Trace Semantics via Determinization

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Trace semantics for (more) coalgebras

- Kleisli trace semantics [HJS’07]

- Traces via the ``generalized powerset construction” --- determinization [SBBR’10]

traces as ``coalgebraic language equivalence”

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GT-coalgebras

T - monad, Eilenberg-Moore category

Needed: TG => GT + final G

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generative

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- Traces via the ``generalized powerset construction” [SBBR’10]

reactive

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Examples: $\mathcal{P}(1 + A \times (-))$ NFA

$\mathcal{D}(1 + A \times (-))$ PTS

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Examples: $2 \times \mathcal{P}^A$ NFA
$S \times \mathcal{M}_S^A$ WTS

What is in common?

Semantics via finality (coinduction)

of coalgebras over Kleisli or EM categories

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Final coalgebra semantics:

$$\begin{tikzcd} X \arrow[r, dashed, "\mathrm{beh}"] \arrow[d, "c"'] & Z \arrow[d, "\cong"] \\ HX \arrow[r, dashed, "H\,\mathrm{beh}"] & HZ \end{tikzcd}$$

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bisimilarity in $\mathbf{Sets}$

(for wpp functors)

trace semantics in $\mathcal{K}\ell(T)$

(for TF-coalgebras)

coalgebraic language eq. in $\mathcal{EM}(T)$

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How do they relate?

The categories via the comparison/extension functor

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[r, "E"] \arrow[dr] & \mathcal{EM}(T) \arrow[d] \\
& \mathbb{C}
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& \mathbb{C} &
\end{tikzcd}$$

$$E(X) = \left(\begin{tikzcd} T^2X \arrow[d, "\mu"] \\ TX \end{tikzcd}\right) \qquad E(f) = \mu \circ T(f)$$

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Kleisli extension

$f : X \to Y$ in $\mathcal{K}\ell(T)$

$f : X \to TY$ in $\mathbb{C}$

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It's all about liftings!

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The big picture

$$\begin{tikzcd}
\mathsf{Coalg}(\hat{F}) \arrow[rr, "\hat{E}"] \arrow[d] & & \mathsf{Coalg}(\hat{G}) \arrow[d] \\
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Eventually we will lift E

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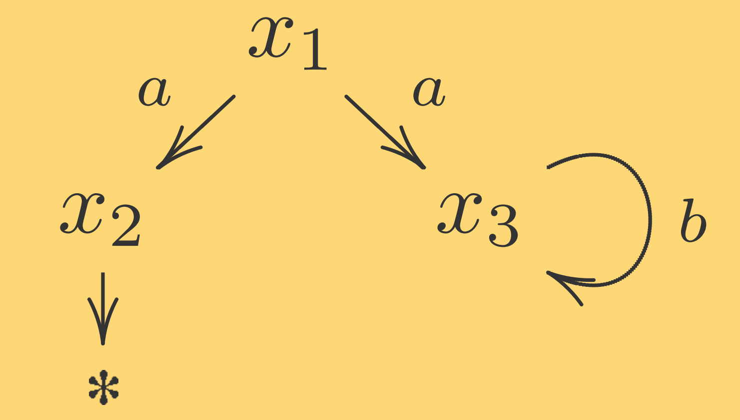
\end{tikzcd}$$

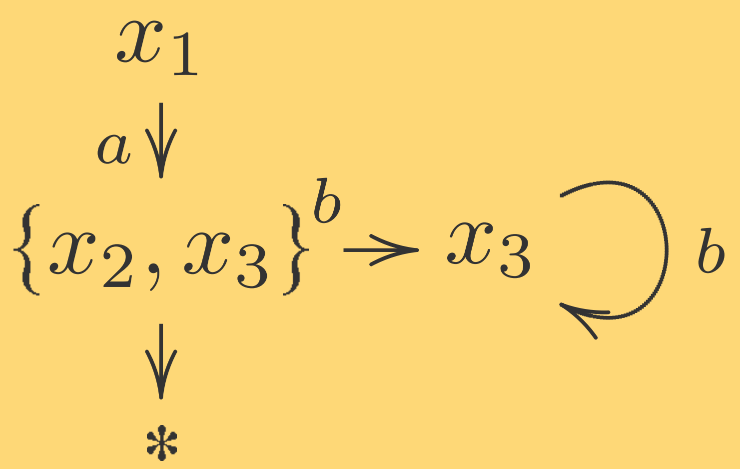
But before that, some intuition...

Determinization of NFA

$\mathcal{P}(1 + A \times (-))$ **NFA**



$2 \times (-)^A$ **DFA**



Determinization of NFA

TF

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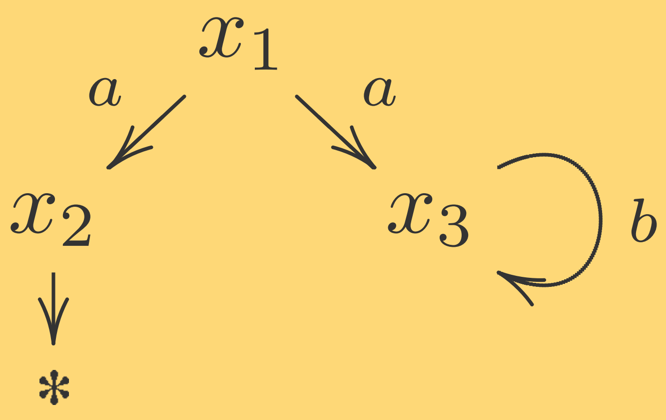

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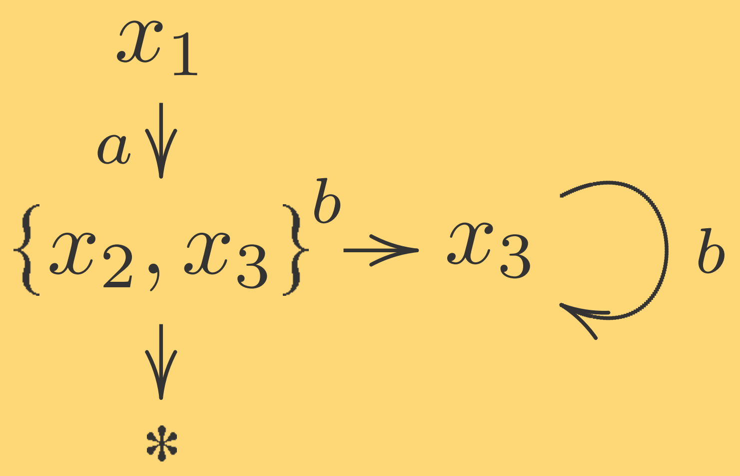

