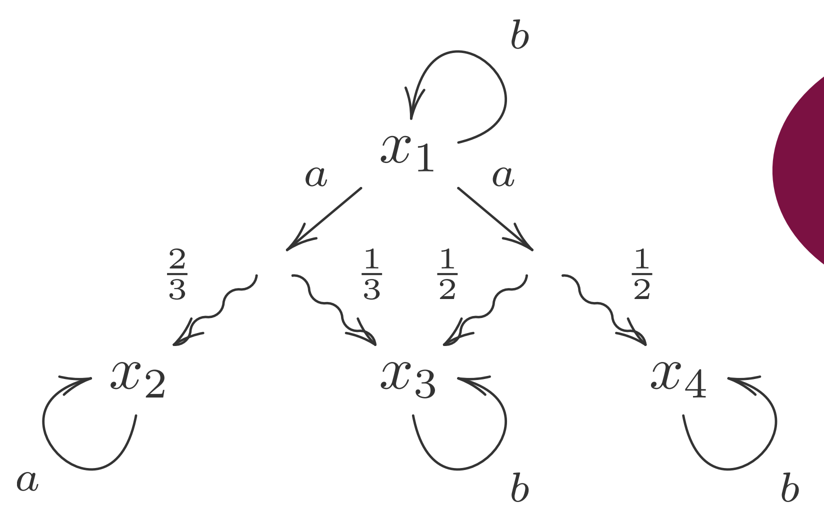


$X \to (\mathcal{P}_c(X)+1)^A$



PA coalgebraically

$X \to (\mathcal{P}\,\mathcal{D}(X))^A$



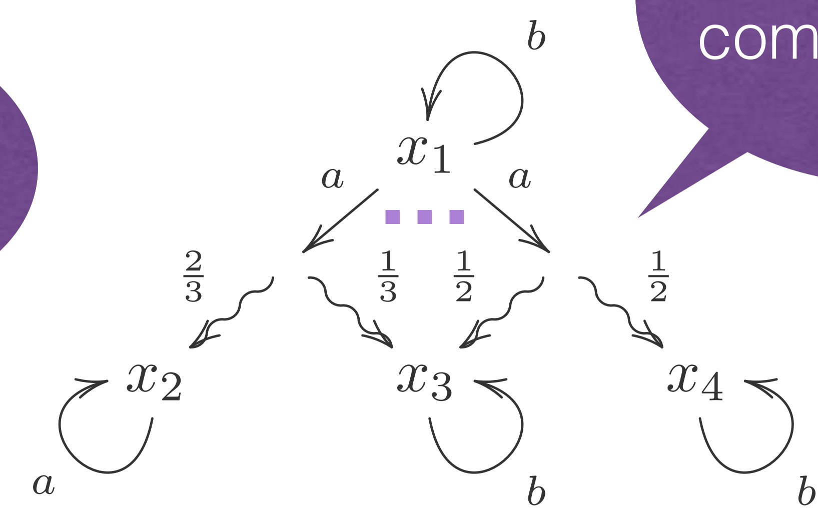
on **Sets**

$\sim \; = \; \approx$

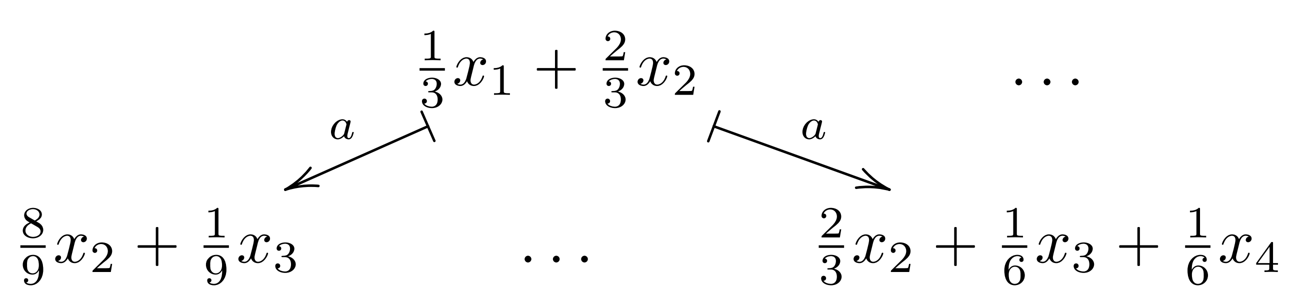
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}(X))^A$

and all convex combinations



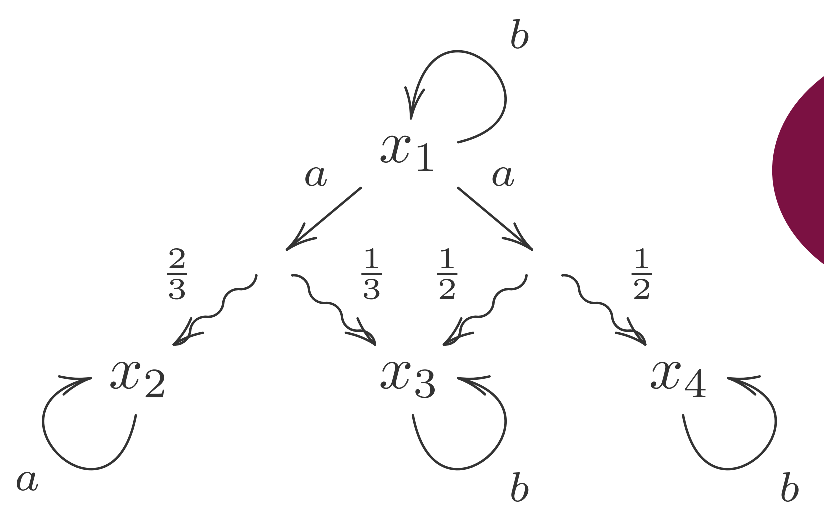
$X \to (\mathcal{P}_c(X)+1)^A$



on convex algebras

PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$



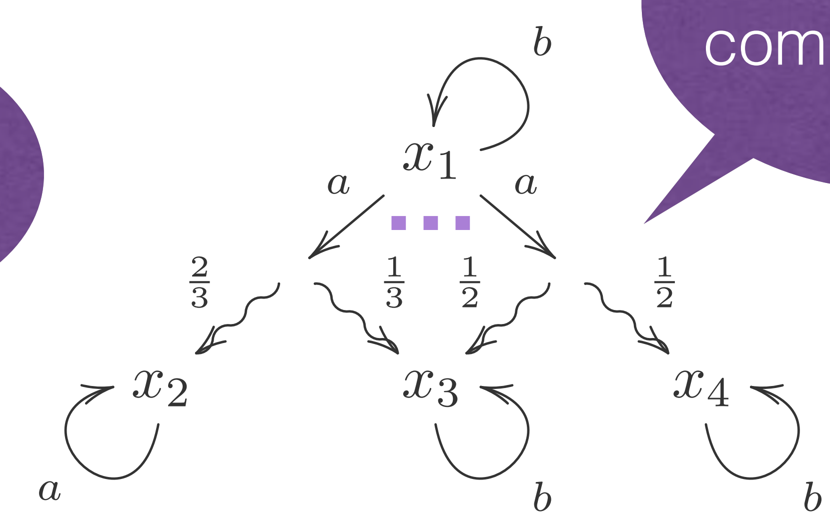
on **Sets**

$\sim \; = \; \approx$

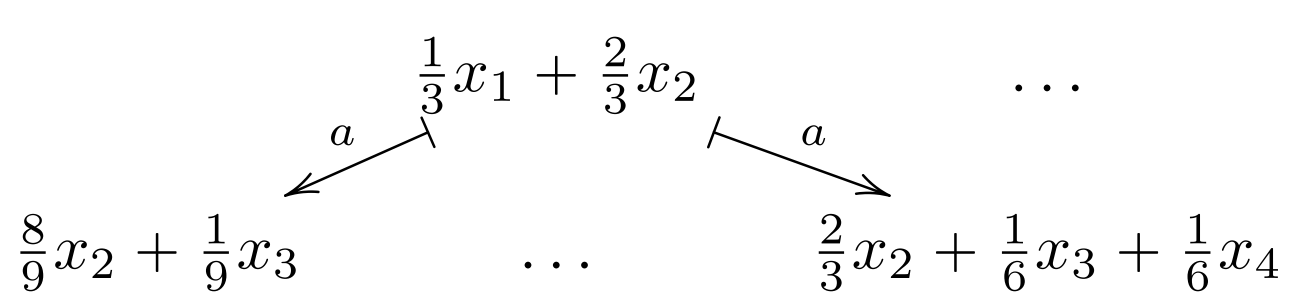
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}(X))^A$

