Trace Semantics via Determinization

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COIN, Radboud University Nijmegen, 6.12.2012

Trace semantics for more coalgebras!

Two approaches for coalgebraic traces

- Kleisli trace semantics [HJS'07]
- Traces via the ``generalized powerset construction'' --- determinization [SBBR'10]

traces as ``coalgebraic language equivalence''

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TF-coalgebras

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T - monad, Kleisli category

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Needed: FT => TF + ...

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  Needed: FT => TF + ...

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  GT-coalgebras

  T - monad, Eilenberg-Moore category

  Needed: TG => GT + final G

Trace semantics for (more) coalgebras

- Kleisli trace semantics [HJS'07]

generative

TF-coalgebras

- Traces via the ``generalized powerset construction" [SBBR'10]

reactive

GT-coalgebras

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Examples: $\mathcal{P}(1 + A \times (-))$ NFA
$\mathcal{D}(1 + A \times (-))$ PTS

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Examples: $2 \times \mathcal{P}^A$ NFA
$S \times \mathcal{M}_S^A$ WTS

What is in common?

Semantics via finality (coinduction)

of coalgebras over Kleisli or EM categories

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Final coalgebra semantics:

$$\begin{tikzcd} X \arrow[r, dashed, "\mathrm{beh}"] \arrow[d, "c"'] & Z \arrow[d, "\cong"] \\ HX \arrow[r, dashed, "H\,\mathrm{beh}"] & HZ \end{tikzcd}$$

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bisimilarity in $\mathbf{Sets}$
(for wpp functors)

trace semantics in $\mathcal{K}\ell(T)$
(for TF-coalgebras)

coalgebraic language eq. in $\mathcal{EM}(T)$
(for GT-coalgebras)

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How do they relate?

The categories via the comparison/extension functor

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[rr, "E"] \arrow[dr] & & \mathcal{EM}(T) \arrow[dl] \\
& \mathbb{C} &
\end{tikzcd}$$

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The categories via the comparison/extension functor

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\mathcal{K}\ell(T) \arrow[rr, "E"] \arrow[dr] & & \mathcal{EM}(T) \arrow[dl] \\
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$$

$$E(X) = \begin{pmatrix} T^2X \\ \downarrow \mu \\ TX \end{pmatrix} \qquad E(f) = \mu \circ T(f)$$

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Kleisli extension

$f\colon X \to Y$ in $\mathcal{K}\ell(T)$

$f\colon X \to TY$ in $\mathbb{C}$

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It's all about liftings!

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The big picture

$$
\begin{tikzcd}
\mathrm{Coalg}(\hat{F}) \arrow[rr, "\hat{E}"] \arrow[d] & & \mathrm{Coalg}(\hat{G}) \arrow[d] \\
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Eventually we will lift E

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\end{tikzcd}$$

But before that, some intuition...

Determinization of NFA

$\mathcal{P}(1 + A \times (-))$ NFA

$$\begin{tikzcd}
& x_1 \arrow[dl, "a"'] \arrow[dr, "a"] & \\
x_2 \arrow[d] & & x_3 \arrow[loop right, "b"] \\
*
\end{tikzcd}$$

$2 \times (-)^A$ DFA

$$\begin{tikzcd}
x_1 \arrow[d, "a"'] & \\
\{x_2, x_3\} \arrow[r, "b"] \arrow[d] & x_3 \arrow[loop right, "b"] \\
*
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Determinization of NFA

TF

$\mathcal{P}(1 + A \times (-))$ **NFA**

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G on T(-)