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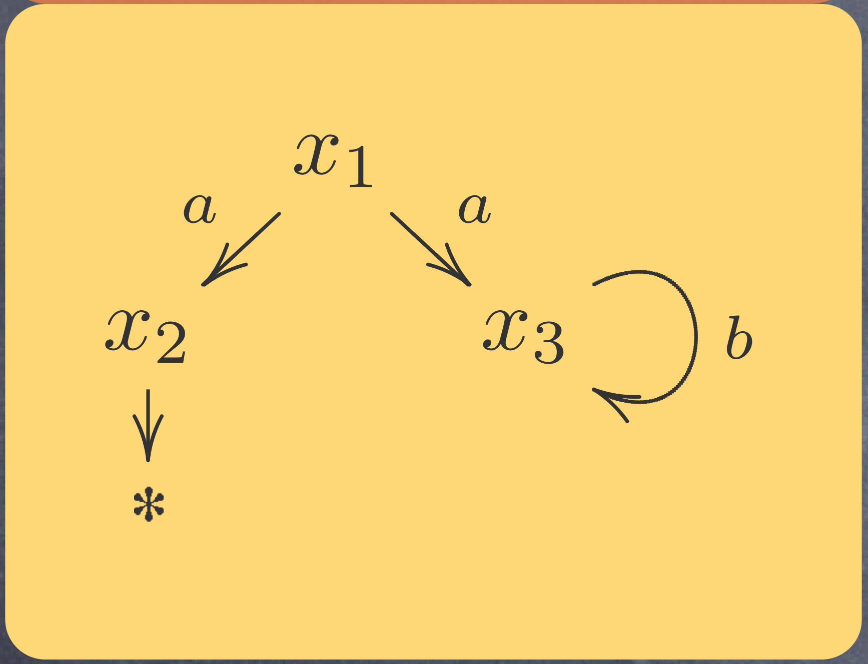
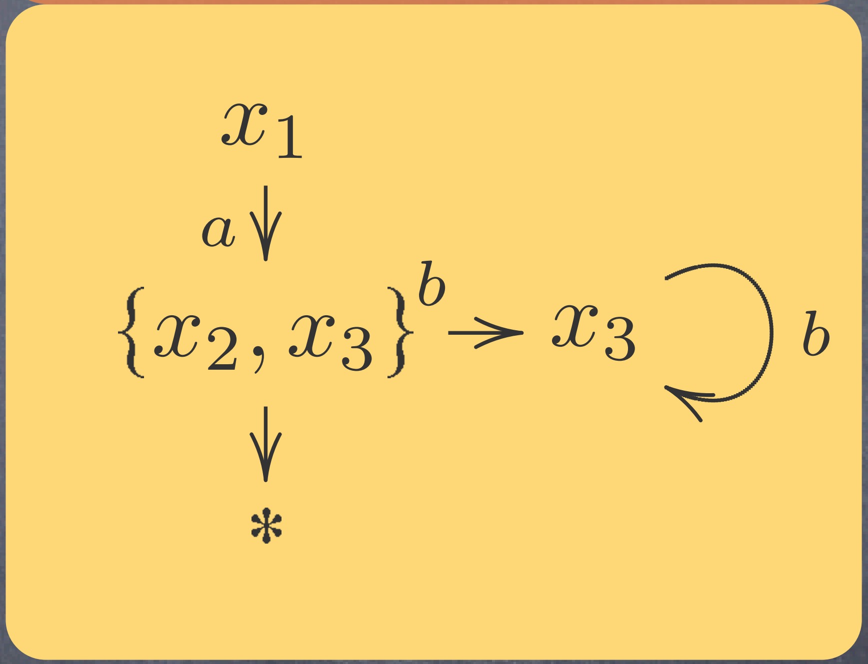


G on T(-)

$2 \times (-)^A$ **DFA**



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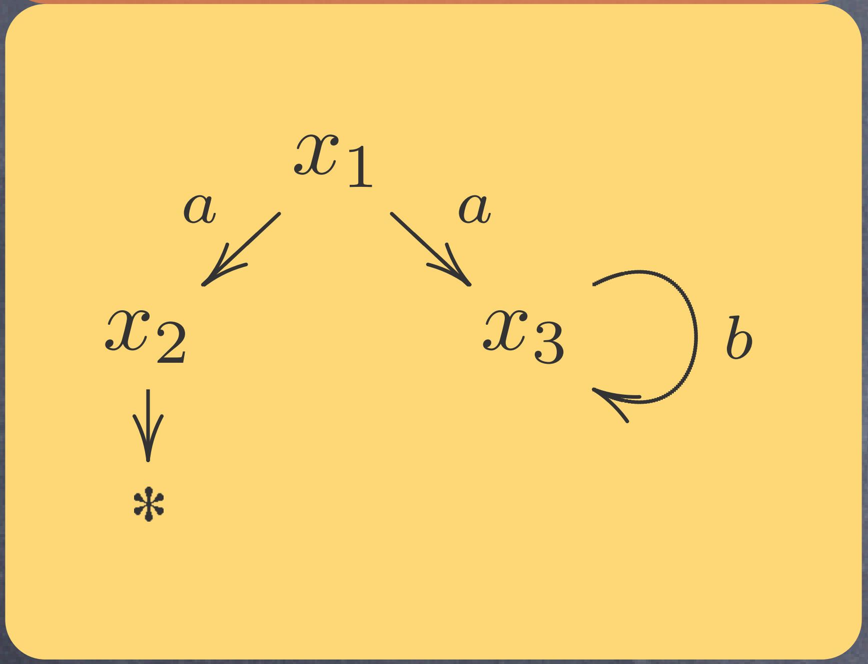

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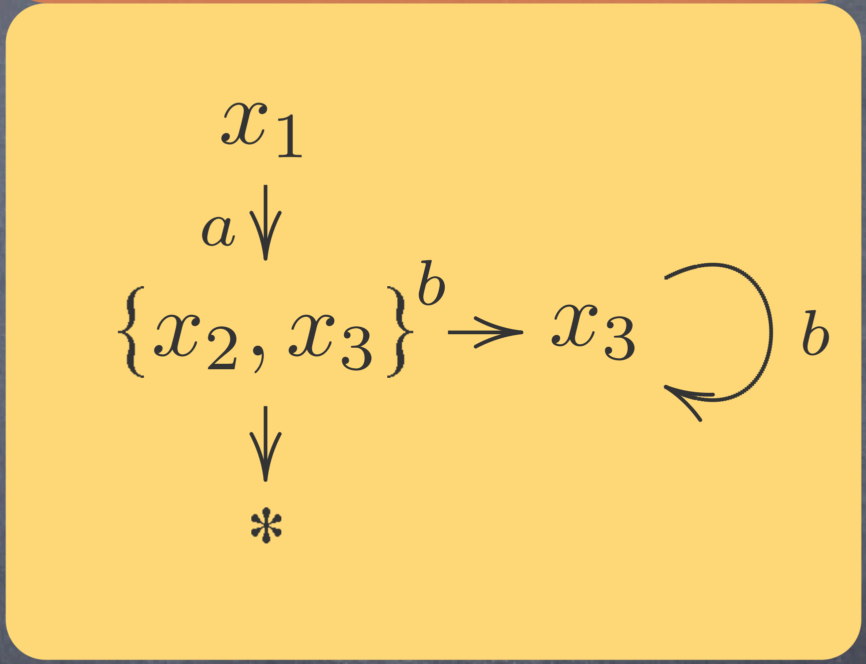