and all convex combinations



$X \to (\mathcal{P}_c(X)+1)^A$

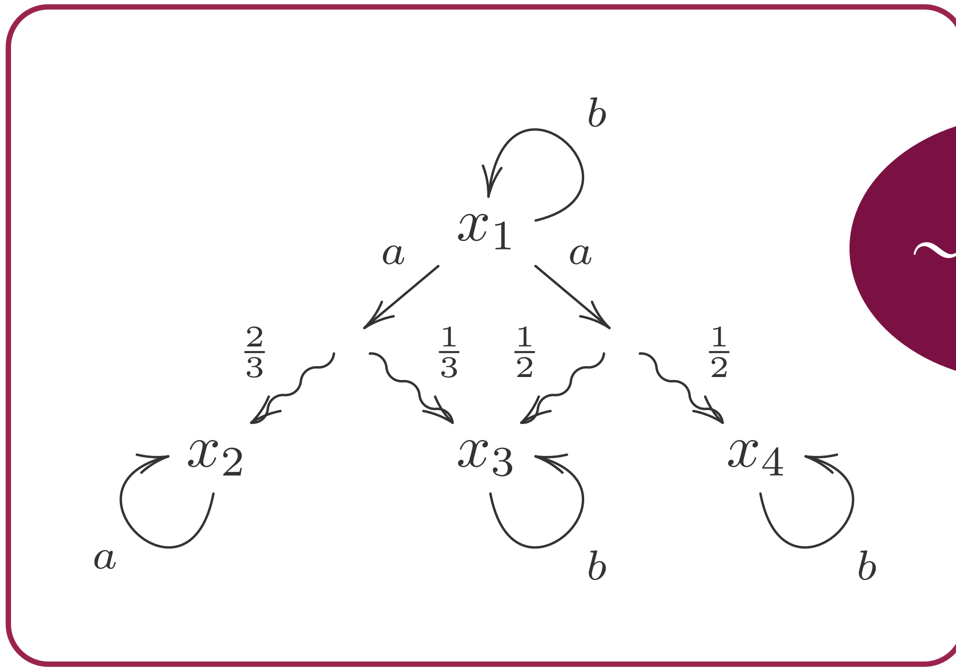


on convex algebras

$\mathcal{EM}(\mathcal{D})$

PA coalgebraically

$X \to (\mathcal{P}\mathcal{D}(X))^A$



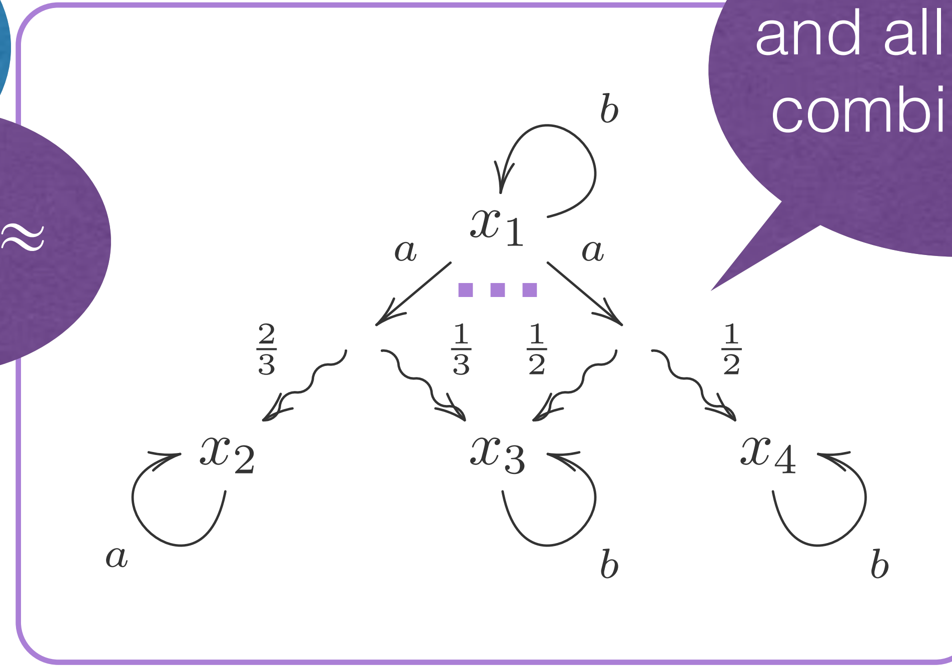
on **Sets**

$\sim \; = \; \approx$

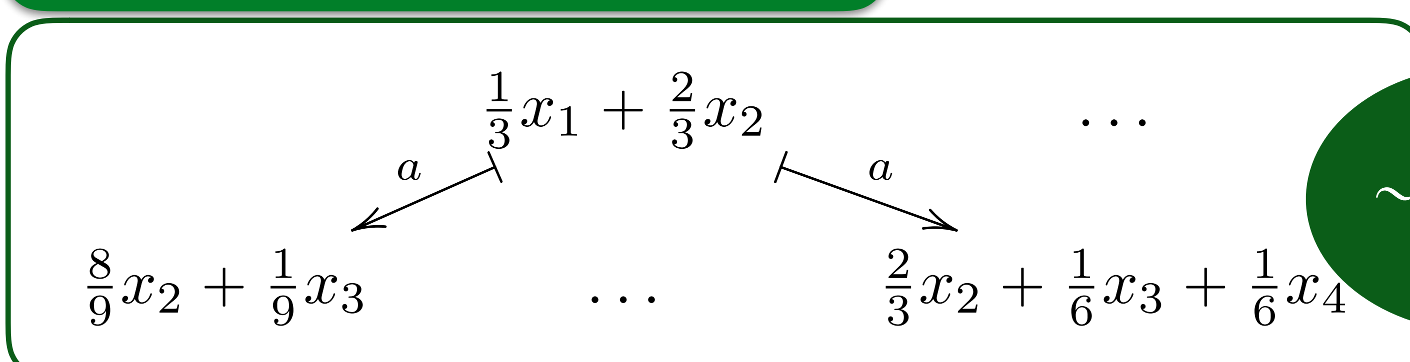
$\sim_c \; = \; \approx$

$X \to (\mathcal{C}(X))^A$

and all convex combinations



$X \to (\mathcal{P}_c(X)+1)^A$



on convex algebras

$\mathcal{EM}(\mathcal{D})$

$\sim_d \; = \; \approx$

Belief-state transformers

coalgebras on
free convex algebras

Belief-state transformers

$\mathbb{D}_S = (\mathcal{D}S, \Sigma)$

free convex algebra

convex combinations

coalgebras on free convex algebras

Belief-state transformers

$\mathbb{D}_S = (\mathcal{D}S, \Sigma)$

free convex algebra

convex combinations

coalgebras on free convex algebras

$$\mathbb{D}_S \to (\mathcal{P}_c(\mathbb{D}_S) + 1)^A$$

constant exponent

nonempty convex powerset

termination

Belief-state transformers

$\mathbb{D}_S = (\mathcal{D}S, \Sigma)$

free convex algebra

convex combinations

coalgebras on free convex algebras

$$\mathbb{D}_S \to (\mathcal{P}_c(\mathbb{D}_S) + 1)^A$$

constant exponent

nonempty convex powerset

termination

$$pA_1 + (1-p)A_2 = \{pa_1 + (1-p)a_2 \mid a_1 \in A_1, a_2 \in A_2\}$$

Belief-state transformers

$$\mathbb{D}_S = (\mathcal{D}S, \Sigma)$$

free convex algebra

convex combinations

coalgebras on free convex algebras

$$\mathbb{D}_S \to (\mathcal{P}_c(\mathbb{D}_S) + 1)^A$$

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termination

$$pA_1 + (1-p)A_2 = \{pa_1 + (1-p)a_2 \mid a_1 \in A_1, a_2 \in A_2\}$$

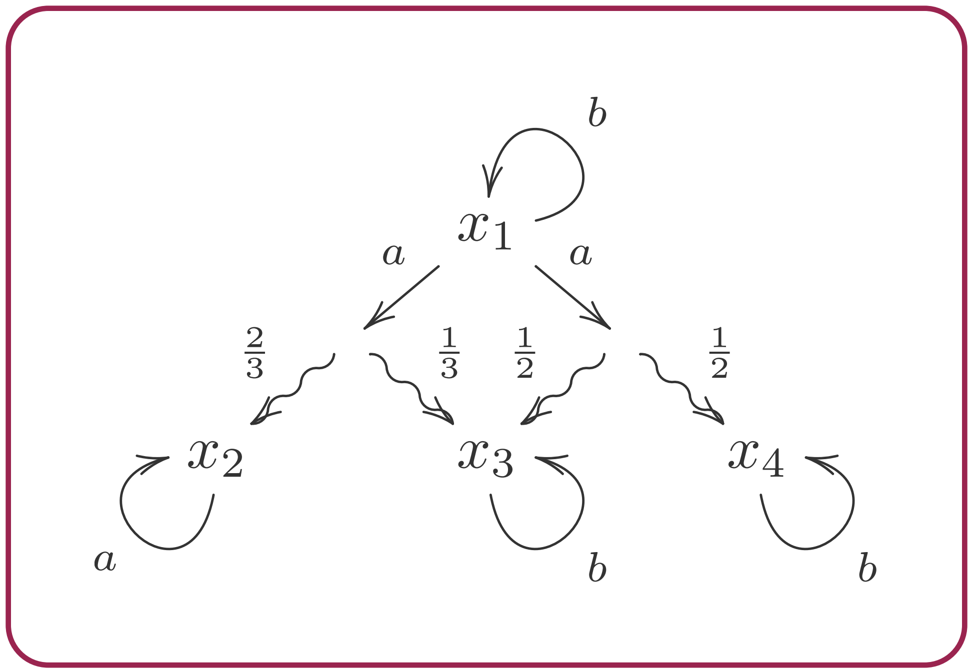
Minkowski sum

Belief-state transformer

foundation ?

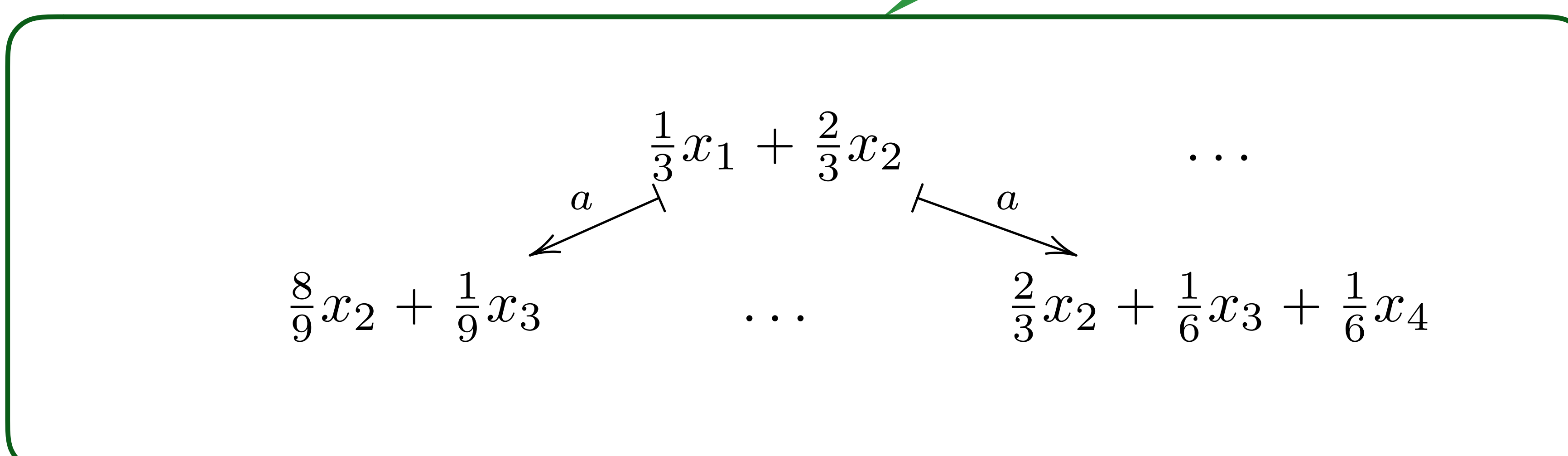
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

what is it?

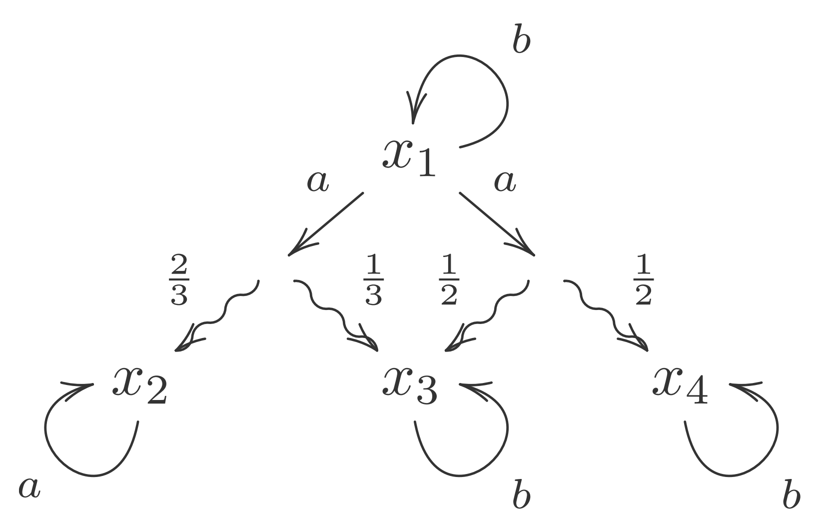


Belief-state transformer

foundation ?

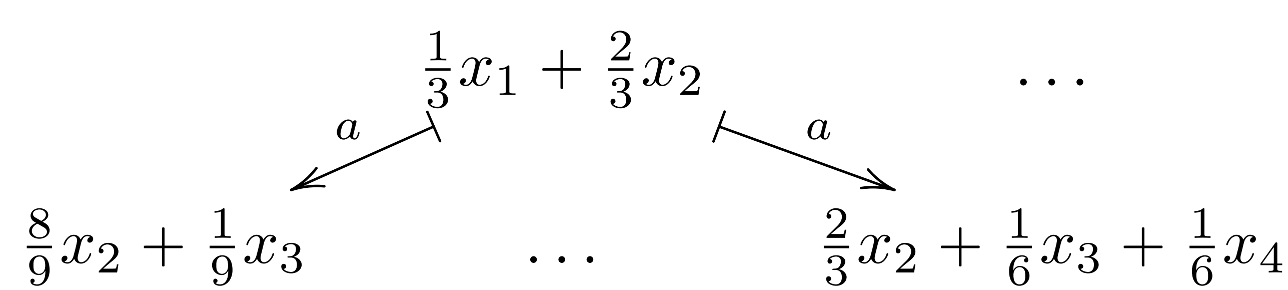
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

coalgebra over free convex algebra



Determinisations I

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations I

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(F)$$

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations I

determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(F)$$

