Probabilistic Systems Coalgebraically

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University of Salzburg

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- their modelling as coalgebras
- coalgebraic results for probabilistic systems

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specific results for probabilistic systems

generic results: "probability" is just a parameter

Major distinction

- Discrete systems
  discrete probability distributions
- Continuous systems
  continuous state space/continuous measures

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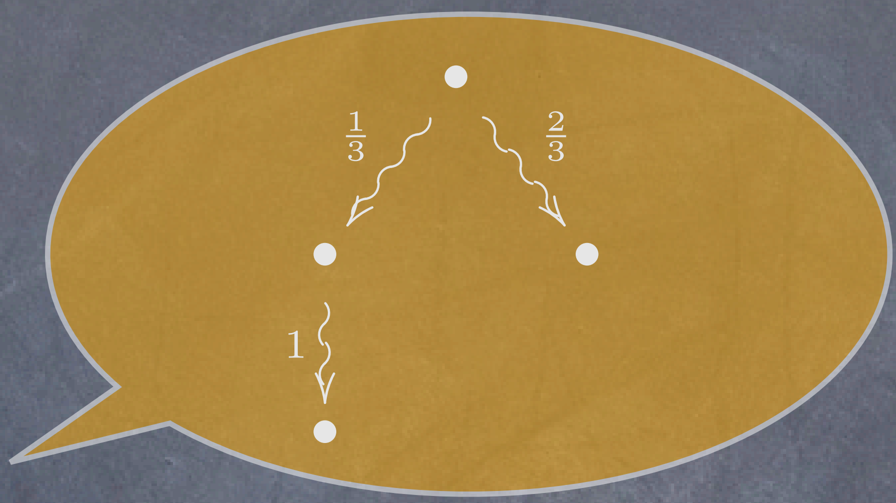

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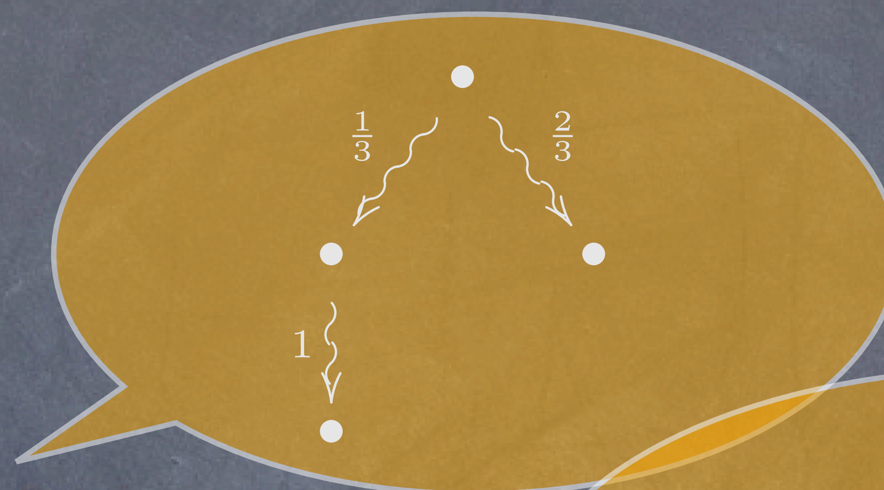

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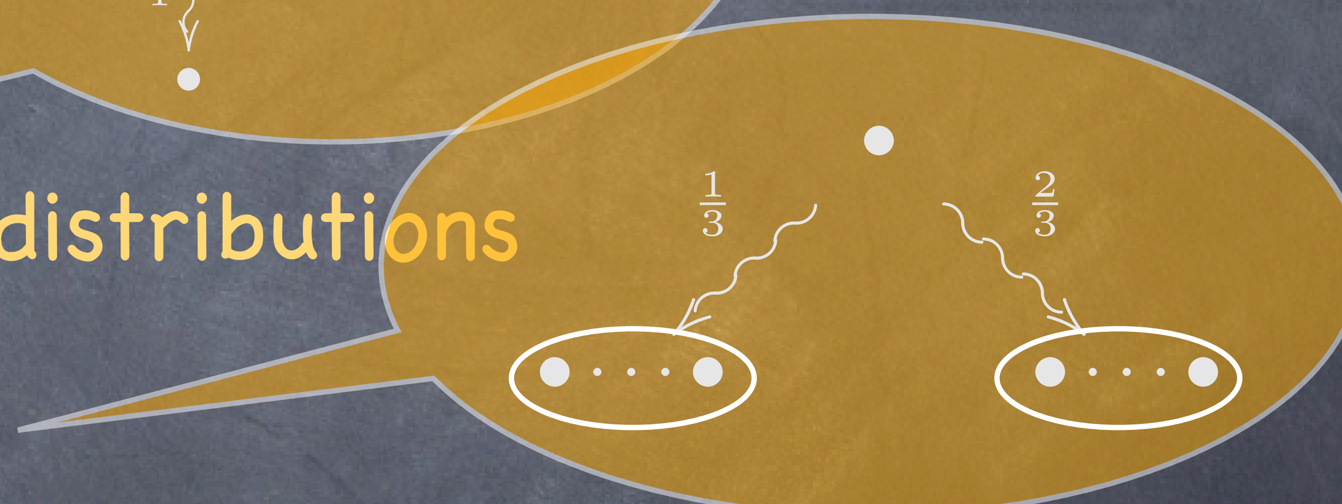

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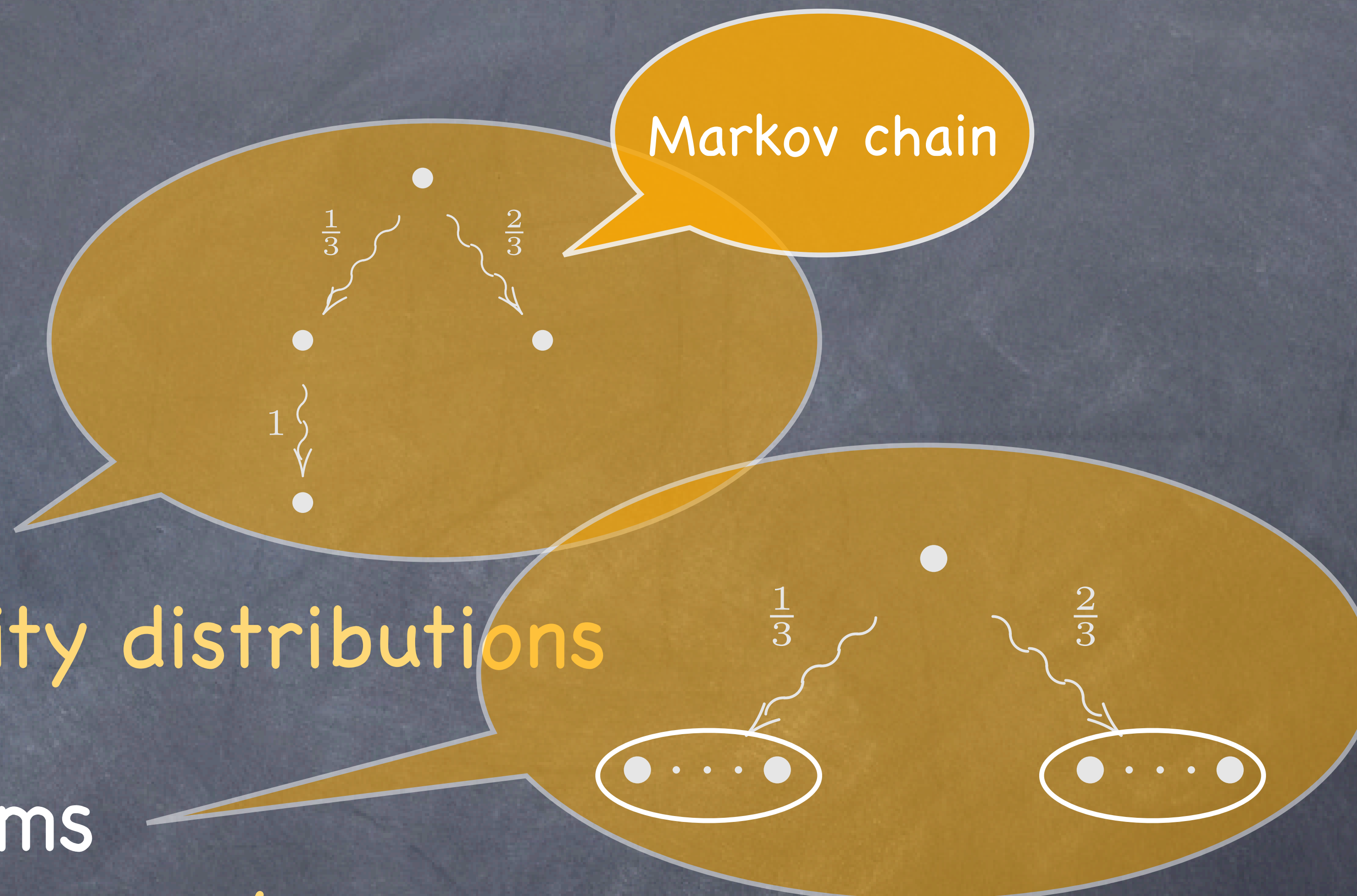

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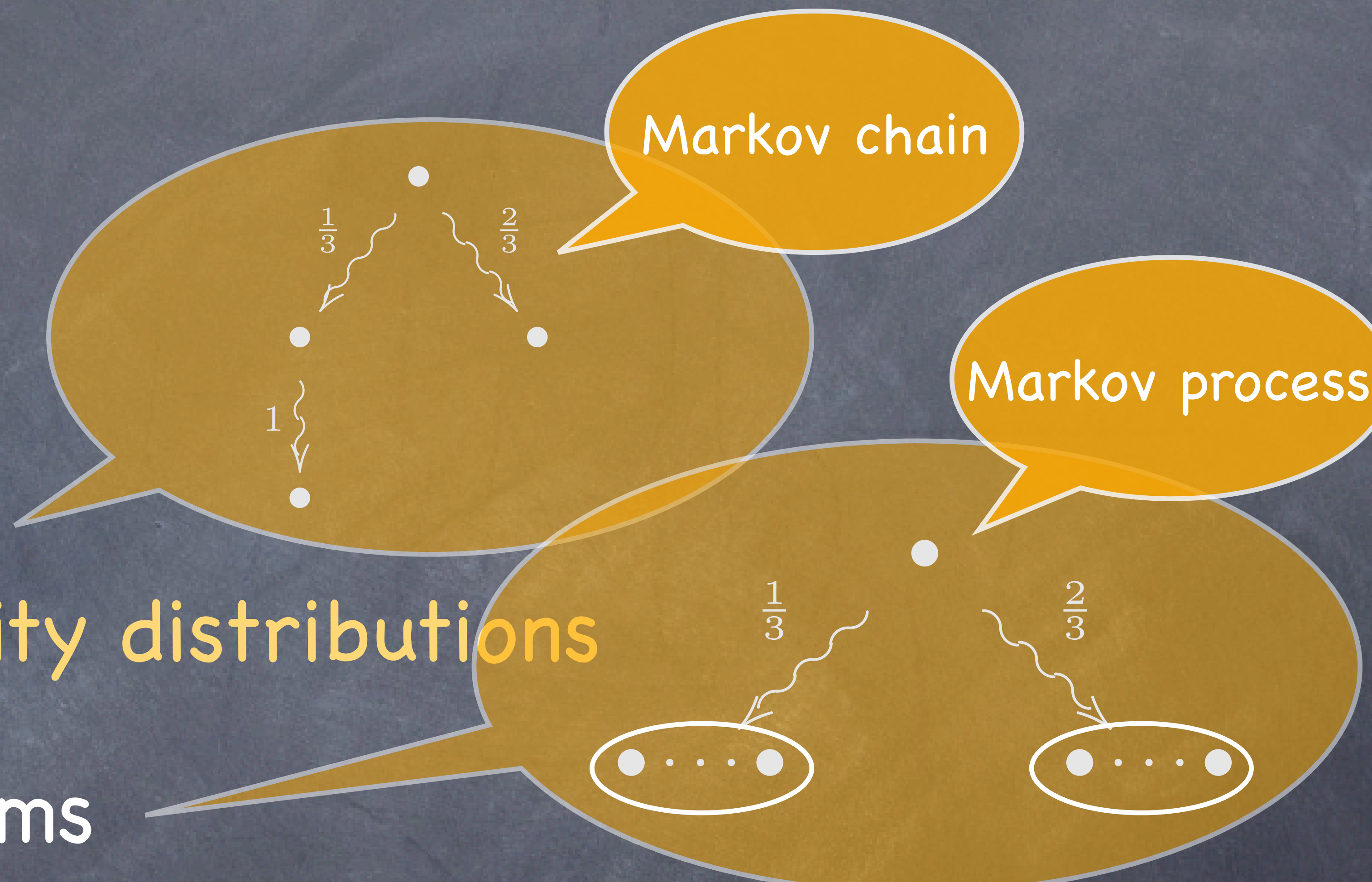