$$\begin{tikzcd}
X \arrow[r, "beh"] \arrow[d, "c"'] & 2^{A^*} \arrow[d, "\cong"] \\
GX \arrow[r, "Gbeh"] & G(2^{A^*})
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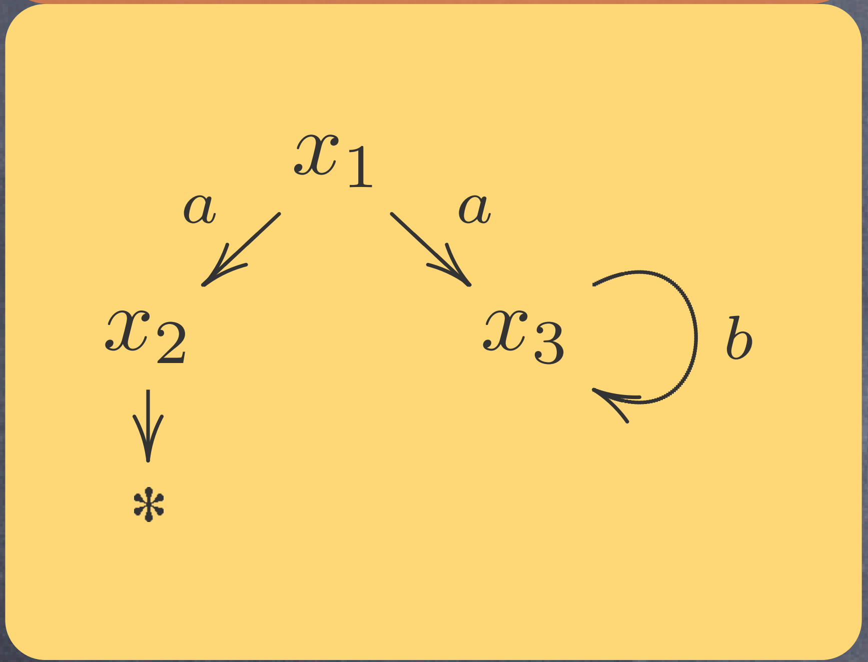

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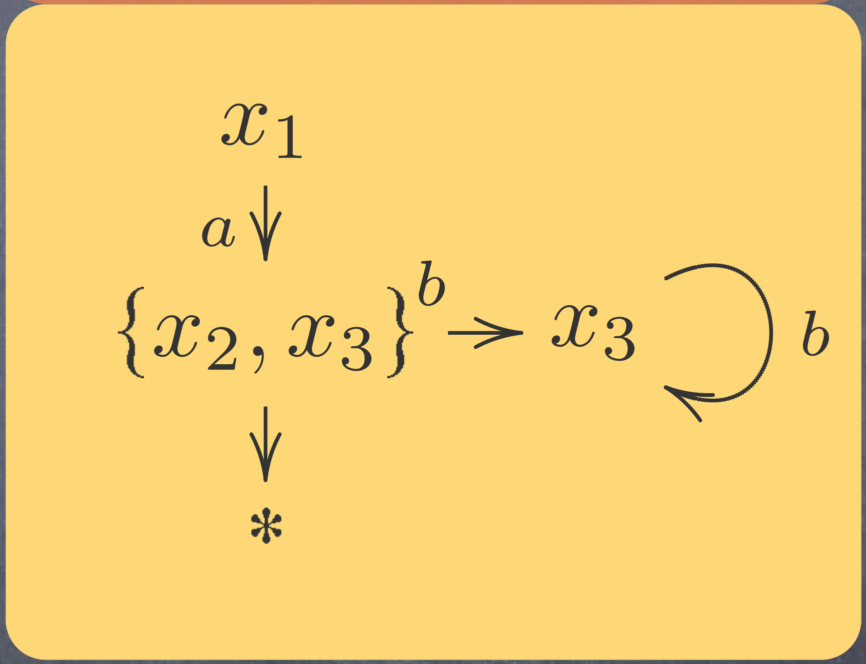

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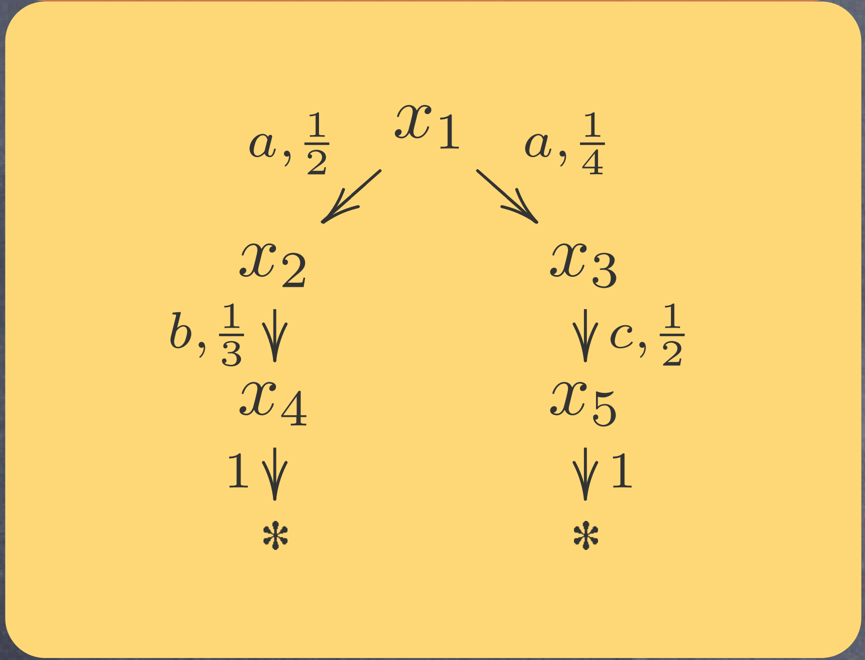
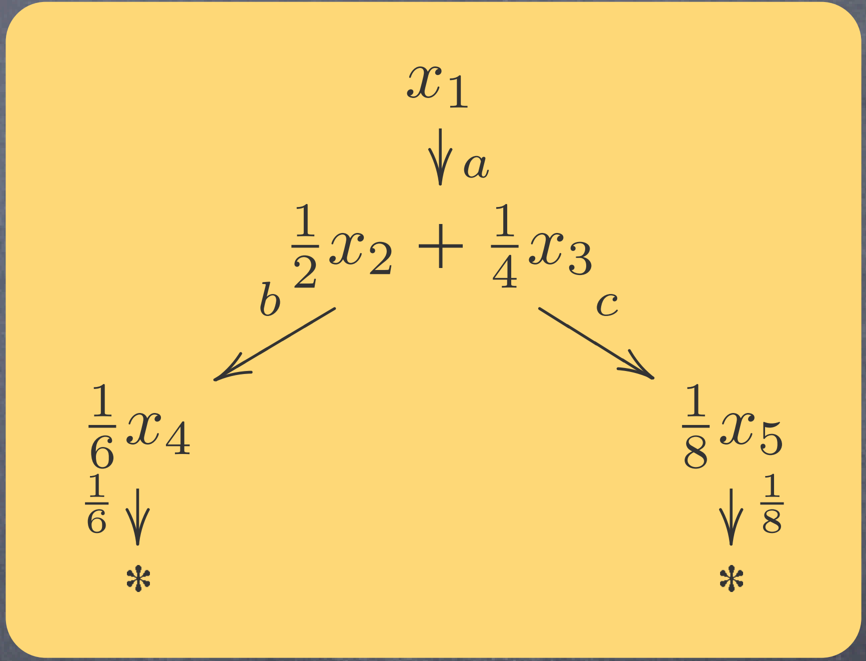
final G-coalgebra