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations I

determinise

forget

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(F)$$

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations I

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F})$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(FT)$

$\mathbf{CoAlg}_{\mathbf{C}}(F)$

needed:
a lifting

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations I

determinise

forget

$$\mathbf{CoAlg}_{\mathbf{C}}(FT) \to \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{F}) \to \mathbf{CoAlg}_{\mathbf{C}}(F)$$

needed:
a lifting

works for NFA

not for generative PTS
not for PA / belief-state
transformer

[Silva, Bonchi, Bonsangue, Rutten, FSTTCS'10]

Determinisations II

[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

Determinisations II

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

[Silva, S. MFPS'11]

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Determinisations II

determinise

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

[Silva, S. MFPS'11]

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Determinisations II

determinise

forget

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$$

$$\mathbf{CoAlg}_{\mathbf{C}}(TF) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

Determinisations II

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G})$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(TF) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

needed:
a lifting

[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

Determinisations II

determinise

$\mathbf{CoAlg}_{\mathbf{C}}(TF) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(\overline{G}) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

forget

needed:
a lifting

works for generative PTS

not for PA / belief-state transformer

[Silva, S. MFPS'11]

[Jacobs, Silva, S JCSS'15]

Determinisations III

Determinisations III

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$$

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \nearrow \qquad \searrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

Determinisations III

determinise

$$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$$

$$\mathbf{CoAlg}_{\mathbf{C}}(F) \longrightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \longrightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$$