Part 1

Discrete probabilistic systems

Modelling discrete probabilistic systems

Probability distribution functor on **Sets**

$$\mathcal{D}(X) = \{\mu : X \to [0,1] \mid \sum_{x \in X} \mu(x) = 1\}$$

for $f : X \to Y$ we have $\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)$

$$\mathcal{D}(f)(\mu)(y) = \sum_{x \in f^{-1}(y)} \mu(x) = \mu(f^{-1}(y))$$

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preserves weak pullbacks

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has a final coalgebra

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Markov chains are $\mathcal{D}$-coalgebras on $\mathbf{Sets}$

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Coincides with Larsen&Skou bisimilarity
de Vink&Rutten '99

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$$F := \_ \mid A \mid \mathcal{D} \mid \mathcal{P} \mid F^A \mid F + F \mid F \times F \mid F \circ F$$

Discrete system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
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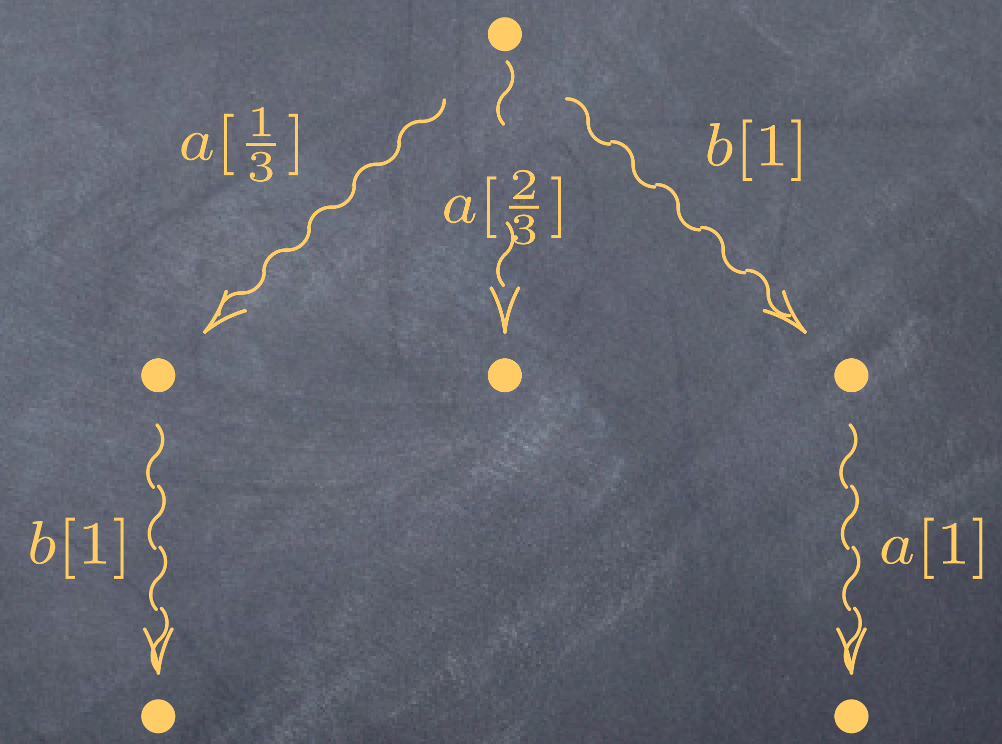

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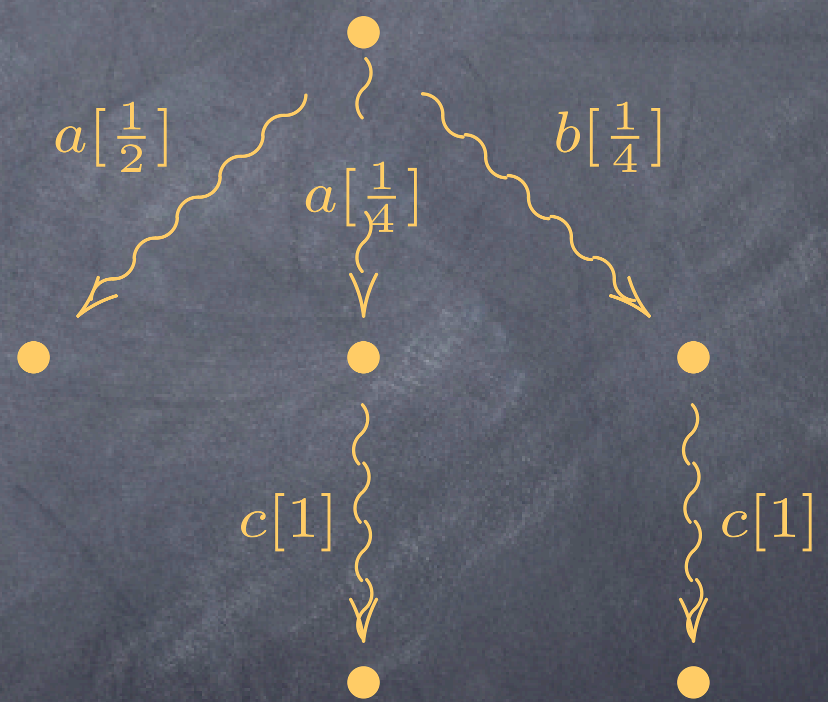

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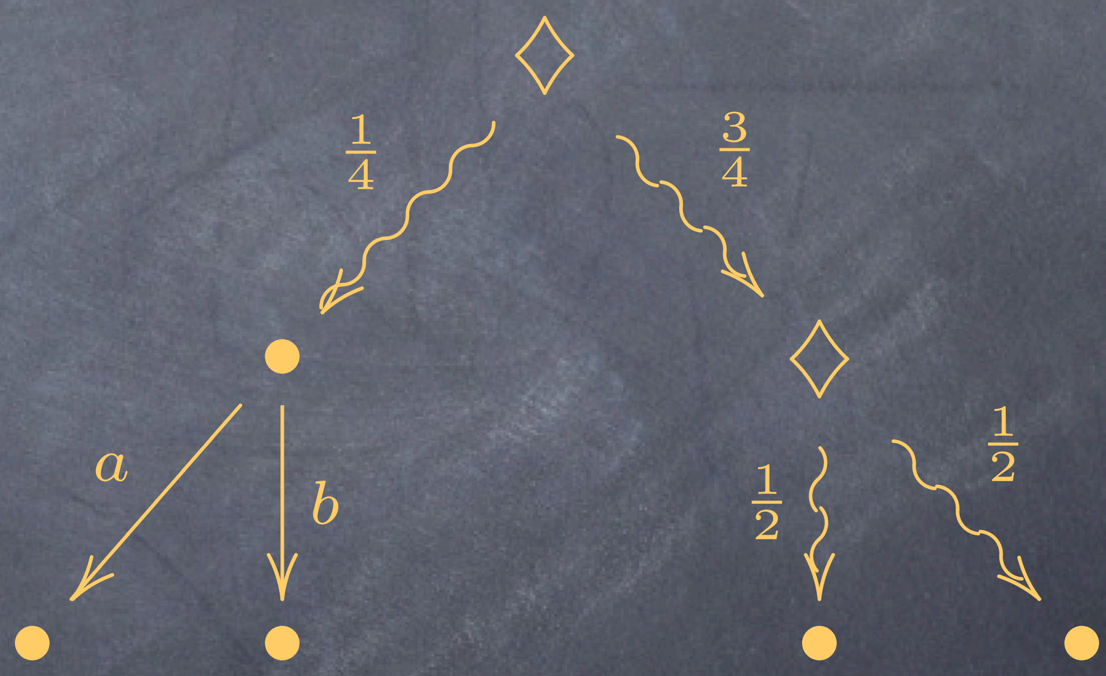

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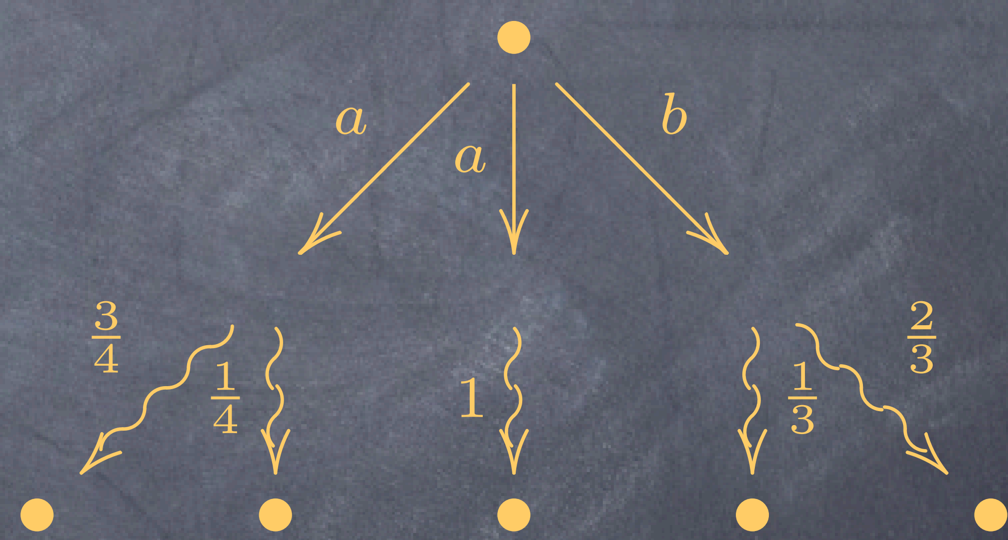

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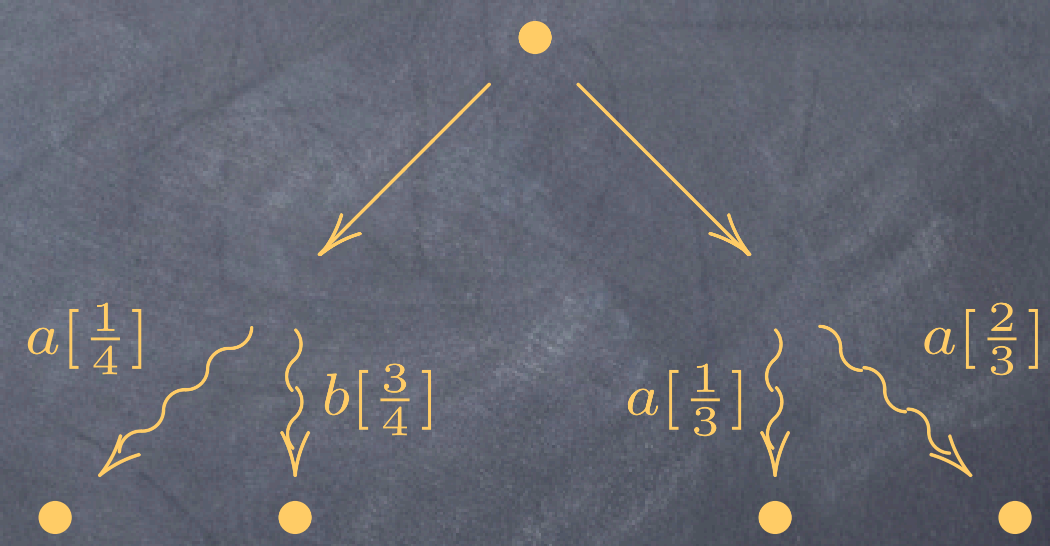