Determinization of PTS

$\mathcal{D}(1 + A \times (-))$ PTS



$[0,1] \times (-)^A$ DFA



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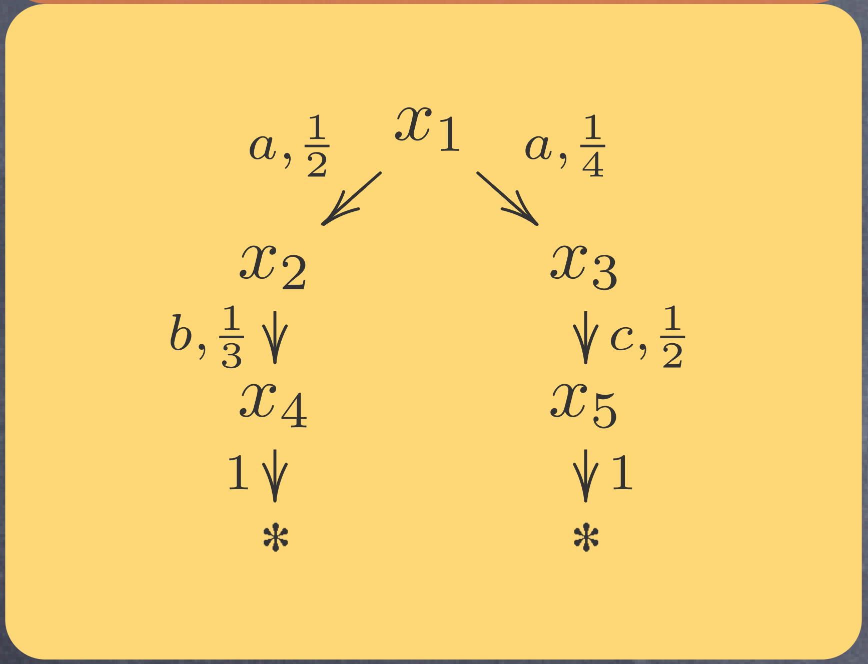

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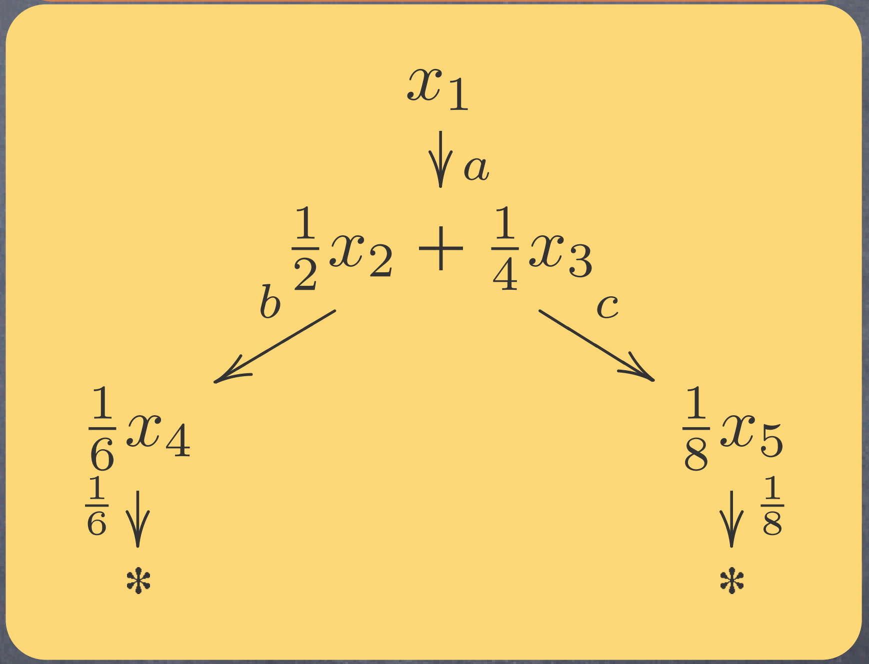

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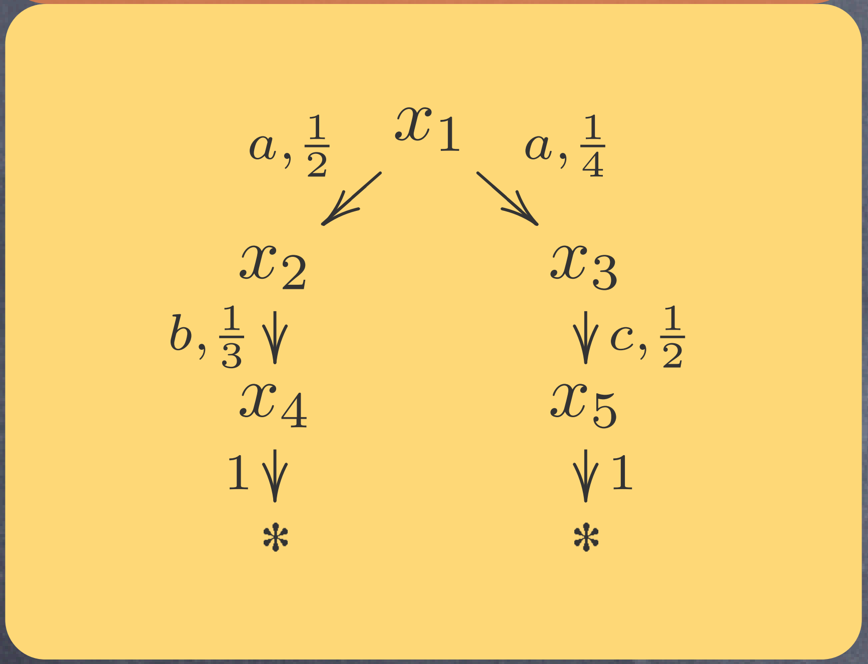

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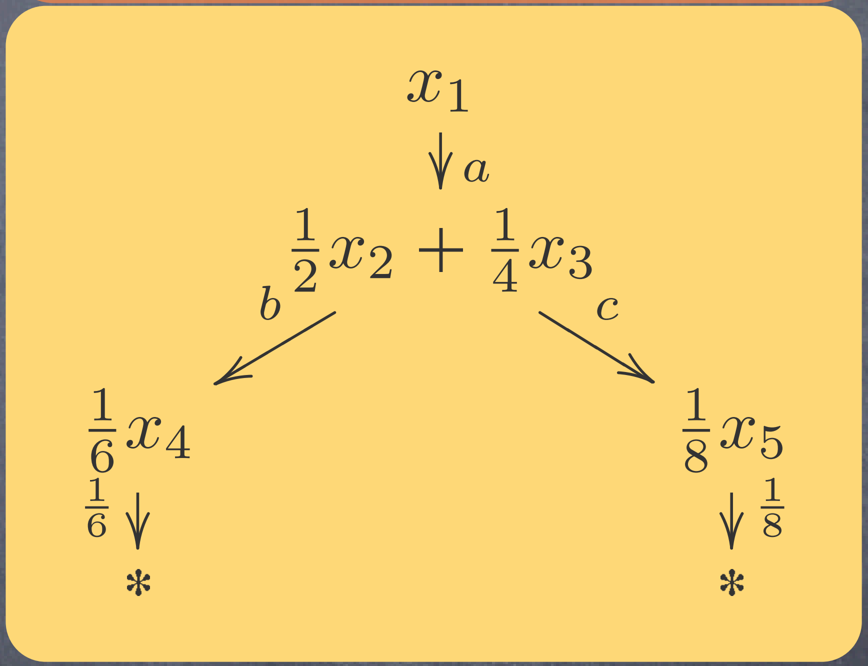