Determinisations III

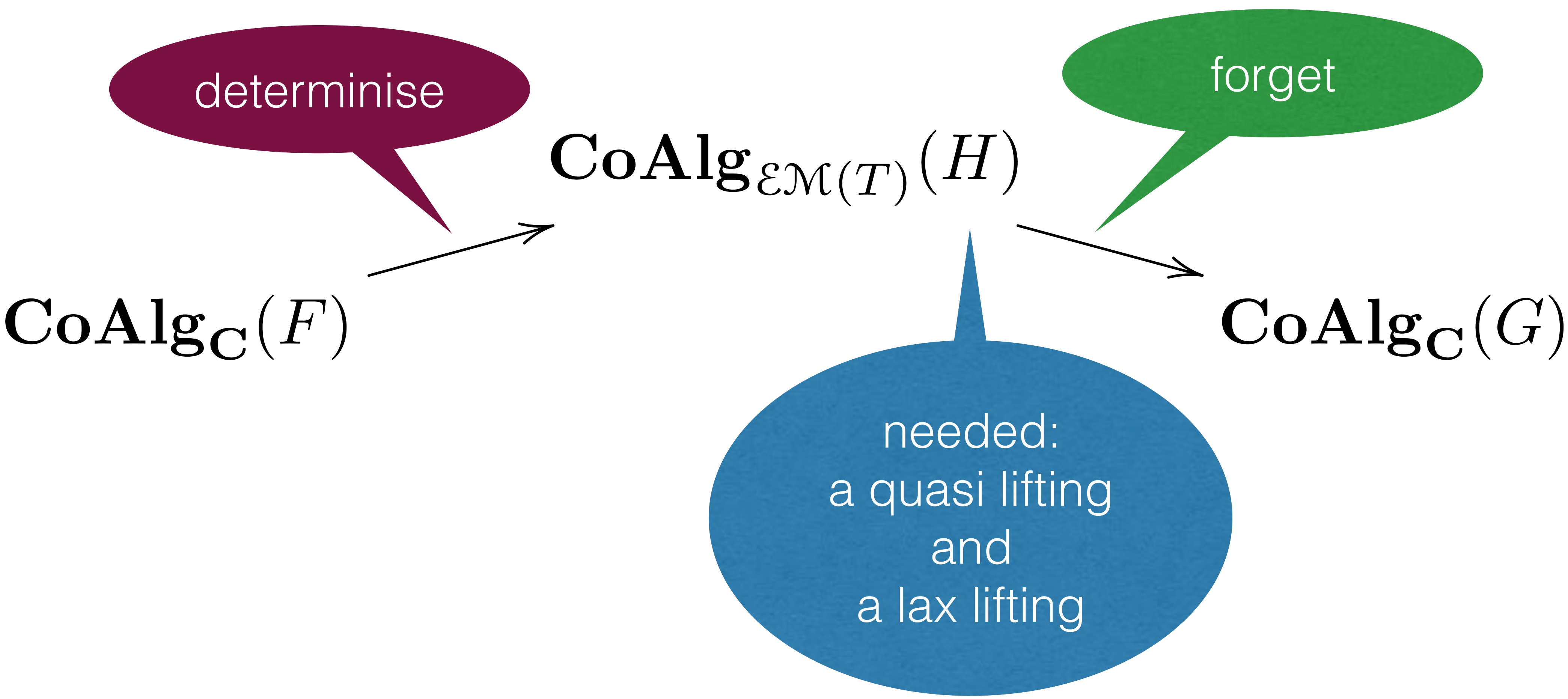
determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(F) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

Determinisations III



Determinisations III

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(F) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

needed:
a quasi lifting
and
a lax lifting

works for PA /belief-state transformer

Determinisations III

determinise

$\mathbf{CoAlg}_{\mathcal{EM}(T)}(H)$

forget

$\mathbf{CoAlg}_{\mathbf{C}}(F) \rightarrow \mathbf{CoAlg}_{\mathcal{EM}(T)}(H) \rightarrow \mathbf{CoAlg}_{\mathbf{C}}(G)$

needed:
a quasi lifting
and
a lax lifting

works for PA /belief-state transformer

$(\mathcal{P}_c+1)^A$ on $\mathcal{EM}(\mathcal{D})$

is a quasi lifting and lax lifting of

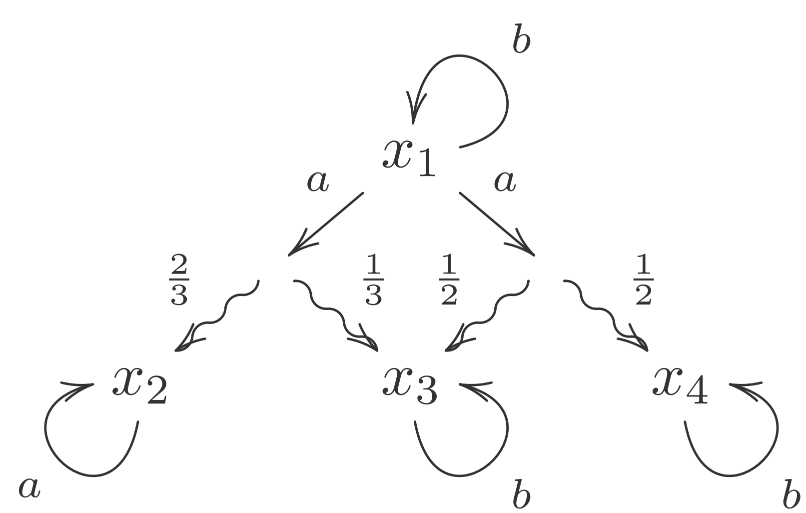
$\mathcal{C}^A$ and $\mathcal{P}^A$ on $\mathbf{Sets}$

Belief-state transformer

foundation ?

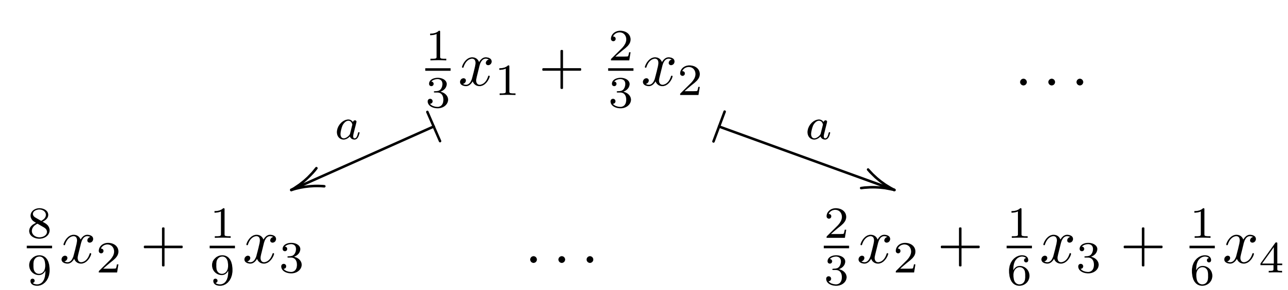
PA

$X \to (\mathcal{PD}(X))^A$



how does it emerge?

coalgebra over free convex algebra

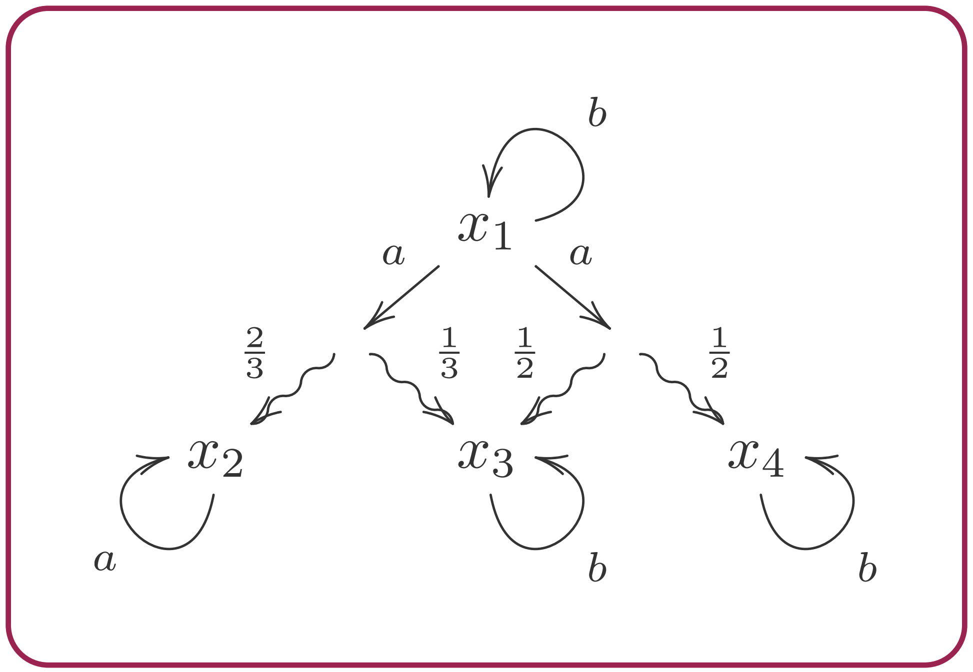


Belief-state transformer

foundation ?