Discrete systems

- enter coalgebra, which provides a unifying framework
- become available as examples for generic coalgebra results
- all concrete probabilistic bisimulations (based on Larsen&Skou bisimulation) coincide with coalgebraic bisimulations
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modular, inductive proof

Bisimilarity for simple Segala automata

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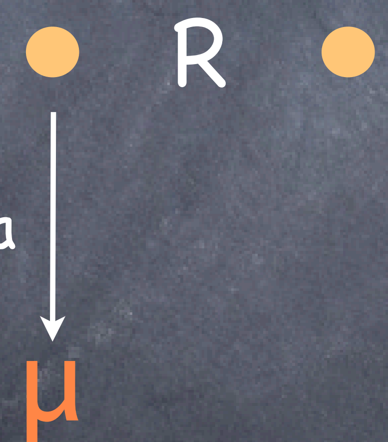

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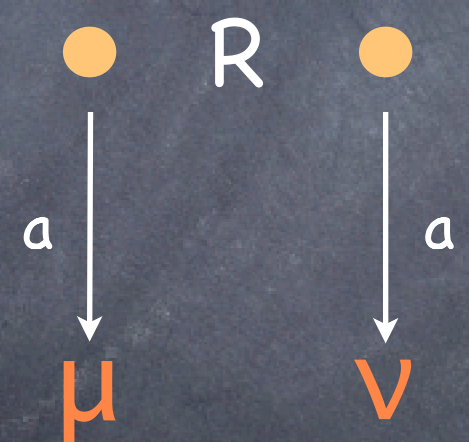

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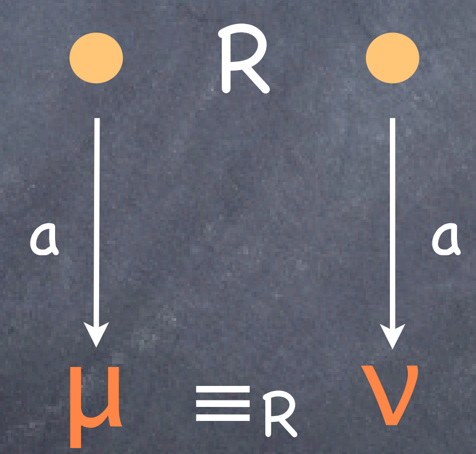

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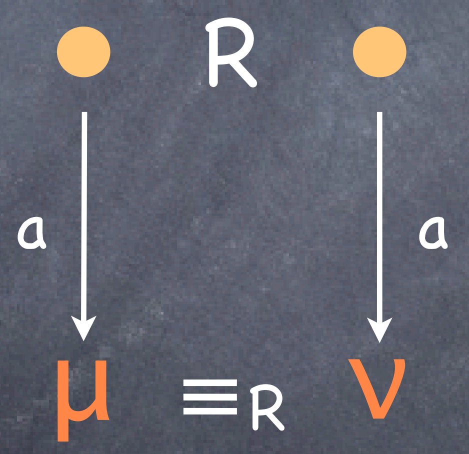

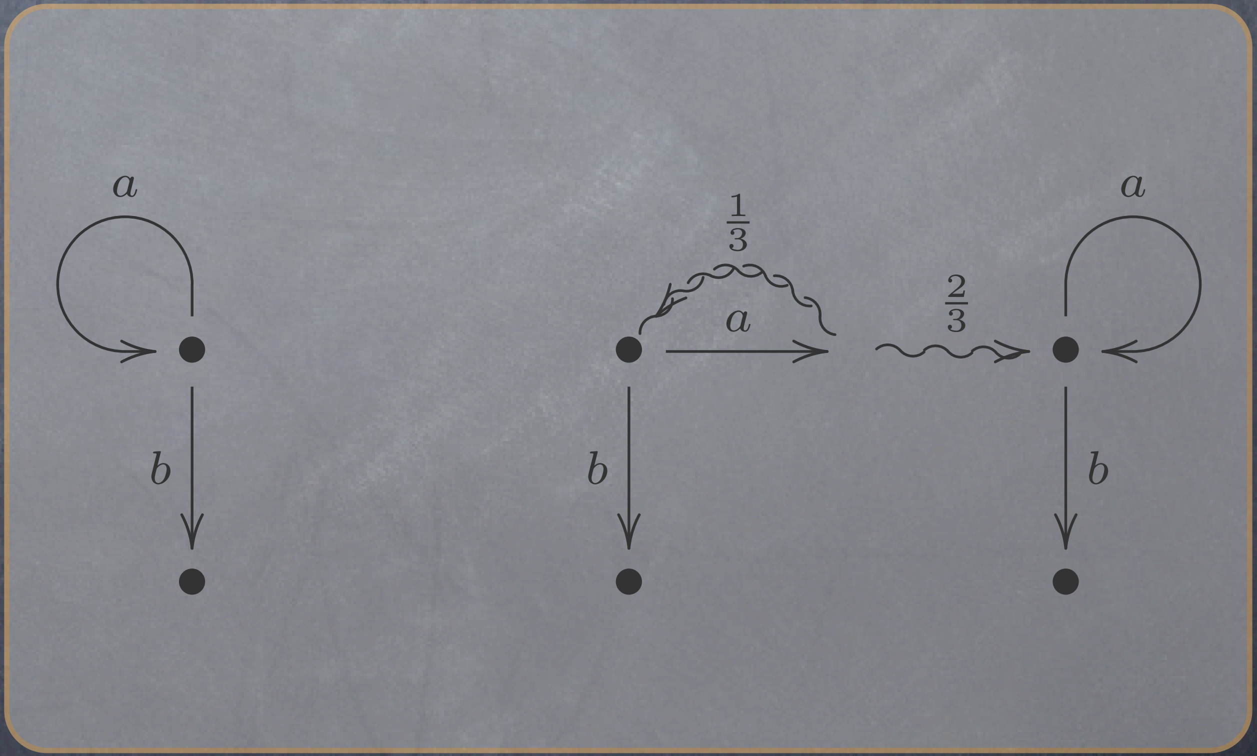

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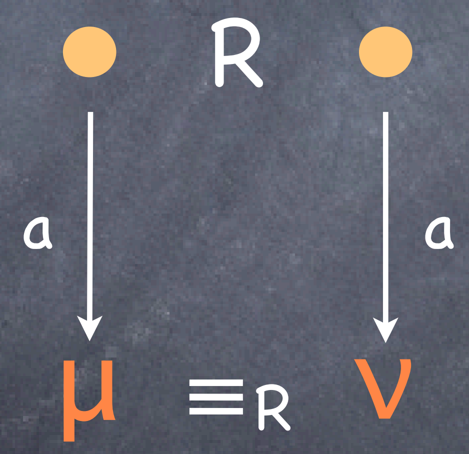

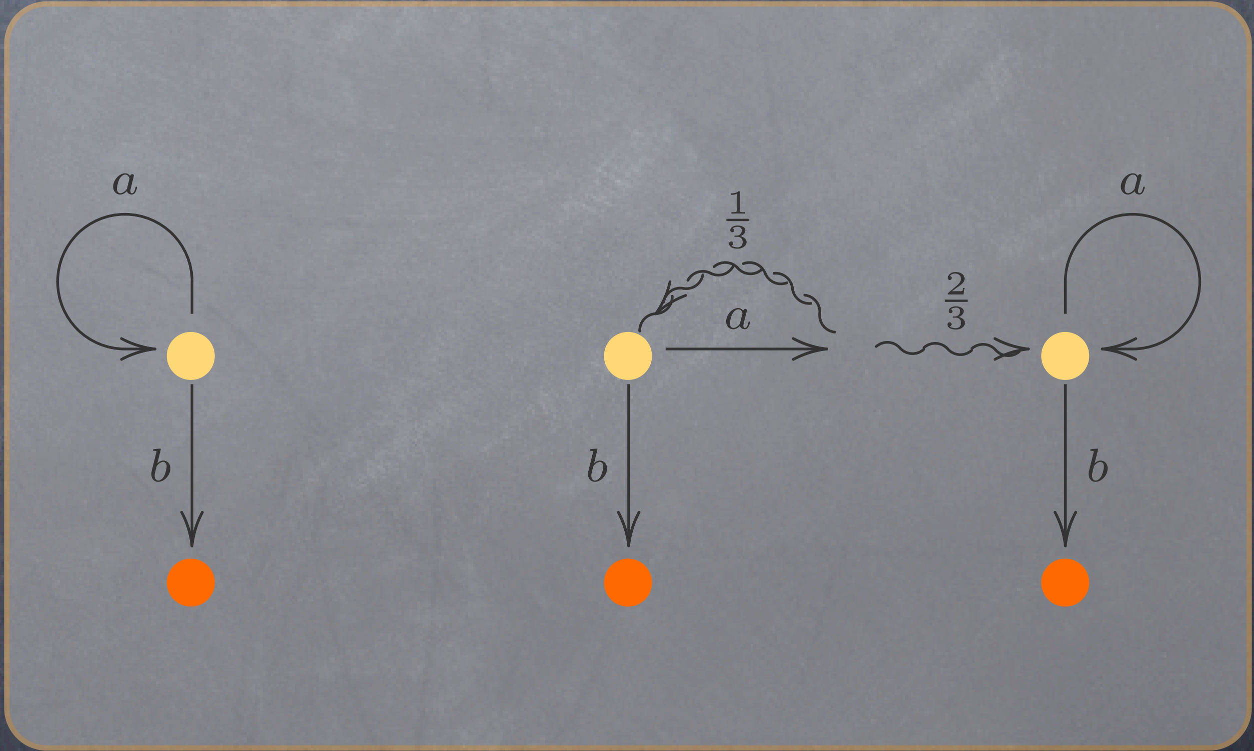

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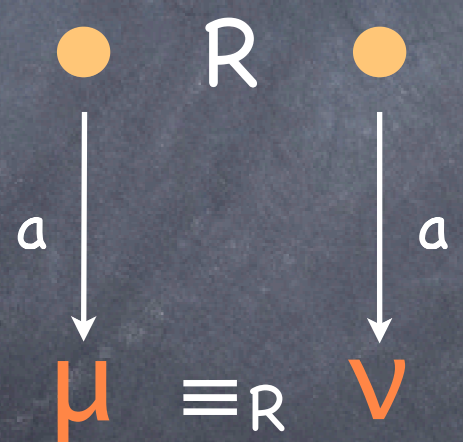