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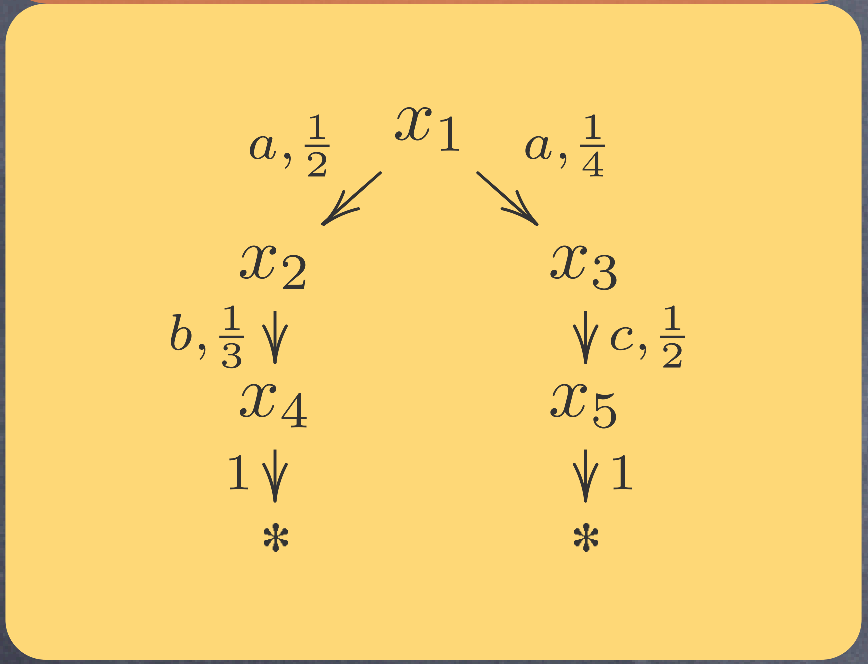

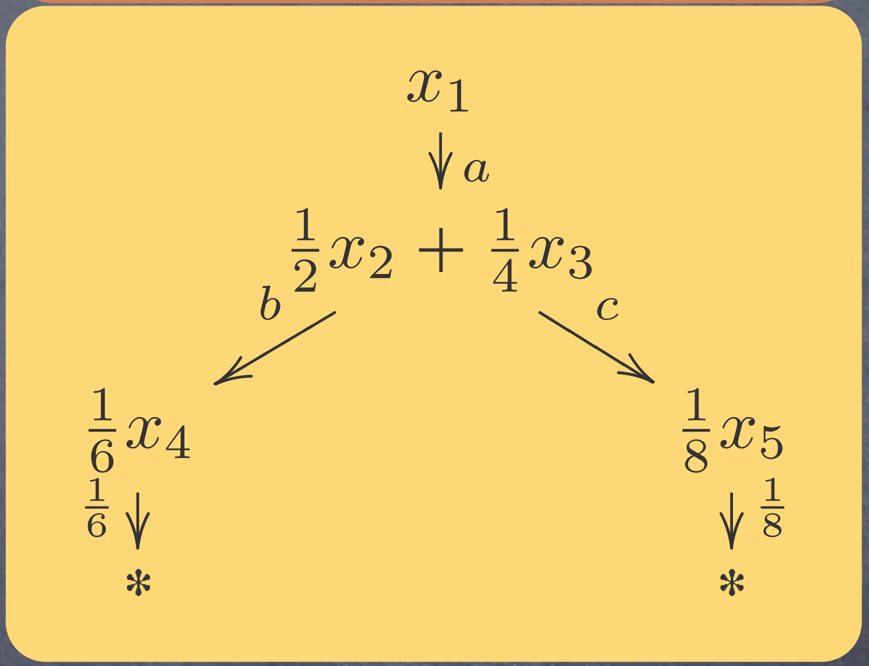

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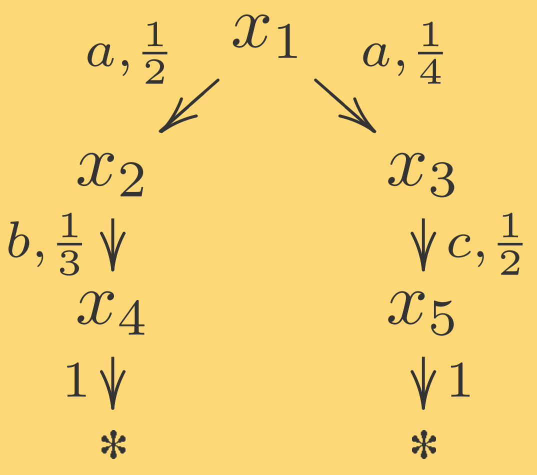

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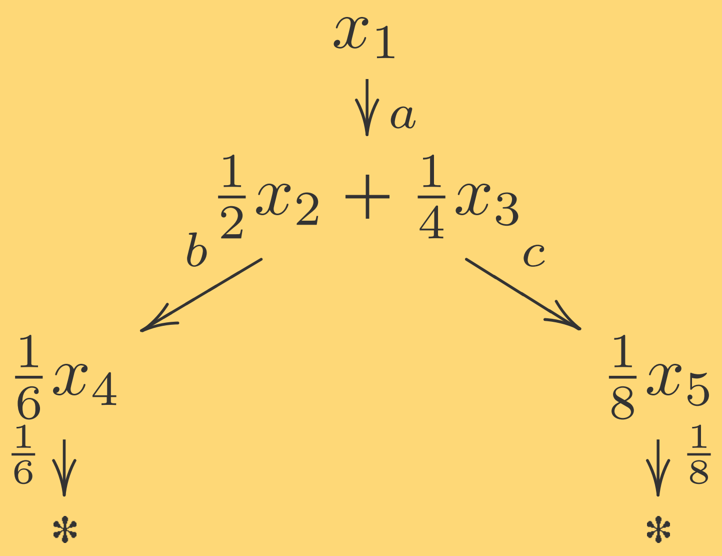

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final G-coalgebra

Laws and liftings

$\mathcal{K}\ell$-law $\lambda\colon FT \Rightarrow TF$

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[r, "\hat{F}"] \arrow[d] & \mathcal{K}\ell(T) \arrow[d] \\
\mathbb{C} \arrow[r, "F"] & \mathbb{C}
\end{tikzcd}$$