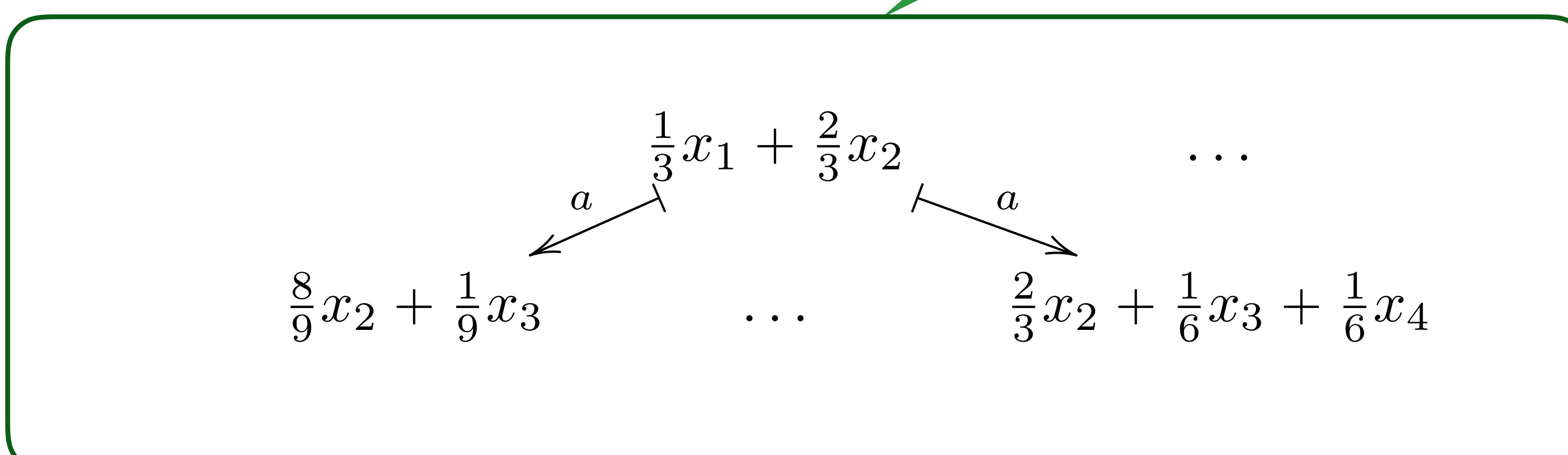
PA

$X \to (\mathcal{PD}(X))^A$



via a generalised determinisation

coalgebra over free convex algebra

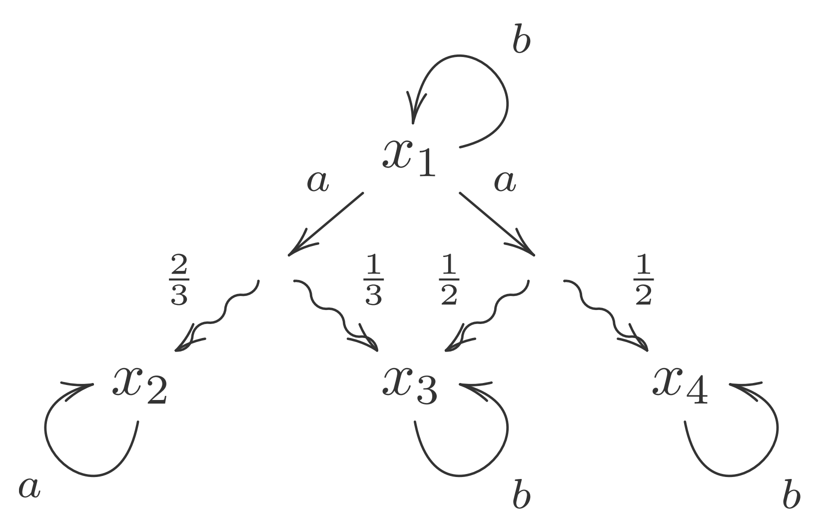


Belief-state transformer

are natural indeed

PA

$X \to (\mathcal{P}\mathcal{D}(X))^A$



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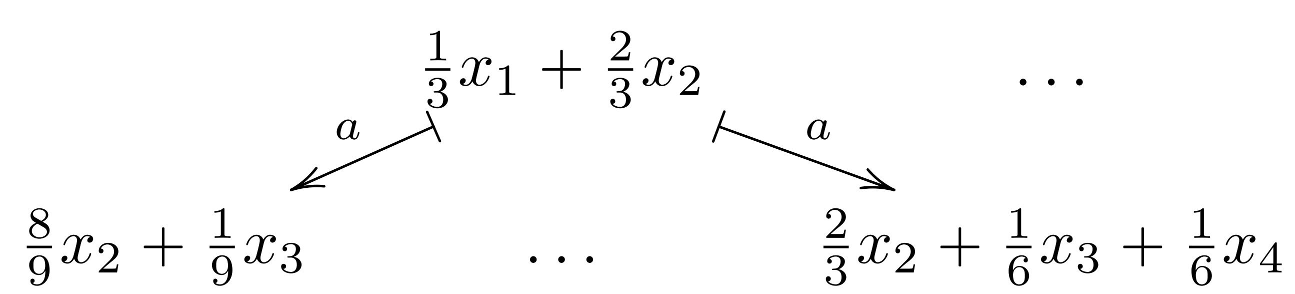


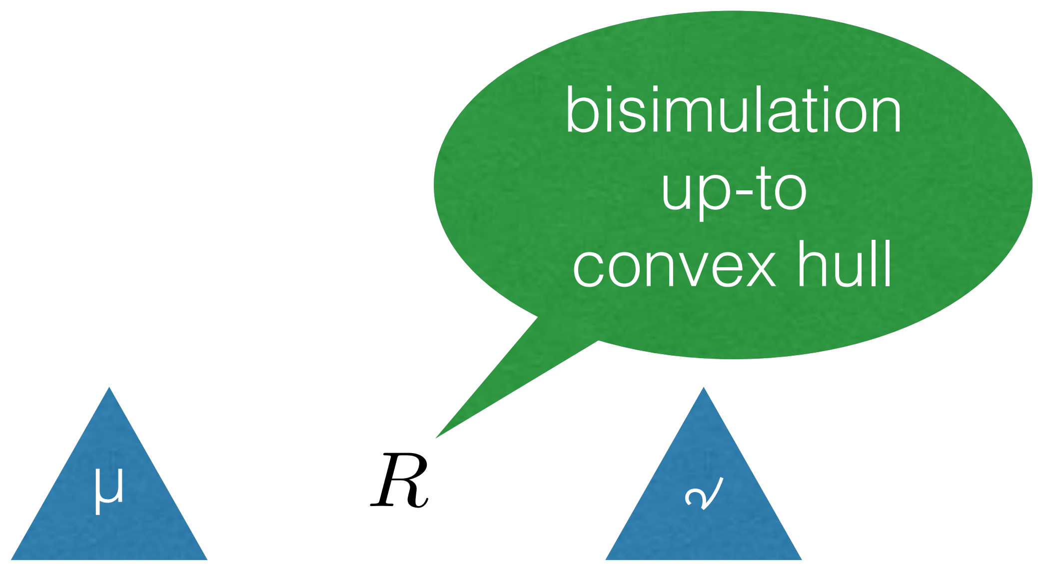
Coinductive proof method for distribution bisimilarity