$$sRt \quad \Rightarrow \quad \left(s \overset{a}{\to} \mu \Rightarrow (\exists \nu)\ t \overset{a}{\to} \nu,\ \mu \equiv_R \nu\right)$$
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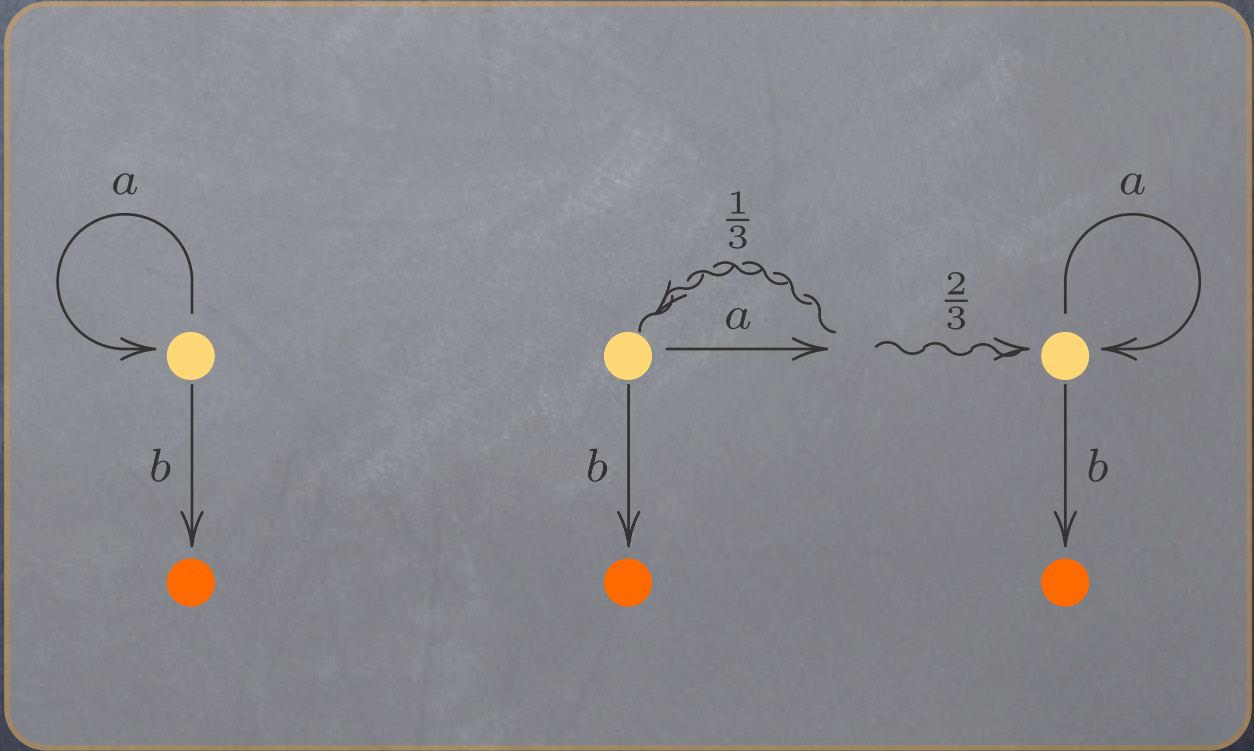

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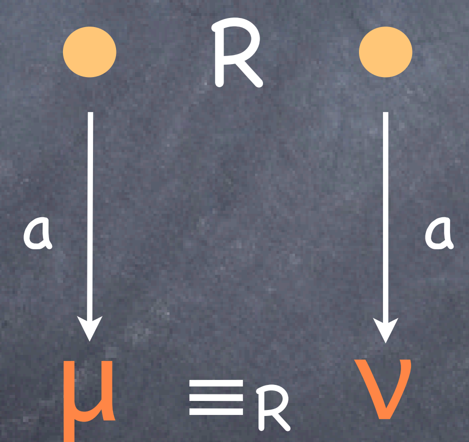

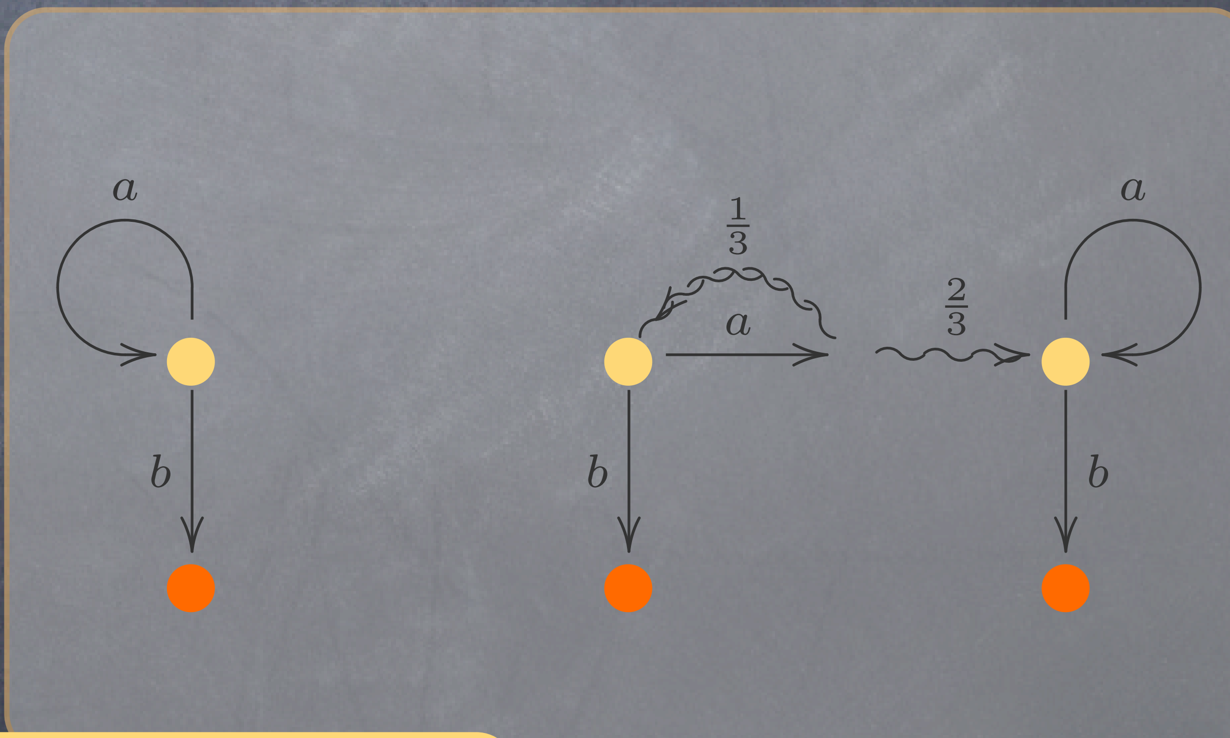

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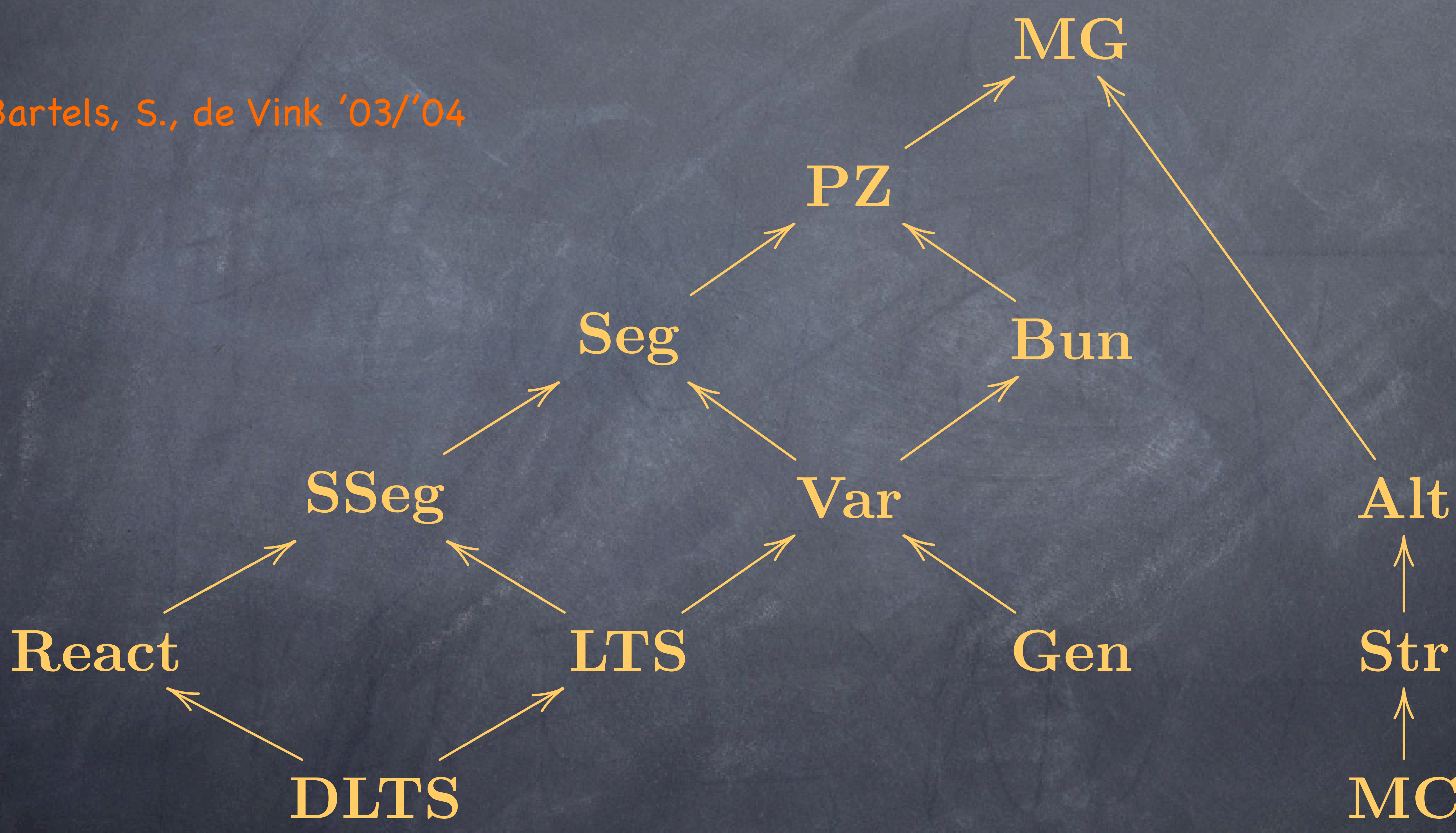
Expressiveness hierarchy

Bartels, S., de Vink '03/'04



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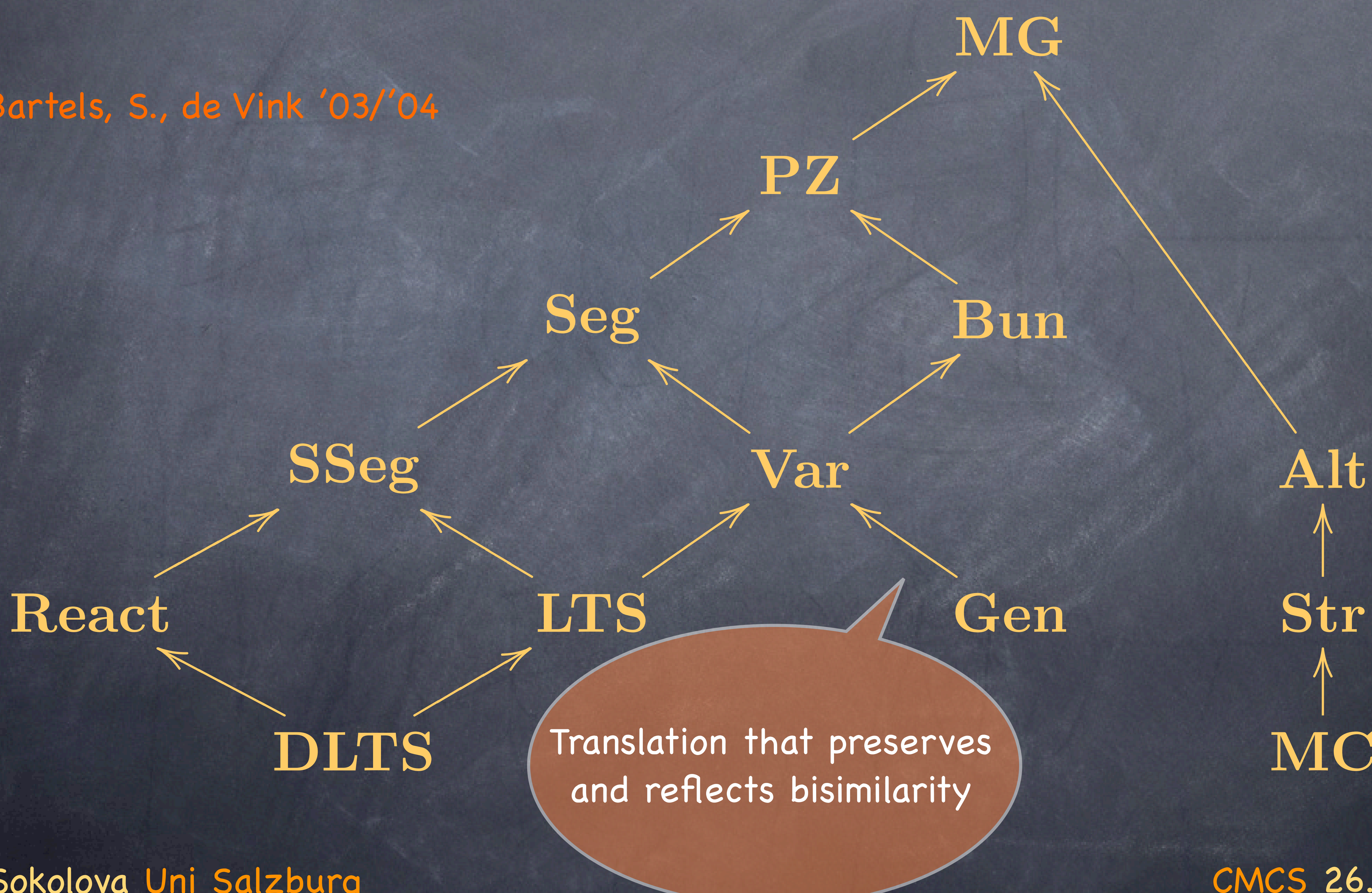