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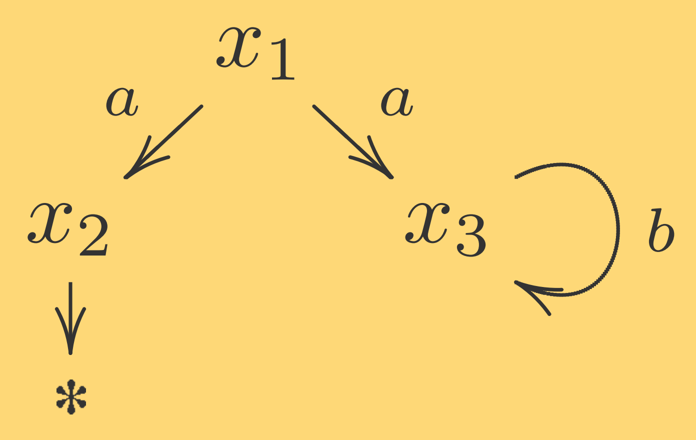
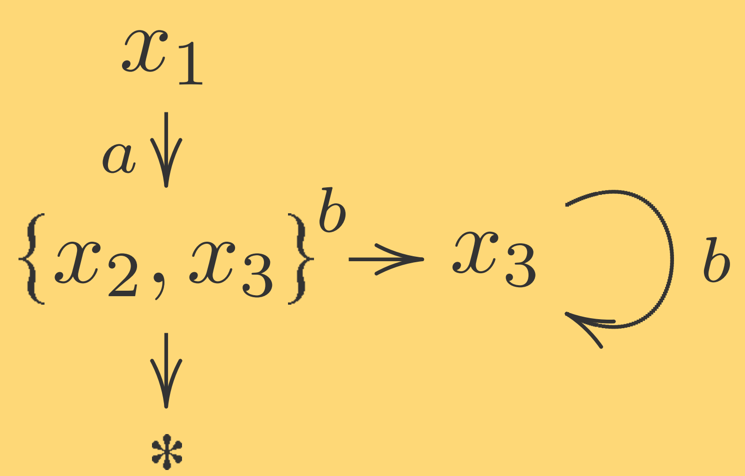


$2 \times (-)^A$ **DFA**



$$\begin{tikzcd} X \arrow[r, "beh"] \arrow[d, "c"'] & 2^{A^*} \arrow[d, "\cong"] \\ GX \arrow[r, "Gbeh"] & G(2^{A^*}) \end{tikzcd}$$

Determinization of NFA

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G on T(-)

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final G-coalgebra

Determinization of PTS

$\mathcal{D}(1 + A \times (-))$ PTS

$$
\begin{tikzcd}[column sep=small]
 & x_1 \arrow[dl, "{a,\frac{1}{2}}"'] \arrow[dr, "{a,\frac{1}{4}}"] & \\
x_2 \arrow[d, "{b,\frac{1}{3}}"'] & & x_3 \arrow[d, "{c,\frac{1}{2}}"] \\
x_4 \arrow[d, "1"'] & & x_5 \arrow[d, "1"] \\
* & & *
\end{tikzcd}
$$

$[0,1] \times (-)^A$ DFA

$$
\begin{tikzcd}[column sep=small]
 & x_1 \arrow[d, "a"] & \\
 & \frac{1}{2}x_2 + \frac{1}{4}x_3 \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\frac{1}{6}x_4 \arrow[d, "\frac{1}{6}"'] & & \frac{1}{8}x_5 \arrow[d, "\frac{1}{8}"] \\
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$$\begin{tikzcd}
X \arrow[r, "beh"] \arrow[d, "c"'] & {[0,1]^{A^*}} \arrow[d, "\cong"] \\
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\end{tikzcd}$$

final
G-coalgebra

Laws and liftings

$\mathcal{K}\ell$-law $\lambda\colon FT \Rightarrow TF$

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\
\mathbb{C} \arrow[r, "F"] & \mathbb{C}
\end{tikzcd}$$

$\mathcal{EM}$-law $\rho\colon TG \Rightarrow GT$

$$\begin{tikzcd}
\mathcal{EM}(T) \arrow[r, "\hat{G}"] \arrow[d] & \mathcal{EM}(T) \arrow[d] \\
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\end{tikzcd}$$

$$\begin{tikzcd}
\mathbf{CoAlg}(GT) \arrow[r, "\mathcal{F}_{\mathcal{EM}}"] \arrow[d] & \mathbf{CoAlg}(\hat{G}) \arrow[d] \\
\mathbb{C} \arrow[loop left, "G"] \arrow[loop below, "T"] \arrow[r, "\mathcal{F}"] & \mathcal{EM}(T) \arrow[loop right, "\hat{G}"]
\end{tikzcd}$$

Laws and liftings

$$\mathcal{K}\ell\text{-law } \lambda\colon FT \Rightarrow TF$$

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\mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\
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\end{tikzcd}$$

$$\mathcal{F}_{\mathcal{EM}}\left(X \xrightarrow{c} GTX\right) = \begin{pmatrix} T^2X \\ \downarrow \mu \\ TX \end{pmatrix} \xrightarrow{G\mu \circ \rho_{TX} \circ T(c)} \hat{G}\begin{pmatrix} T^2X \\ \downarrow \mu \\ TX \end{pmatrix} \qquad \mathcal{F}_{\mathcal{EM}}(f) = T(f)$$