$\mathcal{EM}$-law $\rho\colon TG \Rightarrow GT$

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$$\begin{tikzcd}
\mathbf{CoAlg}(GT) \arrow[r, "\mathcal{F}_{\mathcal{EM}}"] \arrow[d] & \mathbf{CoAlg}(\hat{G}) \arrow[d] \\
\mathbb{C} \arrow[r, "\mathcal{F}"] \arrow[loop left, "G"] \arrow[loop below, "T"] & \mathcal{EM}(T) \arrow[loop below, "\hat{G}"]
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$$\mathcal{F}_{\mathcal{EM}}\left(X \xrightarrow{c} GTX\right) = \begin{pmatrix} T^2X \\ \downarrow\mu \\ TX \end{pmatrix} \xrightarrow{G\mu\circ\rho_{TX}\circ T(c)} \hat{G}\begin{pmatrix} T^2X \\ \downarrow\mu \\ TX \end{pmatrix} \qquad \mathcal{F}_{\mathcal{EM}}(f) = T(f)$$

"Determinization"
(in the GPC)

$$\begin{tikzcd}
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free functor

Laws and liftings

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\end{tikzcd}$$

"Determinization"
(in the GPC)

The final coalgebra also lifts

free functor

GT-coalgebras (GPC)

Assume $TG \Rightarrow GT$ and final $Z \xrightarrow{\cong} GZ$ exists

- Given a coalgebra $X \xrightarrow{c} GTX$
- "Determinize" $\mathcal{F}_{\mathcal{EM}}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$
- Get semantics by

$$\begin{tikzcd}
 & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] \arrow[d, "\mathcal{F}_{\mathcal{EM}}(c)"'] & \hat{G}Z \arrow[d, "\cong"] \\
X \arrow[r, "\eta"] & TX \arrow[r, "beh"] & Z
\end{tikzcd}$$

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GT-coalgebras (GPC)

Assume $TG \Rightarrow GT$ and final $Z \xrightarrow{\cong} GZ$ exists

Works for deterministic automata

$G = T(B) \times (-)^A$

strong

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- "Determinize" $\mathcal{F}_{\mathcal{EM}}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$
- Get semantics by

Determinization

$$\begin{tikzcd}
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Trace semantics

TF-coalgebras?

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$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

TF-coalgebras?

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$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

Extension natural tr.

$\mathfrak{e}\colon TF \Rightarrow GT$

connecting the laws

TF-coalgebras?

$\mathcal{K}\ell$-law $\lambda\colon FT \Rightarrow TF$

$$\begin{tikzcd}
\mathcal{K}\ell(T) \arrow[r, "\hat{F}"] & \mathcal{K}\ell(T) \\
\mathbb{C} \arrow[u] \arrow[r, "F"] & \mathbb{C} \arrow[u]
\end{tikzcd}$$

$\mathcal{EM}$-law $\rho\colon TG \Rightarrow GT$

$$\begin{tikzcd}
\mathcal{EM}(T) \arrow[r, "\hat{G}"] \arrow[d] & \mathcal{EM}(T) \arrow[d] \\
\mathbb{C} \arrow[r, "G"] & \mathbb{C}
\end{tikzcd}$$

$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoAlg}(\hat{G})$$

Extension natural tr.

$\mathfrak{e}\colon TF \Rightarrow GT$

connecting the laws

$$\begin{tikzcd}
\mathsf{Coalg}(\hat{F}) \arrow[r, "\hat{E}"] \arrow[d] & \mathsf{Coalg}(\hat{G}) \arrow[d] \\
\mathcal{K}\ell(T) \arrow[loop left, "\hat{F}"] \arrow[r, "E"] & \mathcal{EM}(T) \arrow[loop right, "\hat{G}"]
\end{tikzcd}$$

TF-coalgebras?

$\mathcal{K}\ell$-law $\lambda\colon FT \Rightarrow TF$

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$$\mathbf{CoAlg}(TF) \xrightarrow{?} \mathbf{CoAlg}(GT) \xrightarrow{\mathcal{F}_{\mathcal{EM}}} \mathbf{CoA}$$

Extension natural tr.

$\mathfrak{e}\colon TF \Rightarrow GT$

connecting the laws

$$\hat{E}(c) = \mathfrak{e} \circ \mu \circ T(c)$$
$$\hat{E}(f) = E(f)$$

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\end{tikzcd}$$

"Determinization"

TF-coalgebras

Assume $FT \Rightarrow TF$ $TG \Rightarrow GT$ $\mathfrak{e}: TF \Rightarrow GT$

and final $Z \xrightarrow{\cong} GZ$ exists

- Given a coalgebra $X \xrightarrow{c} TFX$
- "Determinize" $\hat{E}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$
- Get semantics by

$$\begin{tikzcd}
 & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] \arrow[d, "\hat{E}(c)"'] & \hat{G}Z \arrow[d, "\cong"] \\
X \arrow[r, "\eta"] & TX \arrow[r, "beh"] & Z
\end{tikzcd}$$

TF-coalgebras

Assume $FT \Rightarrow TF \quad TG \Rightarrow GT \quad \mathfrak{e}: TF \Rightarrow GT$
and final $Z \overset{\cong}{\rightarrow} GZ$ exists

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TF-coalgebras

Assume $FT \Rightarrow TF \quad TG \Rightarrow GT \quad \mathfrak{e}: TF \Rightarrow GT$

and final $Z \xrightarrow{\cong} GZ$ exists

Works for
all examples
we have seen

- Given a coalgebra $X \xrightarrow{c} TFX$
- ``Determinize" $\hat{E}(c) = (TX, \mu) \to \hat{G}(TX, \mu)$

Determinization

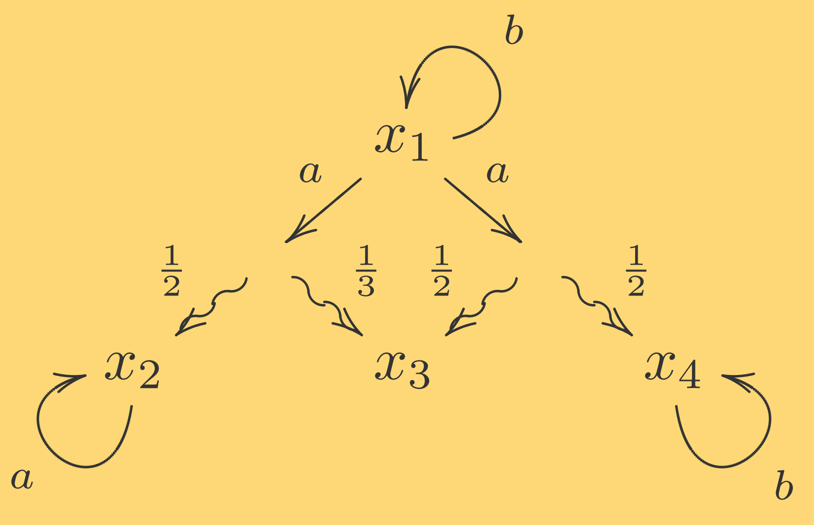
- Get semantics by

$$\begin{tikzcd}
 & \hat{G}(TX) \arrow[r, "\hat{G}(beh)"] & \hat{G}Z \\
X \arrow[r, "\eta"] & TX \arrow[u, leftarrow, "\hat{E}(c)"'] \arrow[r, "beh"] & Z \arrow[u, leftarrow, "\cong"']
\end{tikzcd}$$