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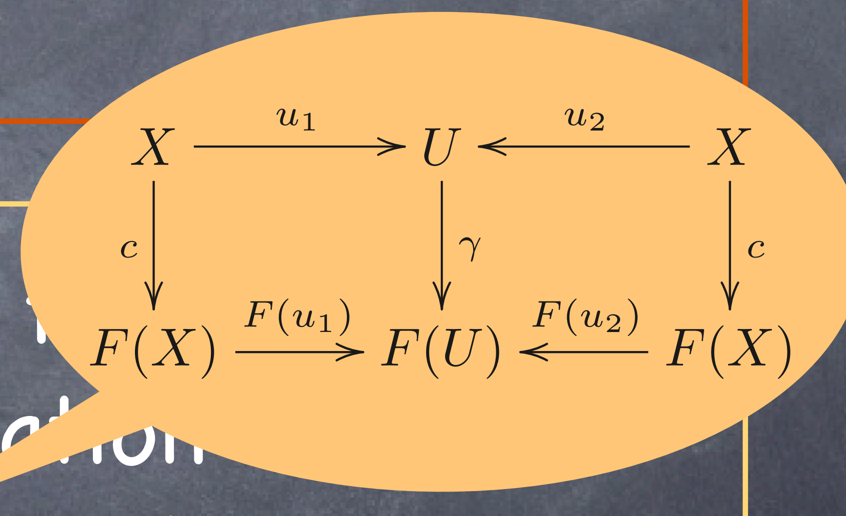

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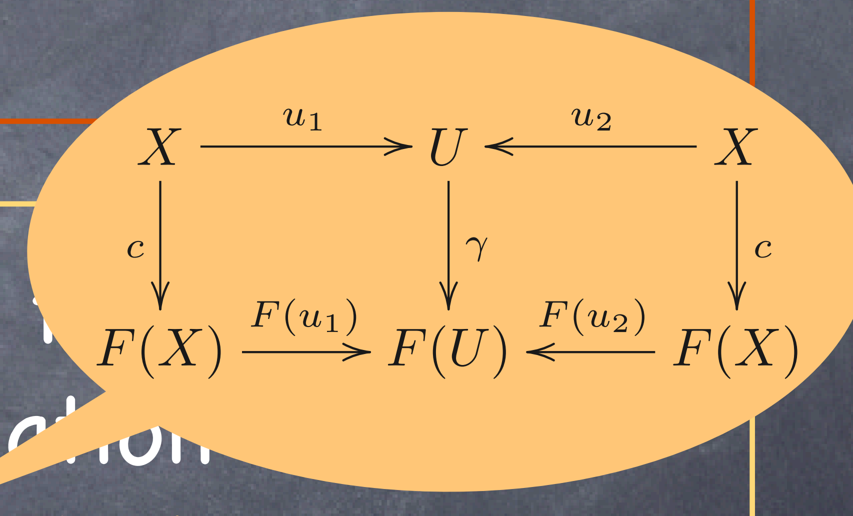

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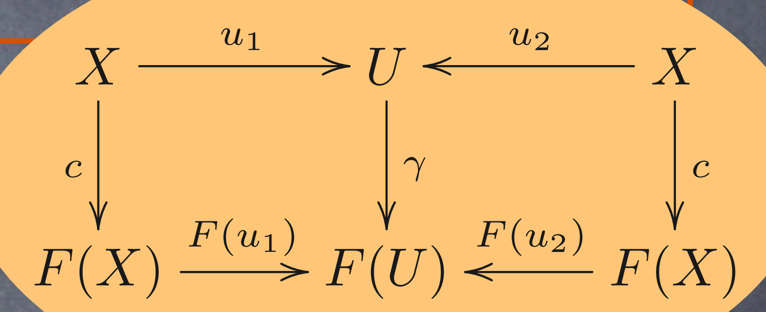

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if not, behaviour equivalence is better

Example embedding

simple Segala system $\longrightarrow$ Segala system

$\mathcal{P}(A \times \mathcal{D})$ $\mathcal{P}\mathcal{D}(A \times \_)$

Example embedding

simple Segala system → Segala system

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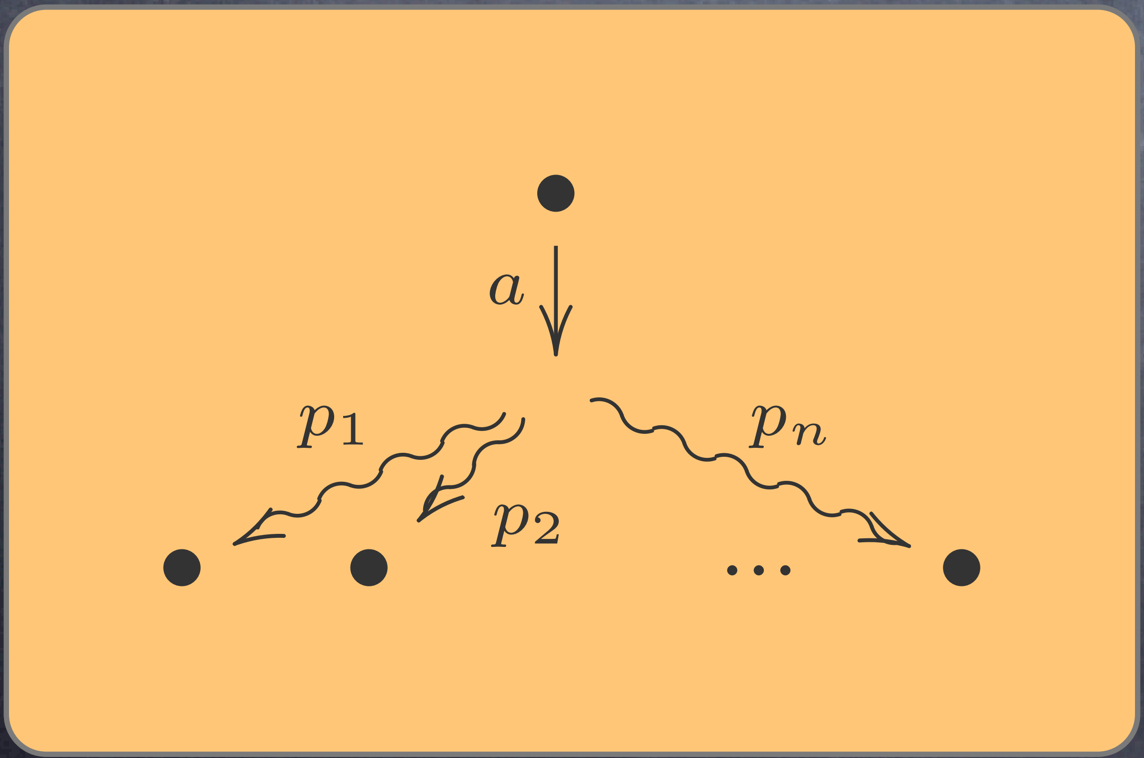
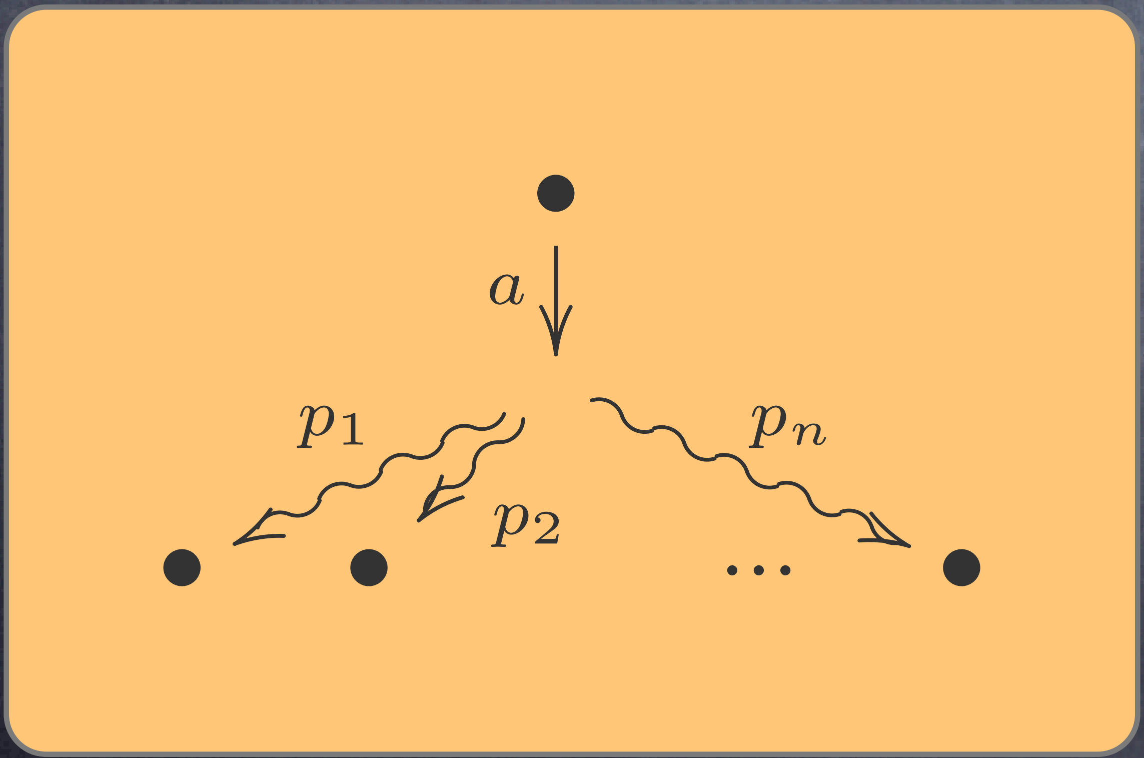
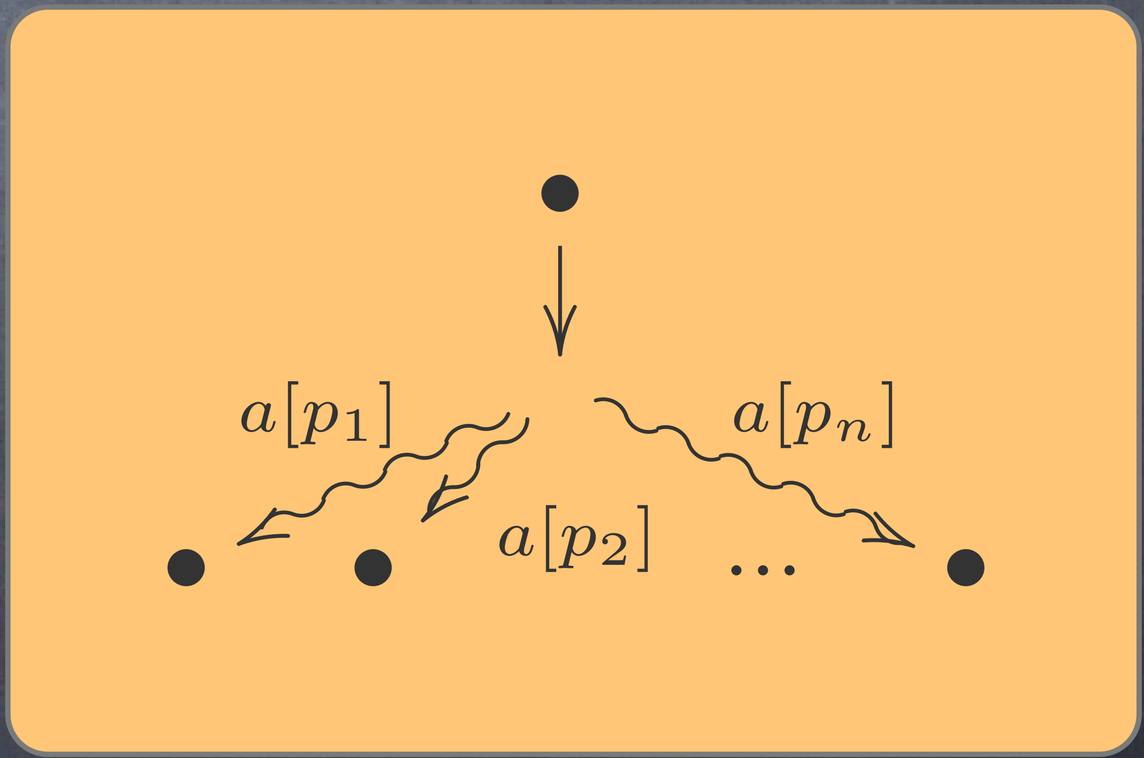