Coinductive proof method for distribution bisimilarity

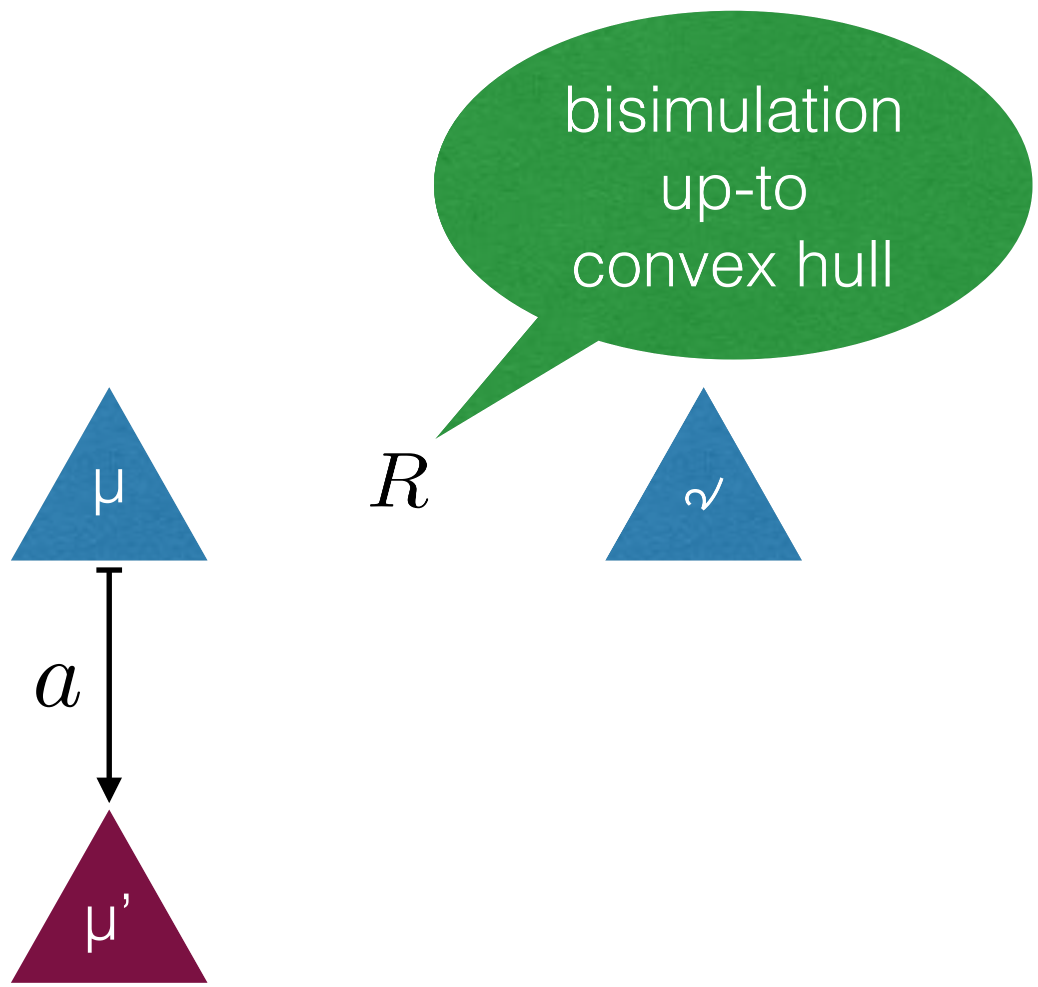
bisimulation up-to convex hull

R

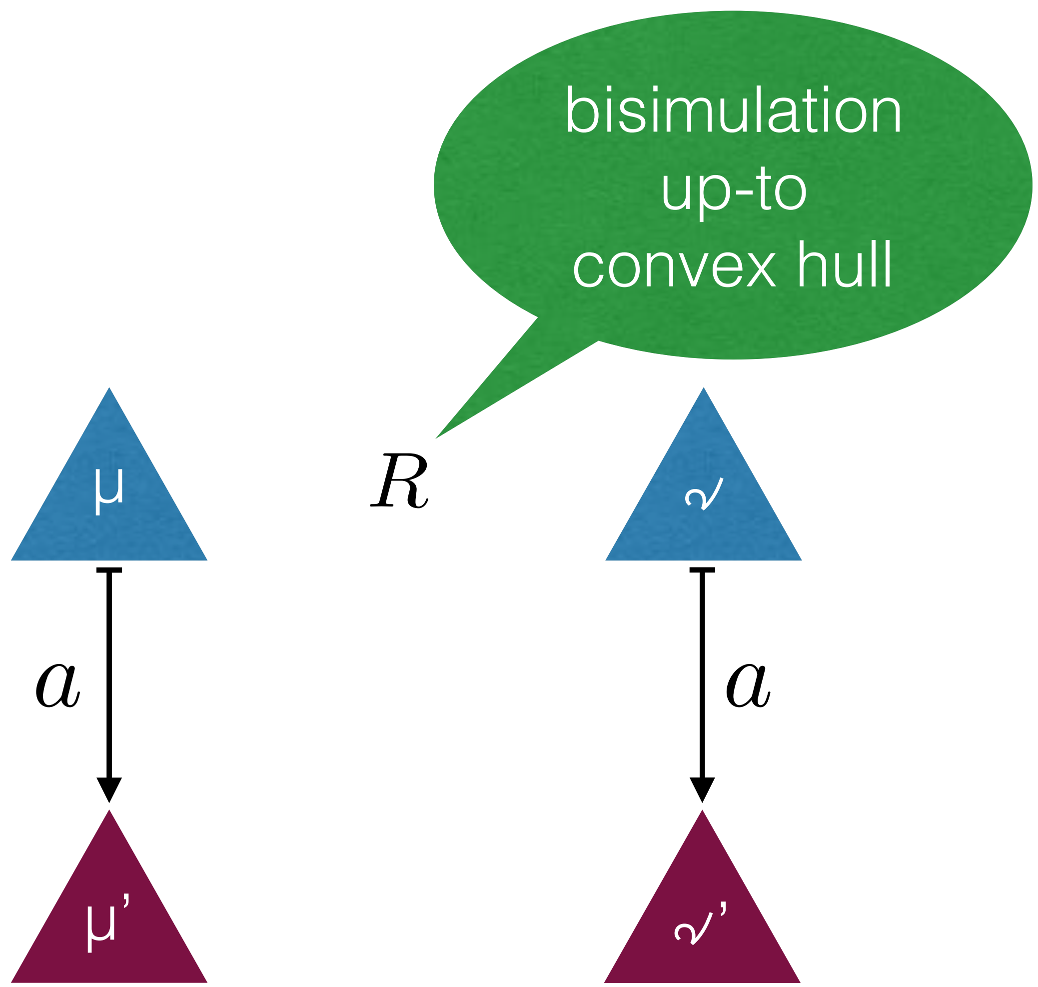
Coinductive proof method for distribution bisimilarity



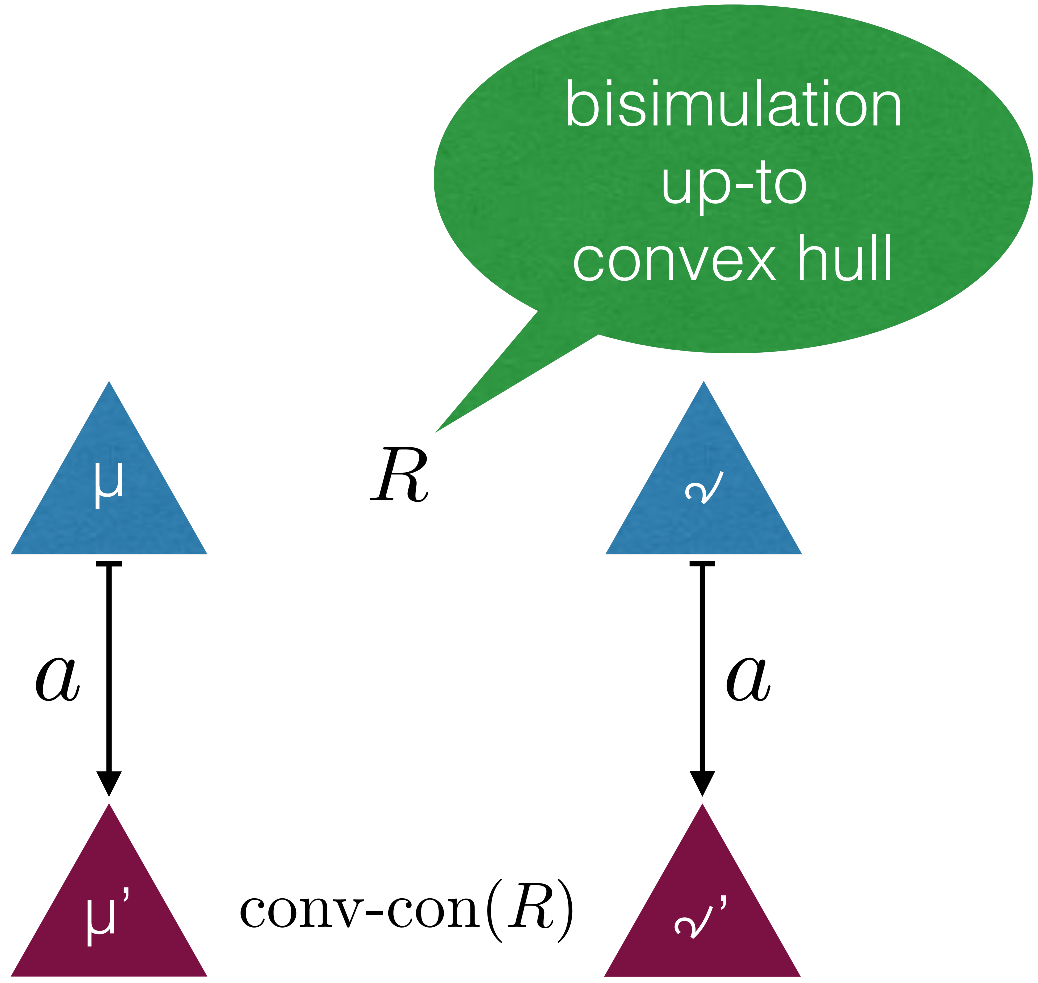
Coinductive proof method for distribution bisimilarity