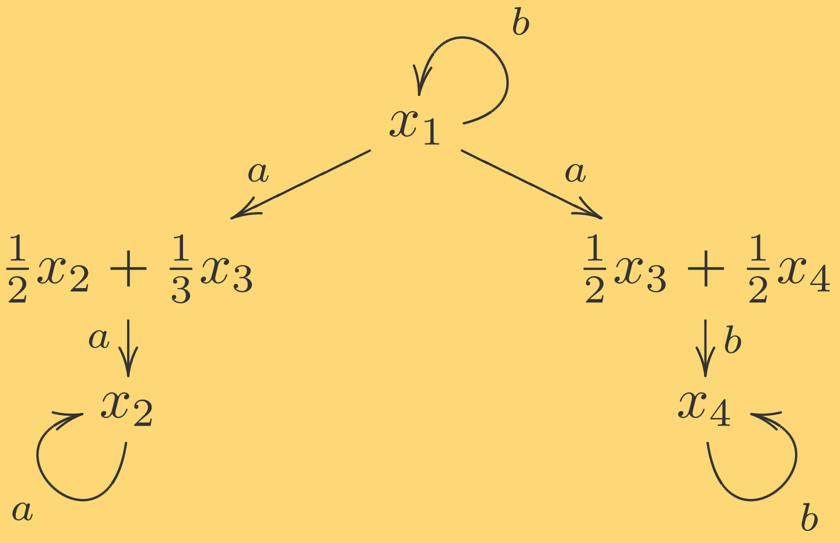
Trace semantics

Non-determinization of simple Segala systems

$\mathcal{P}(A \times \mathcal{D})$ SSeg



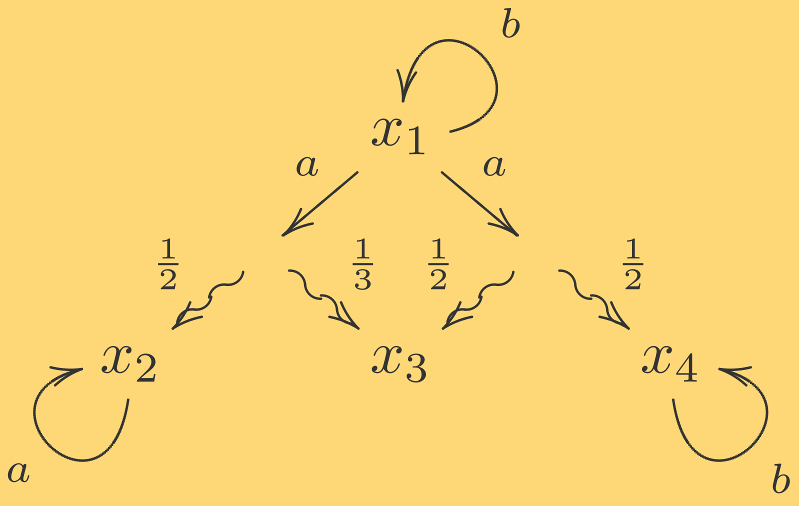
$\mathcal{P}(A \times (-))$ LTS



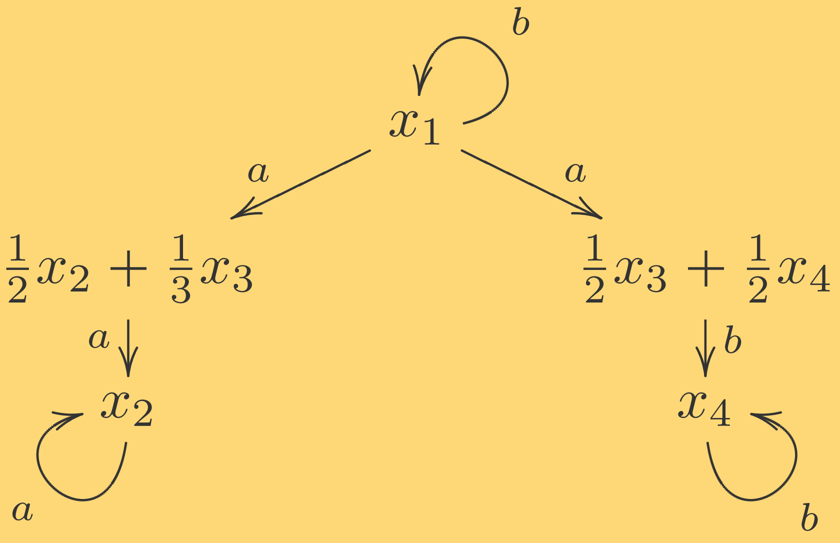
Non-determinization of simple Segala systems

GT

$\mathcal{P}(A \times \mathcal{D})$ SSeg



$\mathcal{P}(A \times (-))$ LTS



Non-determinization of simple Segala systems

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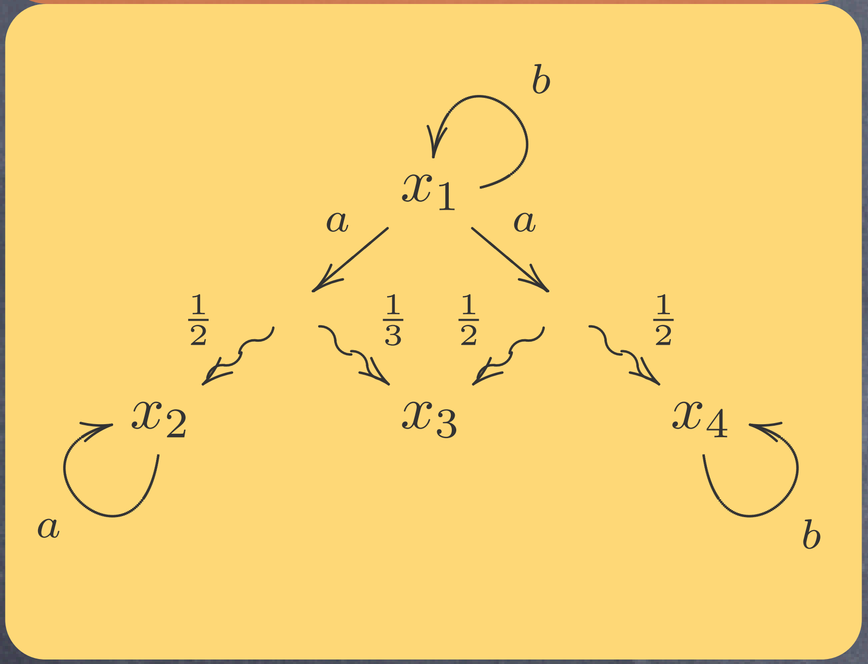
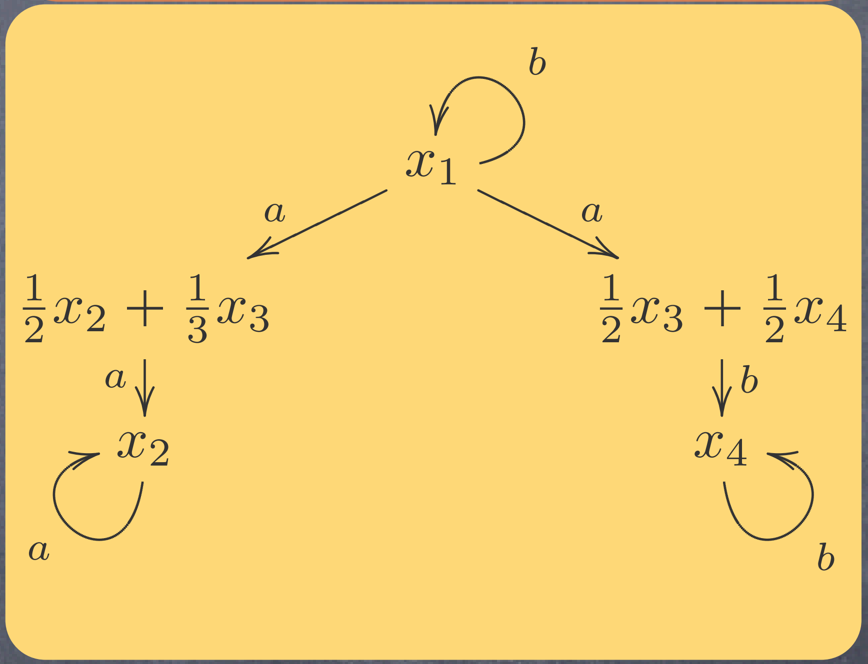