Example embedding

simple Segala system → Segala system

$\mathcal{P}(A \times \mathcal{D})$ $\mathcal{PD}(A \times \_)$





Basic natural transformations

- $\eta : 1 \Rightarrow \mathcal{P}$ with $\eta_X(*) := \emptyset$,
- $\sigma : \_ \Rightarrow \mathcal{P}$ with $\sigma_X(x) := \{x\}$
- $\delta : \_ \Rightarrow \mathcal{D}$ with $\delta_X(x) := \delta_x$ (Dirac),
- $\iota_l : \mathcal{F} \Rightarrow \mathcal{F} + \mathcal{G}$ and $\iota_r : \mathcal{G} \Rightarrow \mathcal{F} + \mathcal{G}$,
- $\phi + \psi : \mathcal{F} + \mathcal{G} \Rightarrow \mathcal{F}' + \mathcal{G}'$ for
  $\phi : \mathcal{F} \Rightarrow \mathcal{F}'$ and $\psi : \mathcal{G} \Rightarrow \mathcal{G}'$ (both with i.c.),
- $\kappa : \mathrm{A} \times \mathcal{P} \Rightarrow \mathcal{P}(\mathrm{A} \times \_)$ with $\kappa_X(a, M) := \{\langle a, x\rangle \mid x \in M\}$,
- $\ldots$

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  Stochastic/weighted GSOS Klin&Sassone'08, Klin'09

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- given a monoid $(M, +, 0)$ and a subset $S \subseteq M$

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additional structure on M
adds structure to the functor
(monad...)

Part 2

Continuous probabilistic systems

Live beyond sets

in $\mathbf{Meas}$

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in $\mathbf{Meas}$

the category of measure spaces and measurable maps

Live beyond sets

in $\mathbf{Meas}$ — the category of measure spaces and measurable maps

objects: measure spaces (X, S_X)

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arrows: measurable maps

$f : X \to Y$ with $f^{-1}(S_Y) \subseteq S_X$

Markov and Giry

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$X \to [0,1]$

$= \varphi(M)$

$$\mathcal{G}(f)\Big(S_X \stackrel{\varphi}{\to} [0,1]\Big) \quad = \quad \Big(S_Y \stackrel{f^{-1}}{\to} S_X \stackrel{\varphi}{\to} [0,1]\Big)$$

Properties, other spaces

- $\mathbf{Meas}$ may not have weak pullbacks

- Analytic spaces have semi-pullbacks
  Edalat '99

- It has a final coalgebra
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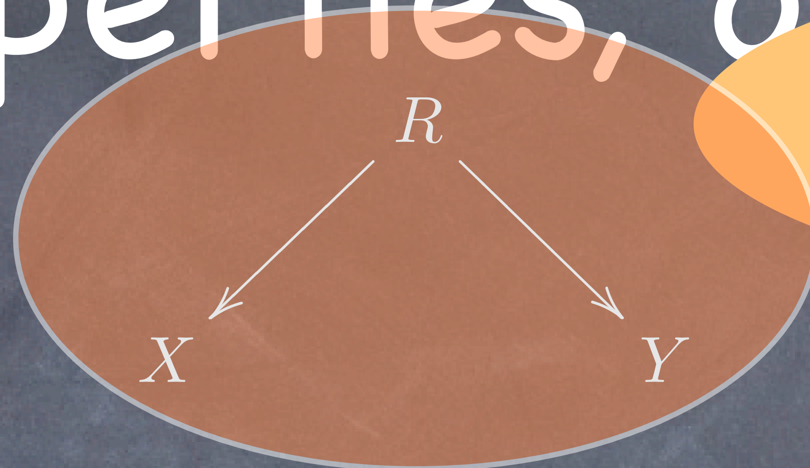

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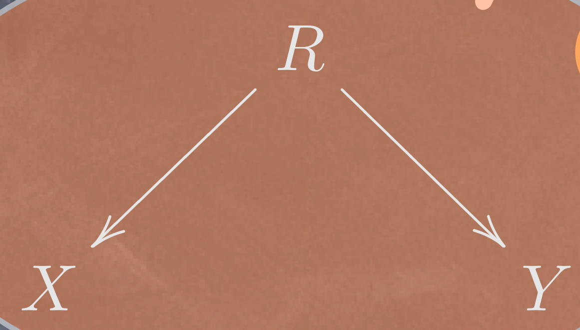

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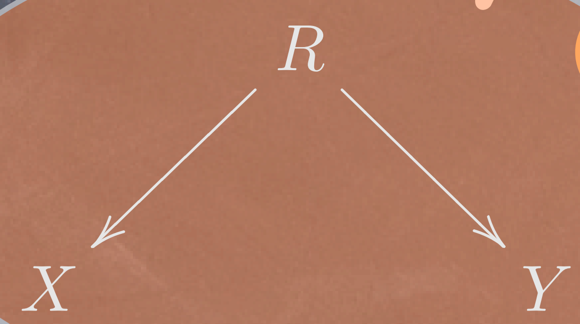

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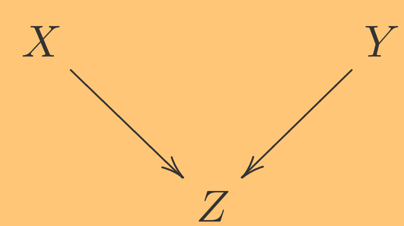

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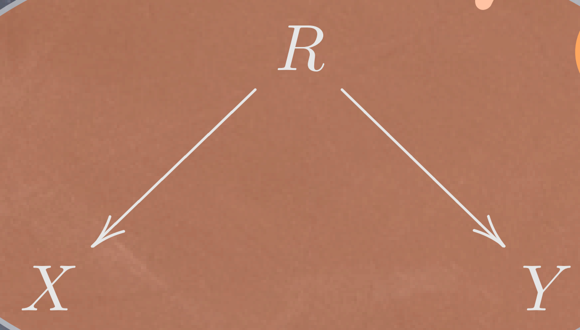

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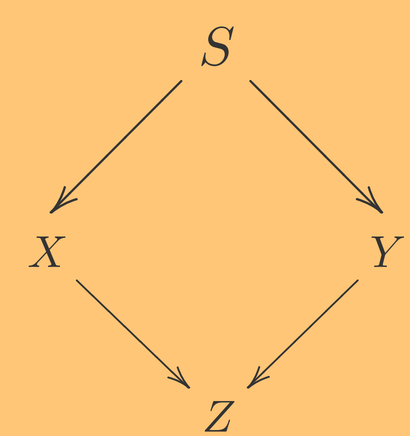

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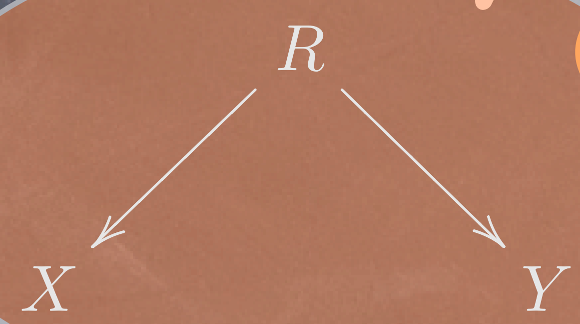

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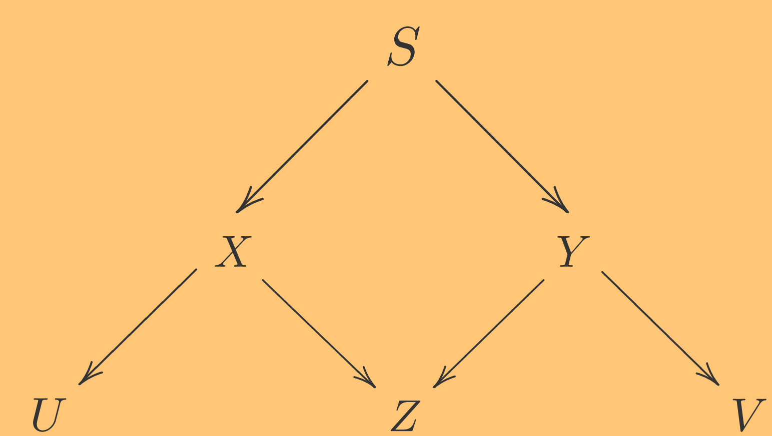

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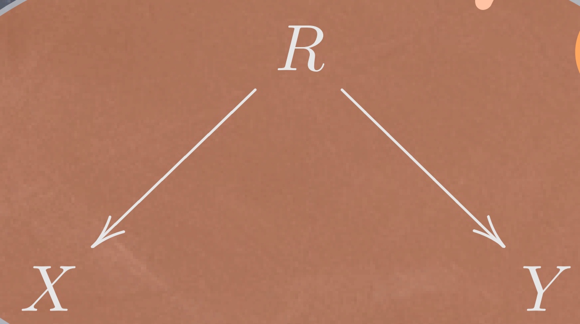

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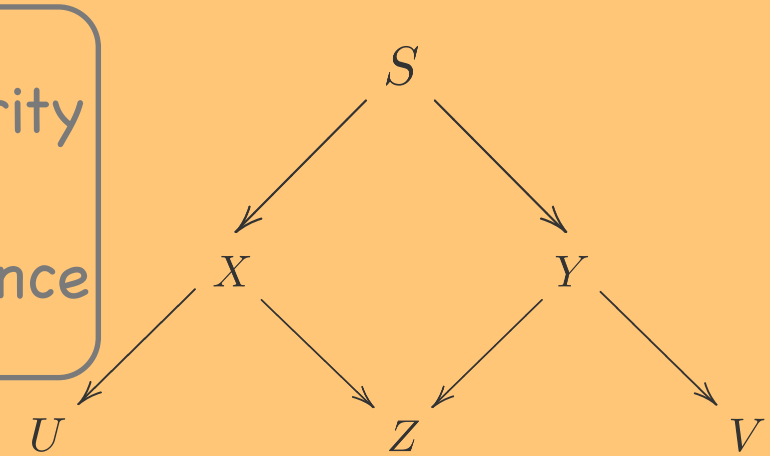