"Determinization"
(in the GPC)

$$\begin{tikzcd}
\mathbf{CoAlg}(GT) \arrow[r, "\mathcal{F}_{\mathcal{EM}}"] \arrow[d] & \mathbf{CoAlg}(\hat{G}) \arrow[d] \\
\mathbb{C} \arrow[r, "\mathcal{F}"] \arrow[loop left, "G"] \arrow[loop below, "T"] & \mathcal{EM}(T) \arrow[loop right, "\hat{G}"]
\end{tikzcd}$$

free functor

Laws and liftings

$\mathcal{K}\ell$-law $\lambda\colon FT \Rightarrow TF$

$$\begin{tikzcd} \mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{EM}(T) \arrow[r, "\hat{G}"] \arrow[d] & \mathcal{EM}(T) \arrow[d] \\ \mathbb{C} \arrow[r, "G"] & \mathbb{C} \end{tikzcd}$$

$$\mathcal{F}_{\mathcal{EM}}\left(X \xrightarrow{c} GTX\right) = \begin{pmatrix} T^2X \\ \downarrow \mu \\ TX \end{pmatrix} \xrightarrow{G\mu \circ \rho_{TX} \circ T(c)} \hat{G}\begin{pmatrix} T^2X \\ \downarrow \mu \\ TX \end{pmatrix} \qquad \mathcal{F}_{\mathcal{EM}}(f) = T(f)$$

"Determinization" (in the GPC)

$$\begin{tikzcd} \mathbf{CoAlg}(GT) \arrow[r, "\mathcal{F}_{\mathcal{EM}}"] \arrow[d] & \mathbf{CoAlg}(\hat{G}) \arrow[d] \\ \mathbb{C} \arrow[r, "\mathcal{F}"] \arrow[loop left, "G"] \arrow[loop below, "T"] & \mathcal{EM}(T) \arrow[loop below, "\hat{G}"] \end{tikzcd}$$

free functor

The final coalgebra also lifts

GT-coalgebras (GPC)

Assume $TG \Rightarrow GT$ and final $Z \stackrel{\cong}{\to} GZ$ exists

- Given a coalgebra $X \stackrel{c}{\to} GTX$
- "Determinize" $\mathcal{F}_{\mathcal{EM}}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$
- Get semantics by

$$\begin{tikzcd} & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] & \hat{G}Z \arrow[d, "\cong"] \\ X \arrow[r, "\eta"] & TX \arrow[u, "\mathcal{F}_{\mathcal{EM}}(c)"'] \arrow[r, "beh"] & Z \end{tikzcd}$$

GT-coalgebras (GPC)

Assume $TG \Rightarrow GT$ and final $Z \overset{\cong}{\to} GZ$ exists

- Given a coalgebra $X \xrightarrow{c} GTX$
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GT-coalgebras (GPC)

Assume $TG \Rightarrow GT$ and final $Z \stackrel{\cong}{\to} GZ$ exists

Works for deterministic automata

$G = T(B) \times (-)^A$

strong

- Given a coalgebra $X \xrightarrow{c} GTX$
- "Determinize" $\mathcal{F}_{\mathcal{EM}}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$

Determinization

- Get semantics by

$$\begin{tikzcd} & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] & \hat{G}Z \\ X \arrow[r, "\eta"] & TX \arrow[u, "\mathcal{F}_{\mathcal{EM}}(c)"] \arrow[r, "beh"] & Z \arrow[u, "\cong"'] \end{tikzcd}$$

Trace semantics

TF-coalgebras?

$$\frac{\mathcal{K}\ell\text{-law } \lambda\colon FT \Rightarrow TF}{\begin{array}{ccc} \mathcal{K}\ell(T) & \xrightarrow{\hat{F}} & \mathcal{K}\ell(T) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array}}$$

$$\frac{\mathcal{EM}\text{-law } \rho\colon TG \Rightarrow GT}{\begin{array}{ccc} \mathcal{EM}(T) & \xrightarrow{\hat{G}} & \mathcal{EM}(T) \\ \downarrow & & \downarrow \\ \mathbb{C} & \xrightarrow{G} & \mathbb{C} \end{array}}$$

$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

TF-coalgebras?

$$\frac{\mathcal{K}\ell\text{-law}\ \lambda\colon FT \Rightarrow TF}{\begin{tikzcd} \mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}}$$

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$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

Extension natural tr.