G on T(-)

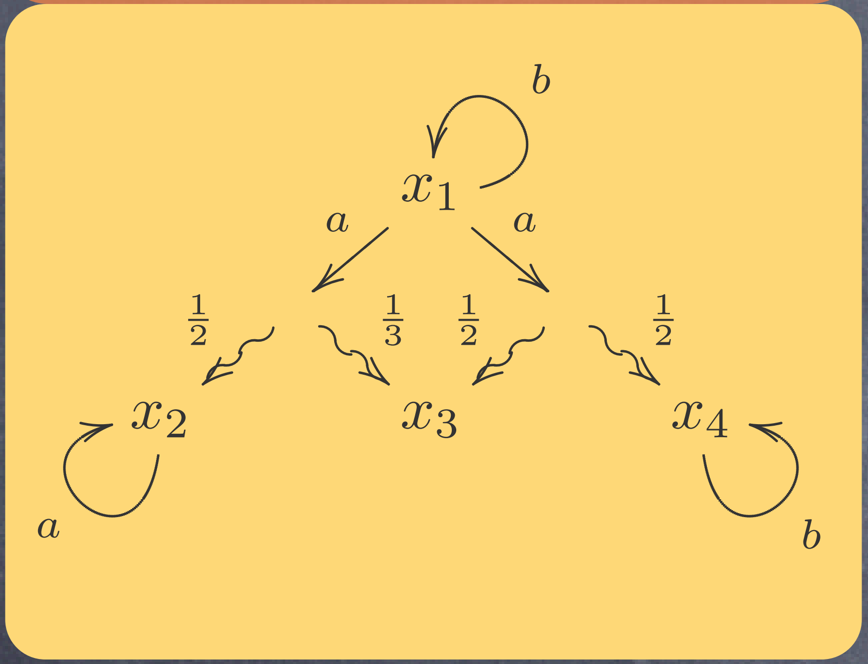
$\mathcal{P}(A \times (-))$ LTS



Non-determinization of simple Segala systems

GT

$\mathcal{P}(A \times \mathcal{D})$ SSeg



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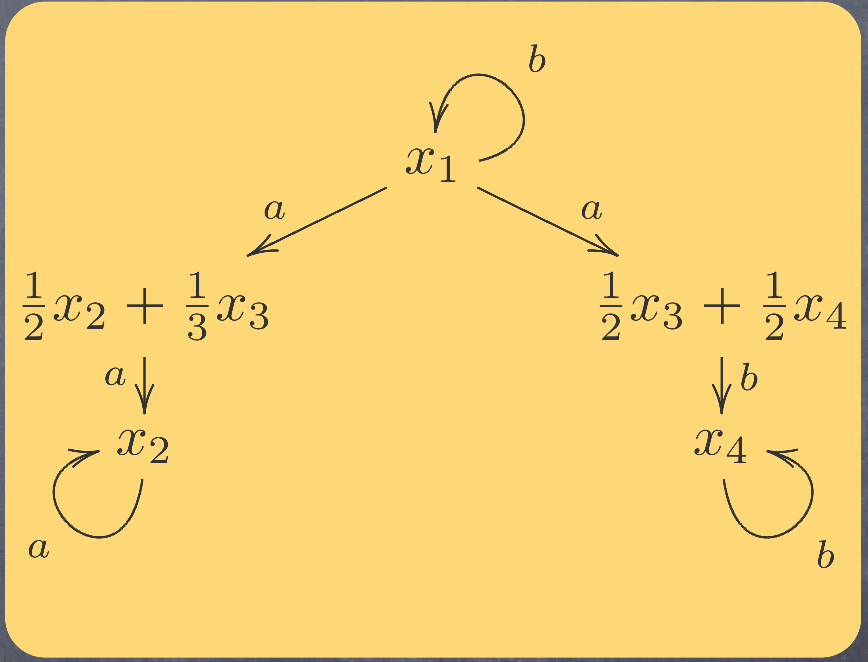

There is a distributive law that provides this non-determinization

LTS-semantics for SSeg

$\mathcal{P}_\omega \quad \mathcal{D}_\omega$

Relation to Kleisli traces

Assume F has an initial algebra $\iota\colon F(W) \xrightarrow{\cong} W$
and $\mathcal{F}(\iota^{-1})\colon W \to \hat{F}(W)$ is final

holds when Kleisli traces exist

- Given a coalgebra $X \xrightarrow{c} TFX$

$$\begin{tikzcd}
 & \hat{G}(TX) \arrow[r] & \hat{G}(TW) \arrow[r, dashed] & \hat{G}(Z) \\
X \arrow[r, "\eta"] \arrow[rr, bend right, two heads, "\mathrm{tr}_{\mathcal{K}\ell}(c)"'] & TX \arrow[u, "\hat{E}(c)"] \arrow[r, two heads, "\hat{E}(\mathrm{tr}_{\mathcal{K}\ell}(c))"] & TW \arrow[u, "\cong"', "\hat{E}(\mathcal{F}(\iota^{-1}))"] \arrow[r, dashed] & Z \arrow[u, "\cong"']
\end{tikzcd}$$

Extension semantics (trace)

Relation to Kleisli traces

Assume F has an initial algebra $\iota\colon F(W) \xrightarrow{\cong} W$ and $\mathcal{F}(\iota^{-1})\colon W \to \hat{F}(W)$ is final

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- Given a coalgebra $X \xrightarrow{c} TFX$

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X \arrow[r, "\eta"] \arrow[rr, bend right, "\mathrm{tr}_{\mathcal{K}\ell}(c)"'] & TX \arrow[u, "\hat{E}(c)"] \arrow[r, "\hat{E}(\mathrm{tr}_{\mathcal{K}\ell}(c))"] & TW \arrow[u, "\cong"', "\hat{E}(\mathcal{F}(\iota^{-1}))"] \arrow[r, dashed] & Z \arrow[u, "\cong"']
\end{tikzcd}$$

Extension semantics (trace)

Conclusions

- Traces via determinization

Kleisli traces

Traces via GPC

- works for both TF and GT coalgebras

in Kleisli and EM

- the semantics relate (often coincide)
- all about coalgebras over algebras

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Thank you !