Main research in continuous systems

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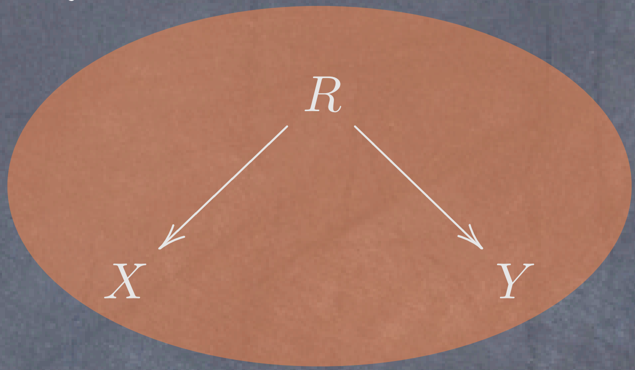

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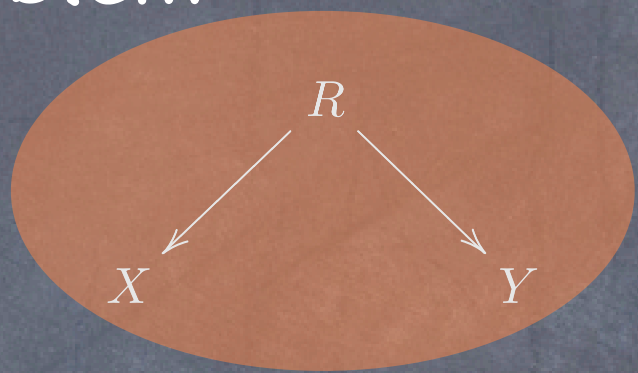

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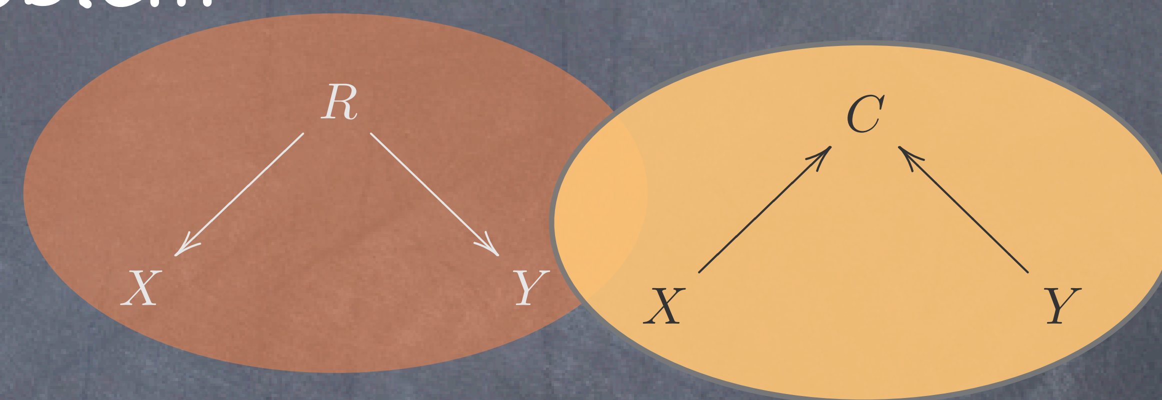

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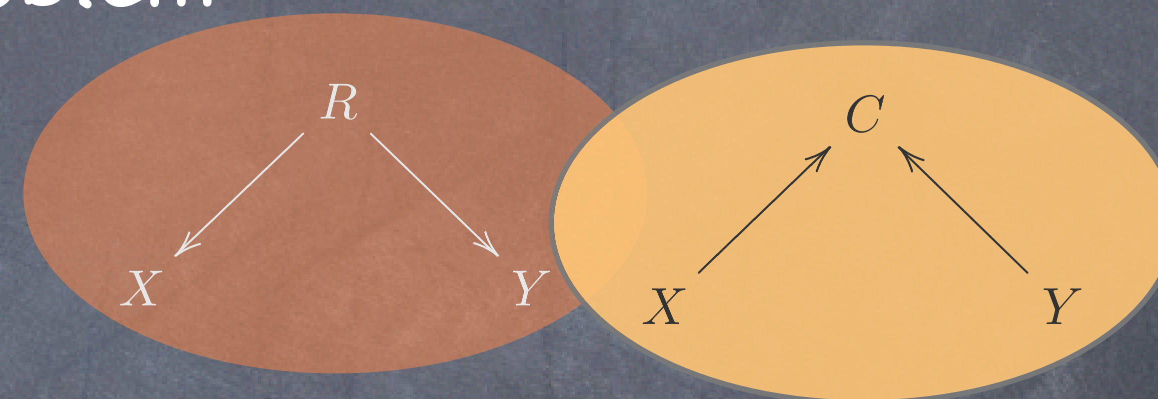

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same logical characterization

all works smoothly

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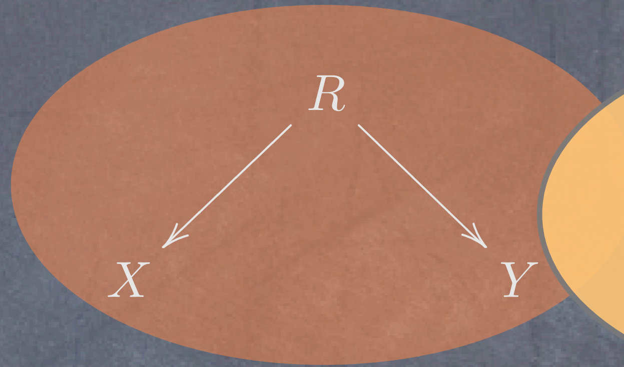

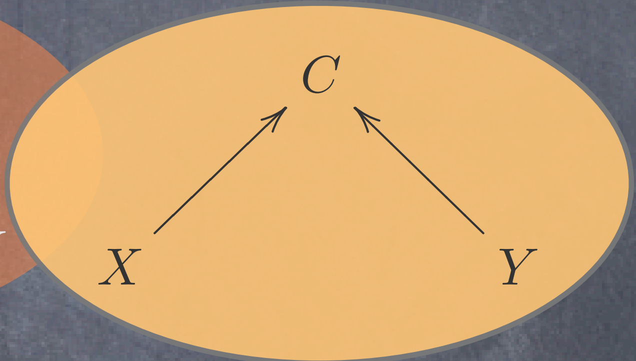

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No need of Polish/ analytic spaces

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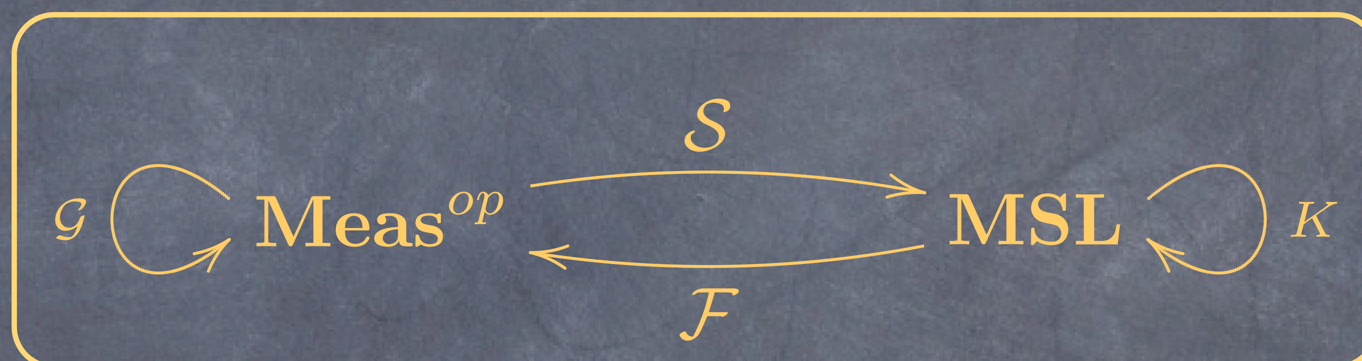
Logical characterization via dual adjunctions

Jacobs&S.'09



Logical characterization via dual adjunctions

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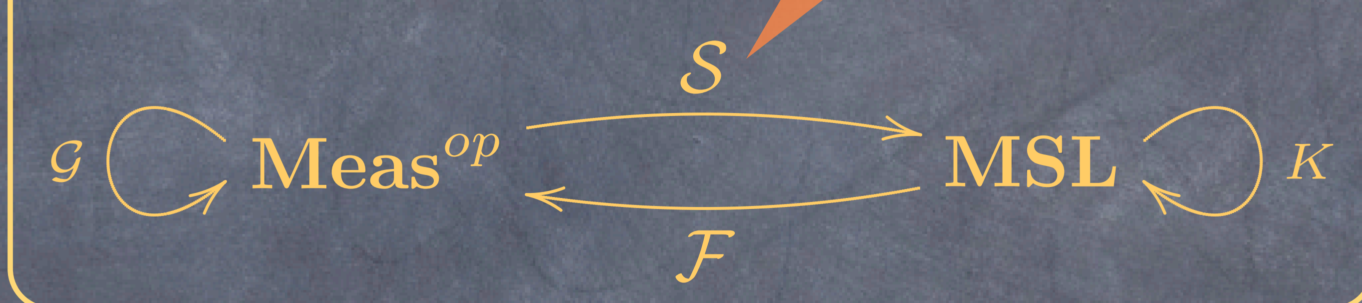

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filters, upsets closed under finite conjunctions, with σ-algebra generated by $\eta(a)$

Logical characterization via dual adjunctions

Jac

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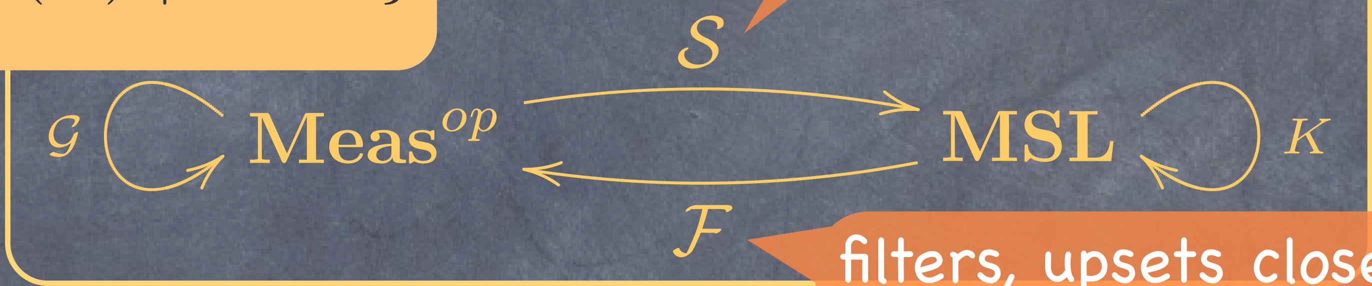

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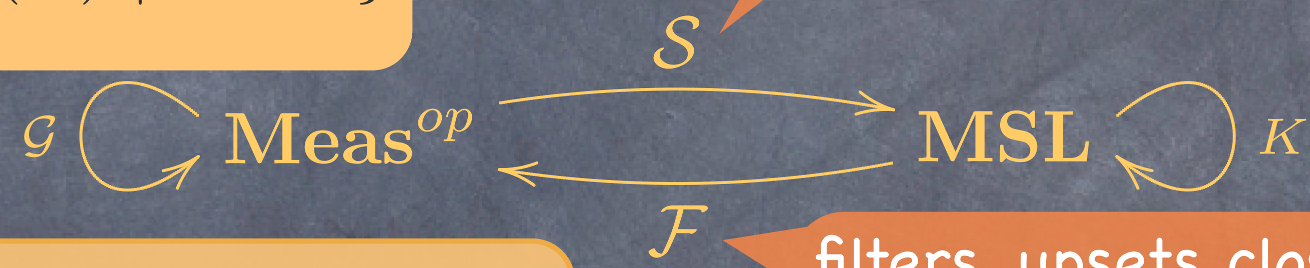