$\mathfrak{e}\colon TF \Rightarrow GT$

connecting the laws

TF-coalgebras?

$$\frac{\mathcal{K}\ell\text{-law } \lambda\colon FT \Rightarrow TF}{\begin{tikzcd} \mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\ \mathbb{C} \arrow[r, "F"] & \mathbb{C} \end{tikzcd}}$$

$$\frac{\mathcal{EM}\text{-law } \rho\colon TG \Rightarrow GT}{\begin{tikzcd} \mathcal{EM}(T) \arrow[r, "\hat{G}"] \arrow[d] & \mathcal{EM}(T) \arrow[d] \\ \mathbb{C} \arrow[r, "G"] & \mathbb{C} \end{tikzcd}}$$

$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

Extension natural tr.

$\mathfrak{e}\colon TF \Rightarrow GT$

connecting the laws

$$\begin{tikzcd} \mathsf{Coalg}(\hat{F}) \arrow[r, "\hat{E}"] \arrow[d] & \mathsf{Coalg}(\hat{G}) \arrow[d] \\ \mathcal{K}\ell(T) \arrow[r, "E"] \arrow[loop left, "\hat{F}"] & \mathcal{EM}(T) \arrow[loop right, "\hat{G}"] \end{tikzcd}$$

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$$\hat{E}(c) = \mathfrak{e} \circ \mu \circ T(c)$$
$$\hat{E}(f) = E(f)$$

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"Determinization"

TF-coalgebras

Assume $FT \Rightarrow TF \quad TG \Rightarrow GT \quad \mathfrak{e}: TF \Rightarrow GT$

and final $Z \stackrel{\cong}{\Rightarrow} GZ$ exists

- Given a coalgebra $X \xrightarrow{c} TFX$
- "Determinize" $\hat{E}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$
- Get semantics by

$$
\begin{tikzcd}
 & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] & \hat{G}Z \\
X \arrow[r, "\eta"] & TX \arrow[u, leftarrow, "\hat{E}(c)"'] \arrow[r, "beh"] & Z \arrow[u, leftarrow, "\cong"']
\end{tikzcd}
$$

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TF-coalgebras

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Works for
all examples
we have seen

- Given a coalgebra $X \xrightarrow{c} TFX$
- "Determinize" $\hat{E}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$

Determinization

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 & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] & \hat{G}Z \\
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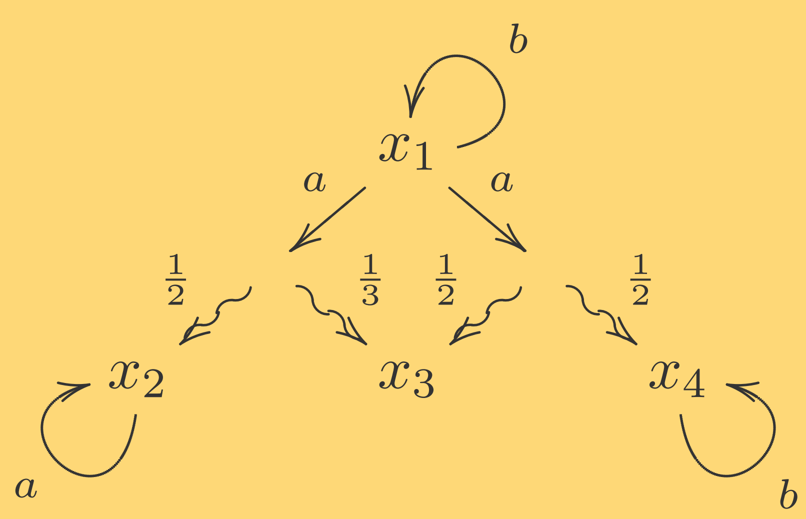
\end{tikzcd}$$