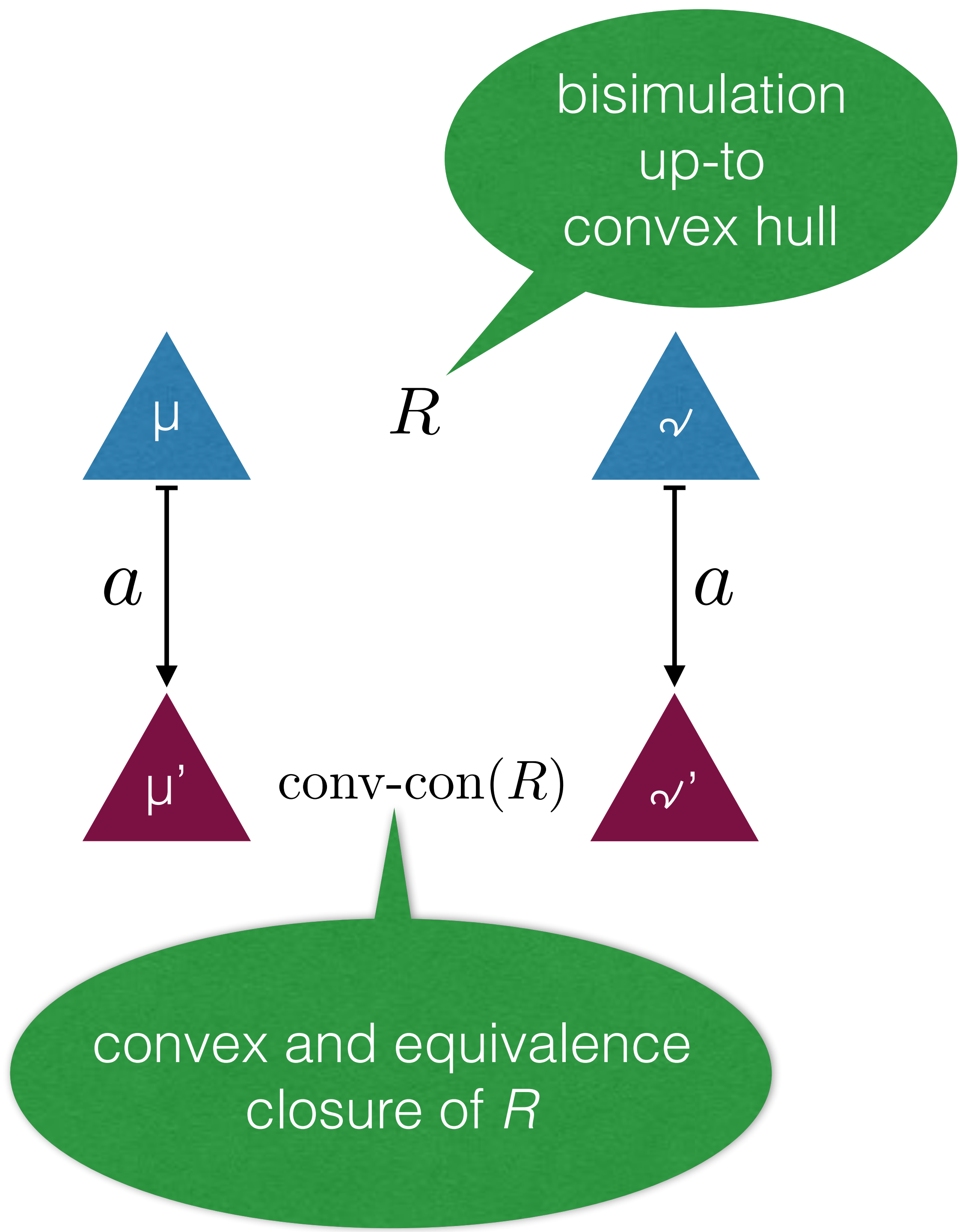
Coinductive proof method for distribution bisimilarity



Coinductive proof method for distribution bisimilarity

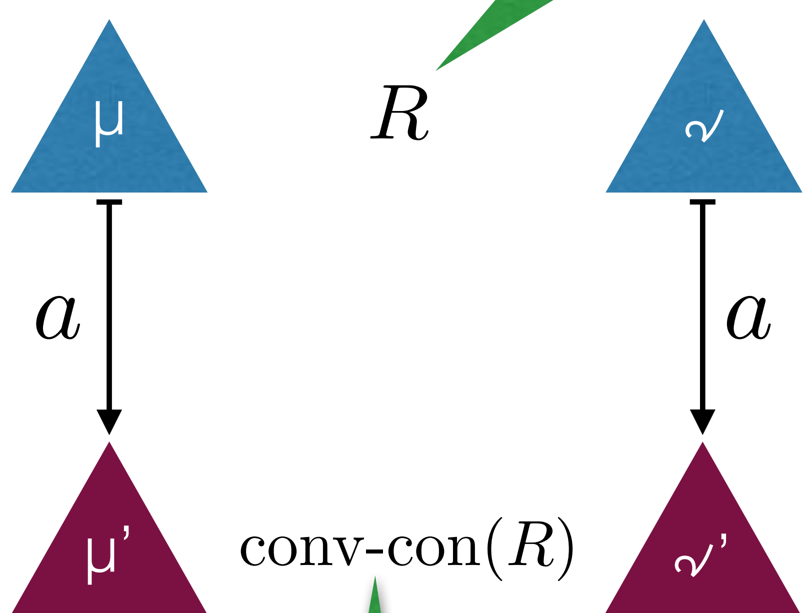


Coinductive proof method for distribution bisimilarity



Coinductive proof method for distribution bisimilarity

bisimulation up-to convex hull



convex and equivalence closure of *R*

to prove $\mu \sim_d \nu$ it suffices to find a bisimulation up-to convex hull *R* with $\mu \; R \; \nu$

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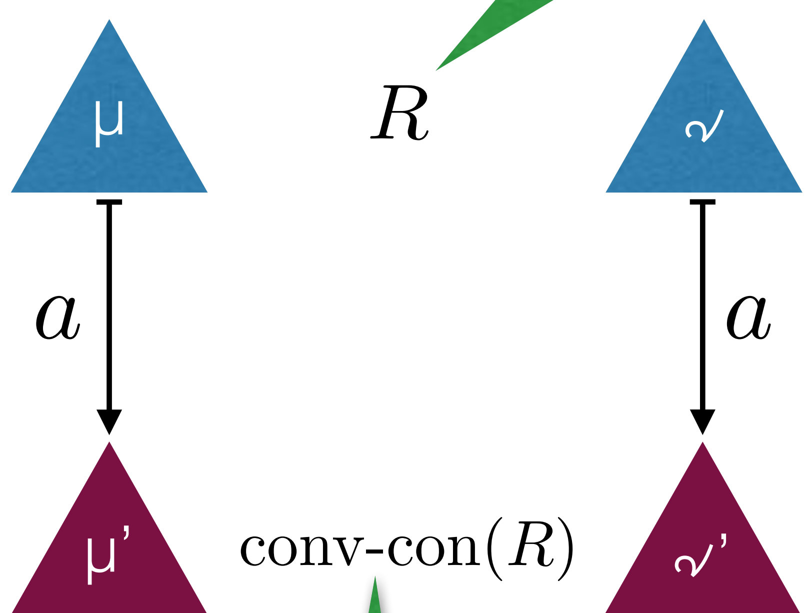

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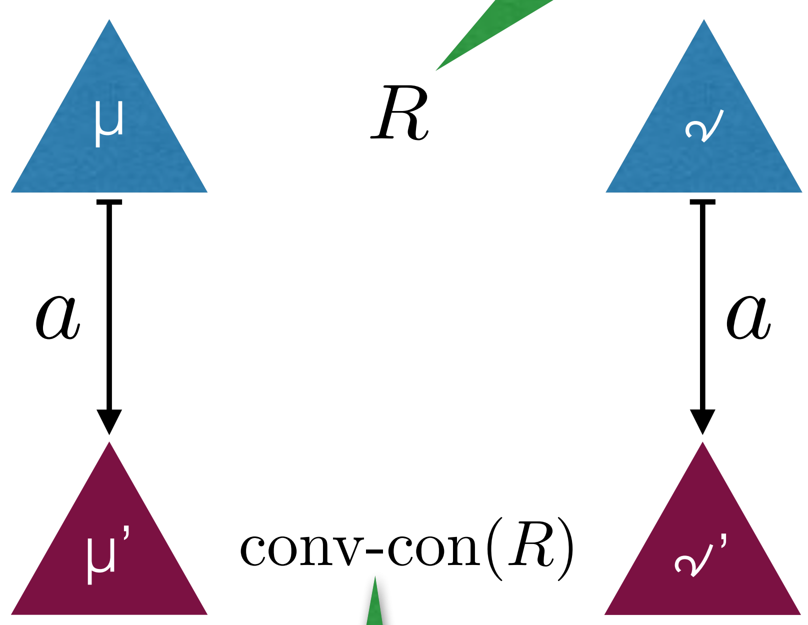

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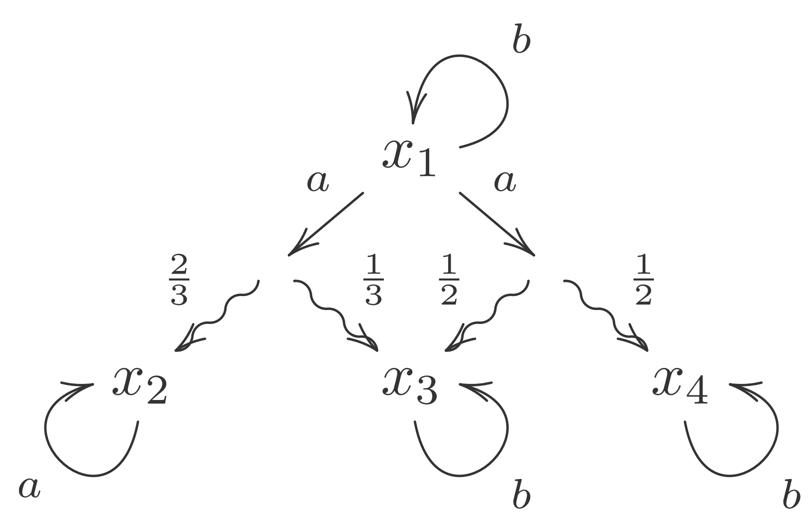
[S., Woracek JPAA'15]

Belief-state transformer

are natural indeed

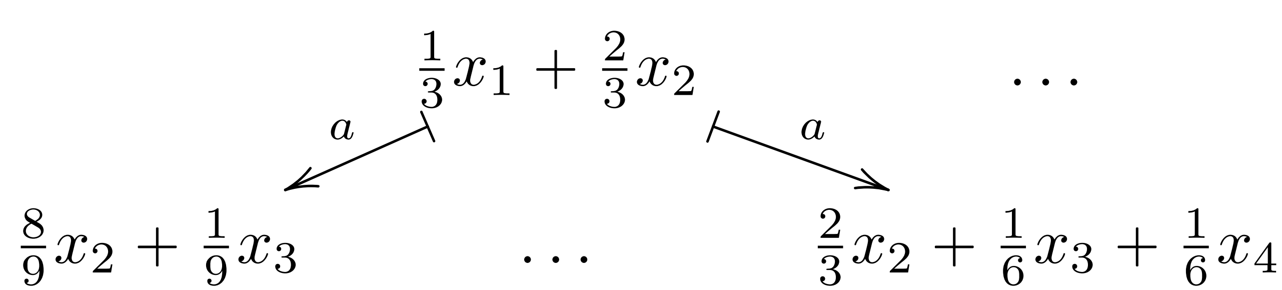
PA

X → ($\mathcal{PD}$(X))^A



via a generalised determinisation

a coalgebra over free convex algebra



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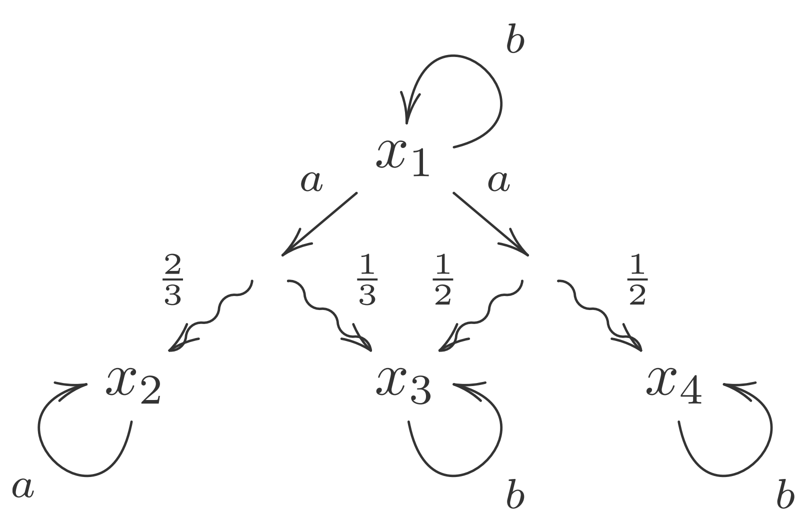

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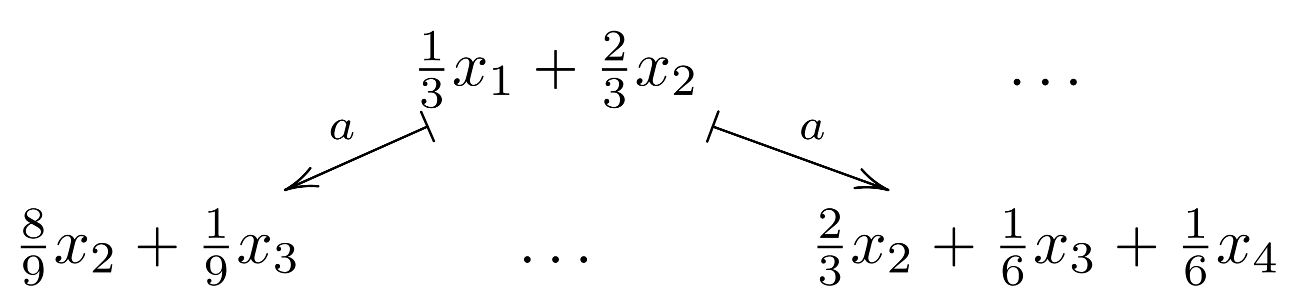
sound proof method for distribution bisimilarity



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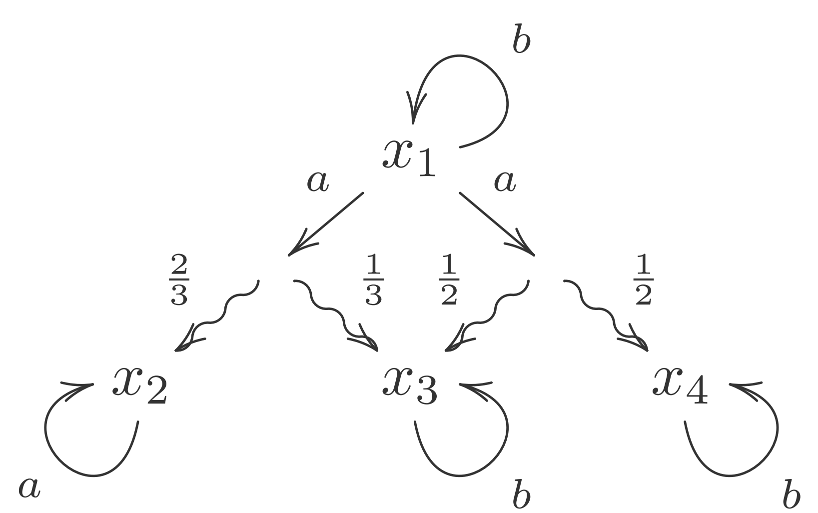

Thank You!

sound proof method for distribution bisimilarity