- Get semantics by $X \xrightarrow{\eta} TX \xrightarrow{beh} Z$

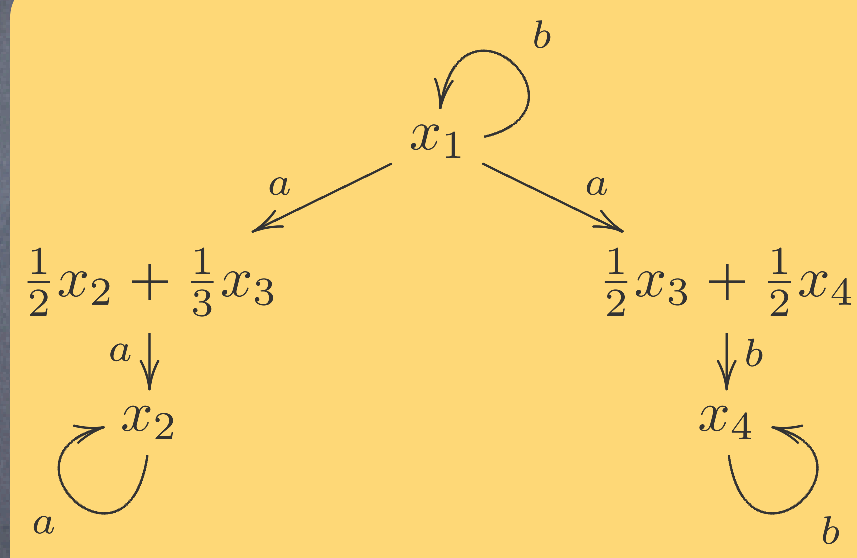
Trace semantics

Non-determinization of simple Segala systems

$\mathcal{P}(A \times \mathcal{D})$ SSeg



$\mathcal{P}(A \times (-))$ LTS



Non-determinization of simple Segala systems

GT

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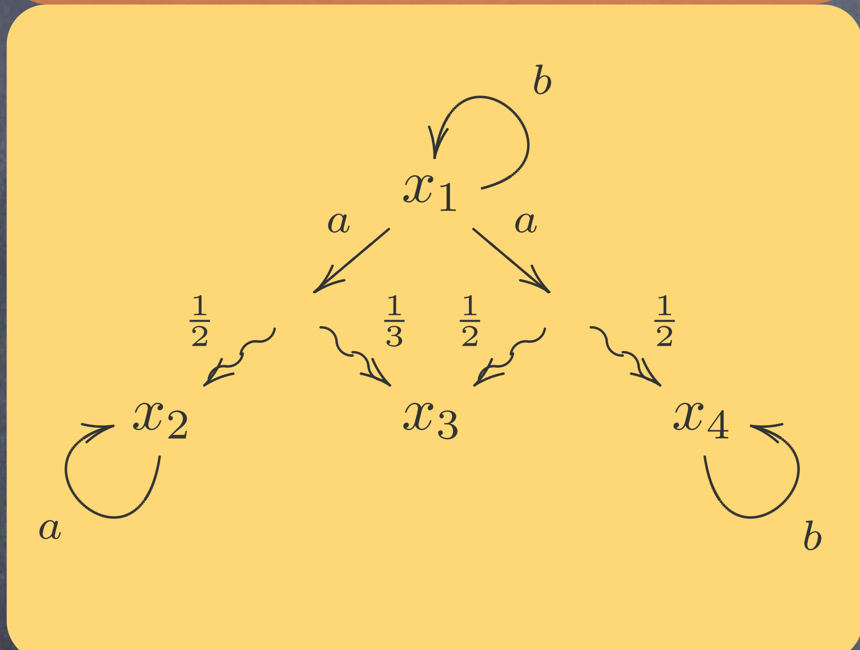
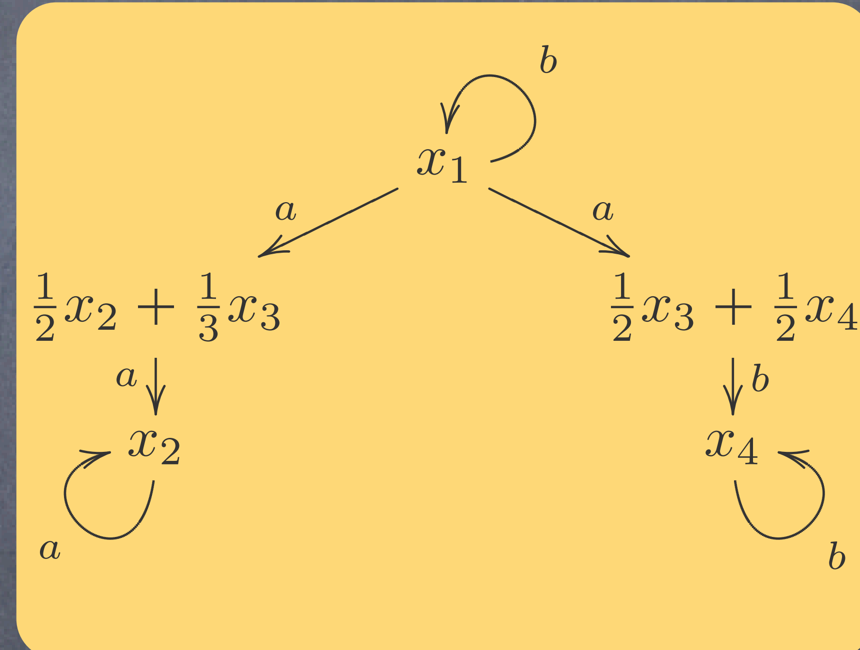


$\mathcal{P}(A \times (-))$ LTS



Non-determinization of simple Segala systems

GT

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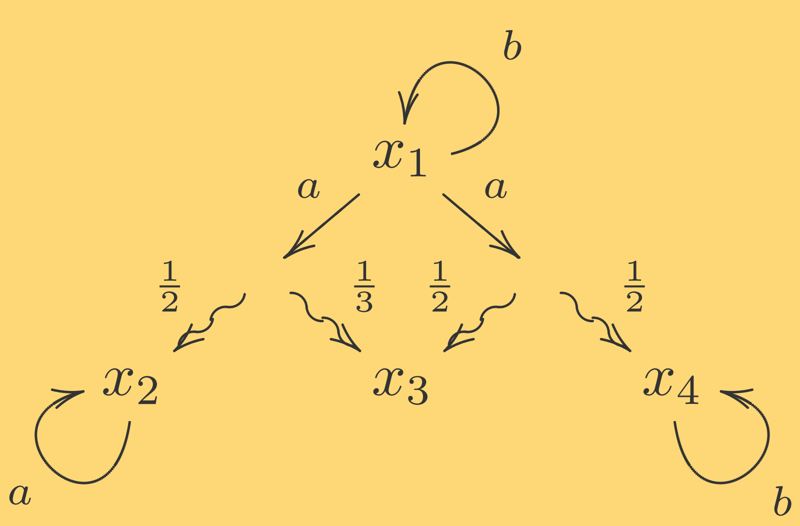
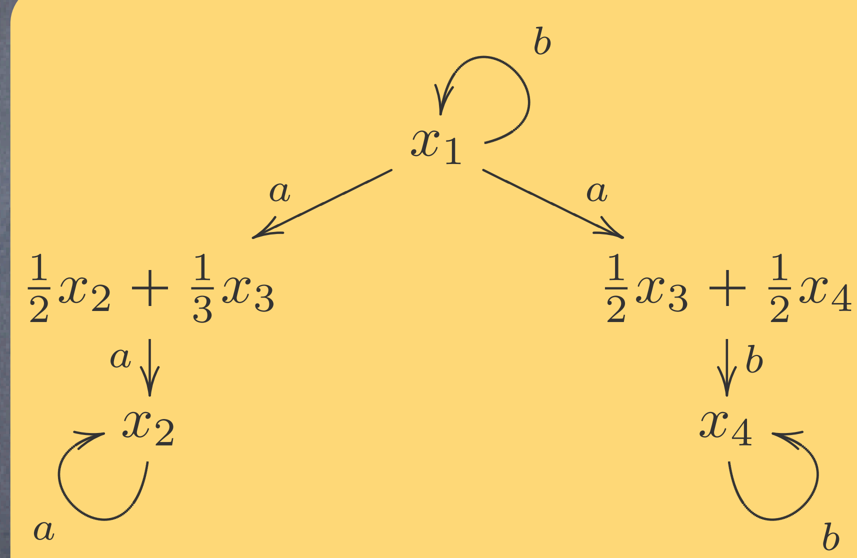


G on T(-)

$\mathcal{P}(A \times (-))$ LTS



Non-determinization of simple Segala systems

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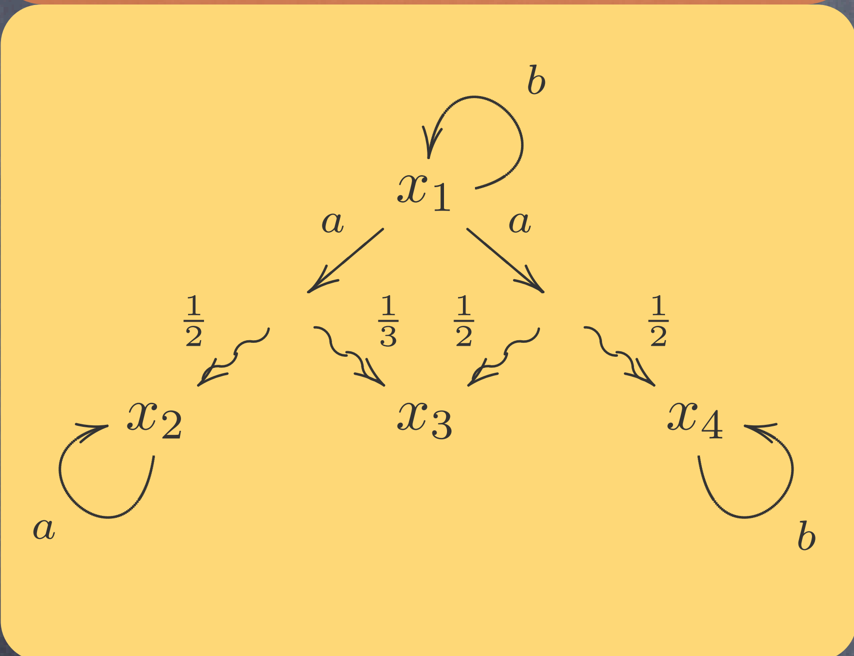

G on T(-)

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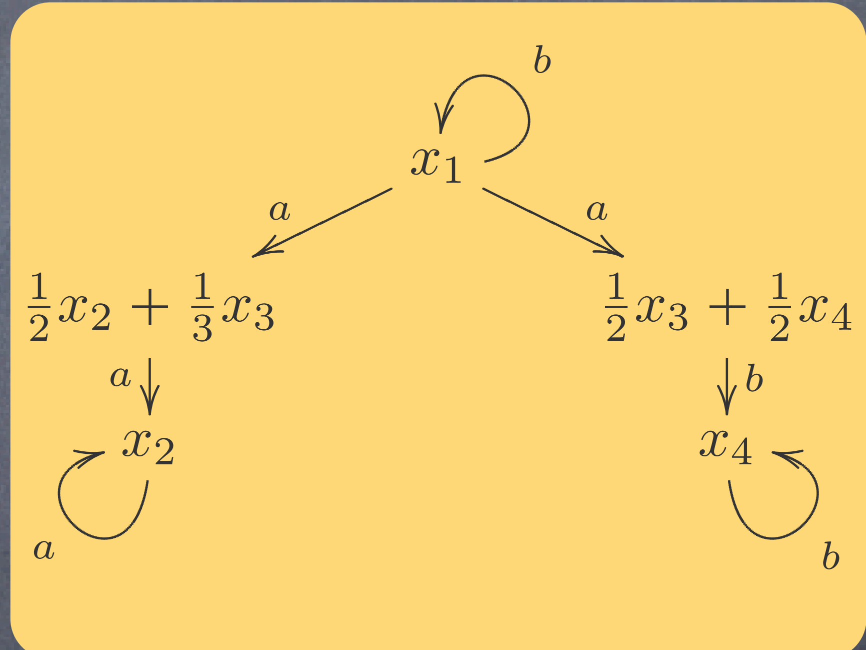

There is a distributive law that provides this non-determinization

LTS-semantics for SSeg

$\mathcal{P}_\omega \quad \mathcal{D}_\omega$

Relation to Kleisli traces

Assume F has an initial algebra $\iota\colon F(W) \xrightarrow{\cong} W$
and $\mathcal{F}(\iota^{-1})\colon W \to \hat{F}(W)$ is final

holds when Kleisli traces exist

- Given a coalgebra $X \xrightarrow{c} TFX$

$$
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 & \hat{G}(TX) \arrow[r] & \hat{G}(TW) \arrow[r, dashed] & \hat{G}(Z) \\
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\end{tikzcd}
$$

Extension semantics (trace)

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\end{tikzcd}$$

Extension semantics (trace)

Conclusions

- Traces via determinization

Kleisli traces

Traces via GPC

- works for both TF and GT coalgebras

in Kleisli and EM

- the semantics relate (often coincide)
- all about coalgebras over algebras

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Thank you !