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filters, upsets closed under finite conjunctions, with σ-algebra generated by $\eta(a)$

$$\frac{f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \mathbf{Meas}}{g : A \longrightarrow \mathcal{S}(X) \quad \text{in } \mathbf{MSL}}$$

via $\dfrac{a \in f(x)}{x \in g(a)}$

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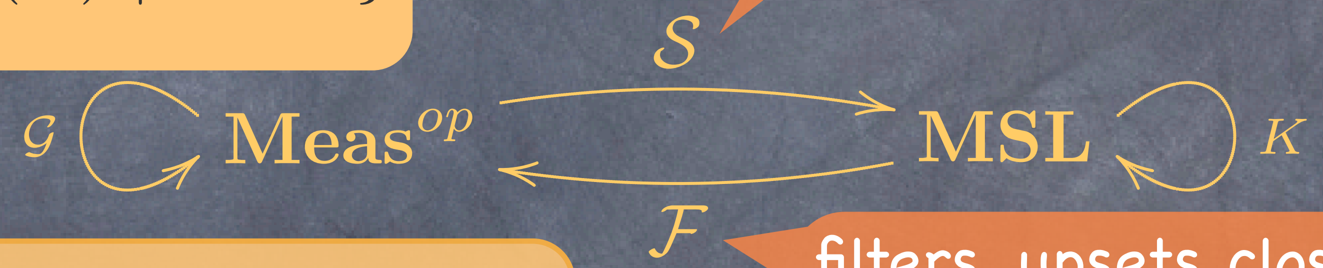

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Monday, March 29, 2010

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$$\eta(a) = \{\alpha \in \mathcal{F}(A) \mid a \in \alpha\}$$

maps a measure space to its σ-algebra

$\mathcal{G}$ ⟲ $\mathbf{Meas}^{op}$ $\underset{\mathcal{F}}{\overset{\mathcal{S}}{\rightleftarrows}}$ $\mathbf{MSL}$ ⟲ K

monotone modal operators

filters, upsets closed under finite conjunctions, with σ-algebra generated by $\eta(a)$

$$\frac{f : X \longrightarrow \mathcal{F}(A) \quad \text{in } \mathbf{Meas}}{g : A \longrightarrow \mathcal{S}(X) \quad \text{in } \mathbf{MSL}}$$

via $\dfrac{a \in f(x)}{x \in g(a)}$

expressivity

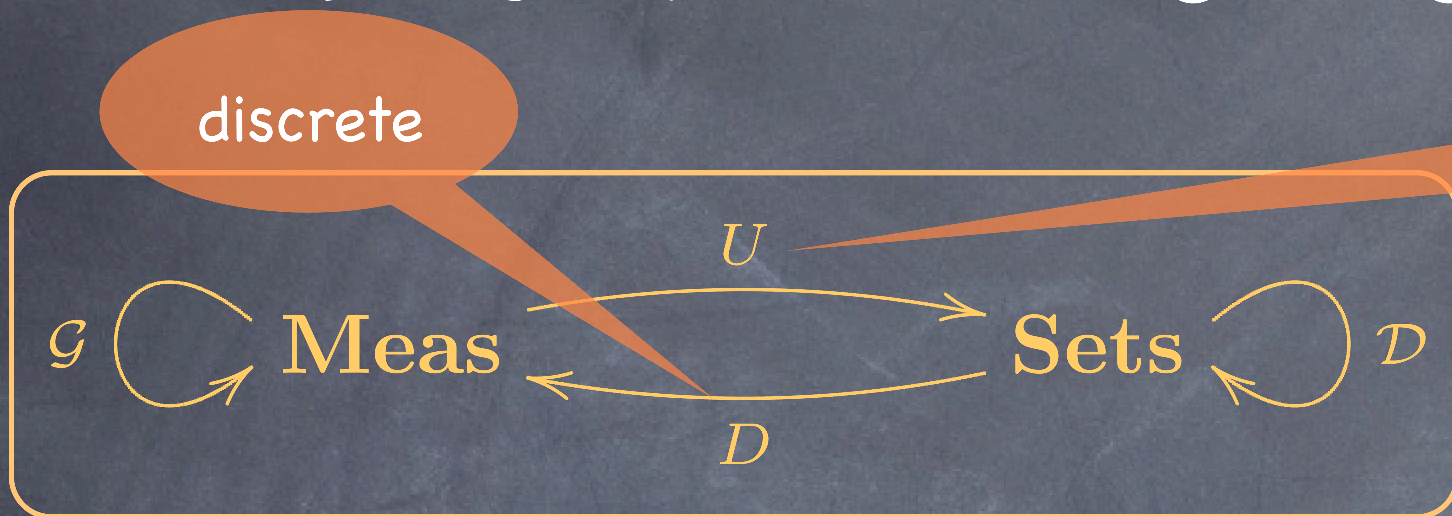
Discrete to continuous

$$\mathcal{G} \circlearrowright \mathbf{Meas} \underset{D}{\overset{U}{\rightleftarrows}} \mathbf{Sets} \circlearrowleft \mathcal{D} \qquad \text{with} \quad D \dashv U$$

Discrete to continuous

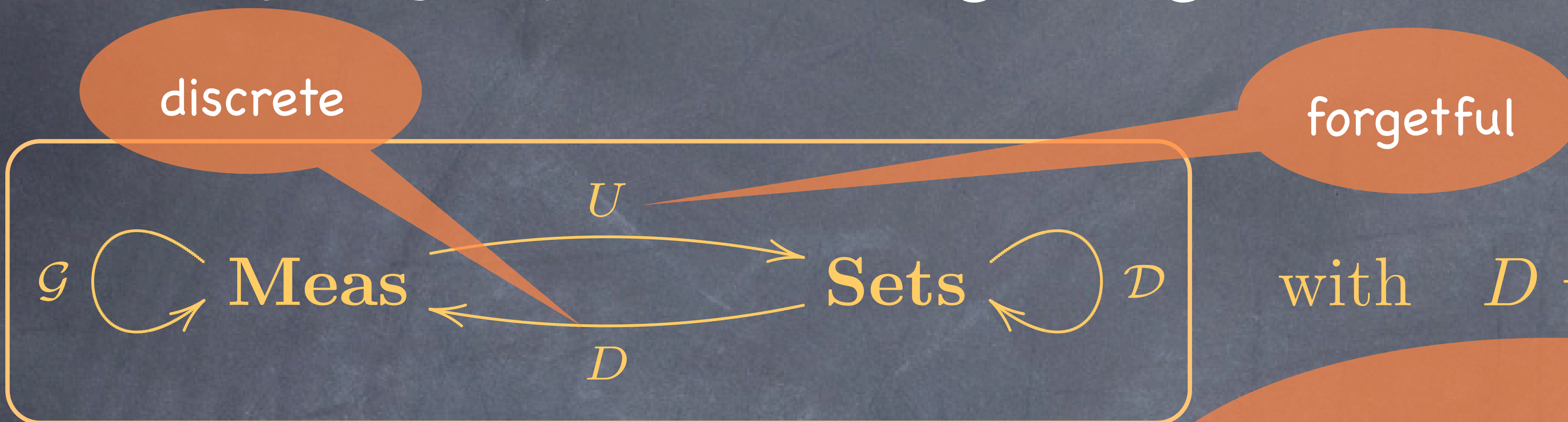
with $D \dashv U$

Discrete to continuous



with $D \dashv U$

Discrete to continuous

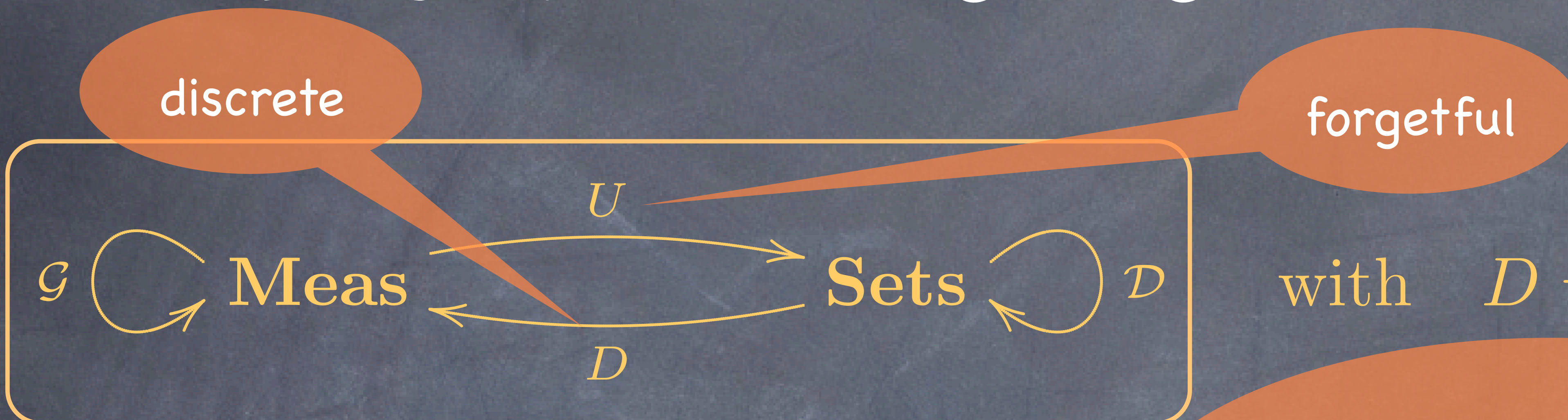


with $D \dashv U$

obvious natural transformation

$$\rho : \mathcal{D}U \Rightarrow U\mathcal{G}$$

Discrete to continuous



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obvious natural transformation

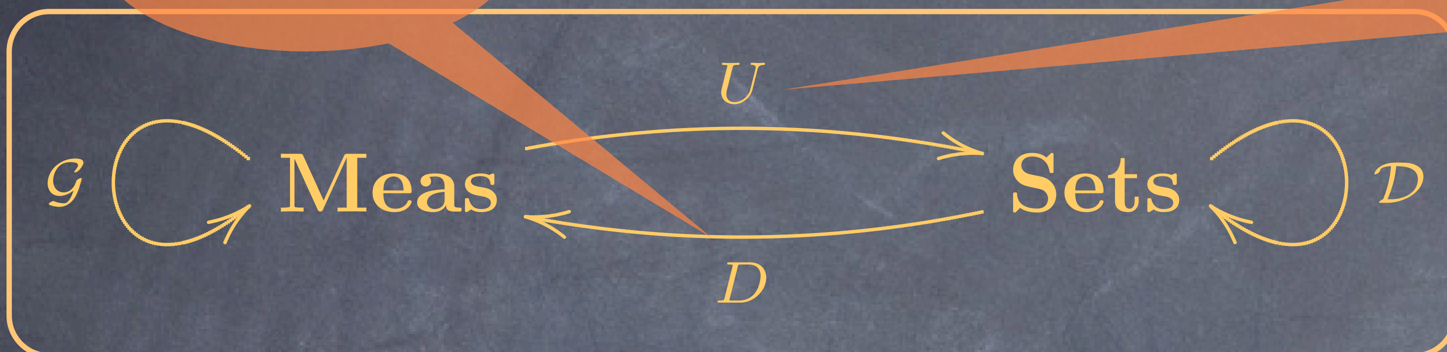
$\rho : \mathcal{D}U \Rightarrow U\mathcal{G}$

We can translate chains into processes:

Discrete to continuous

discrete

forgetful



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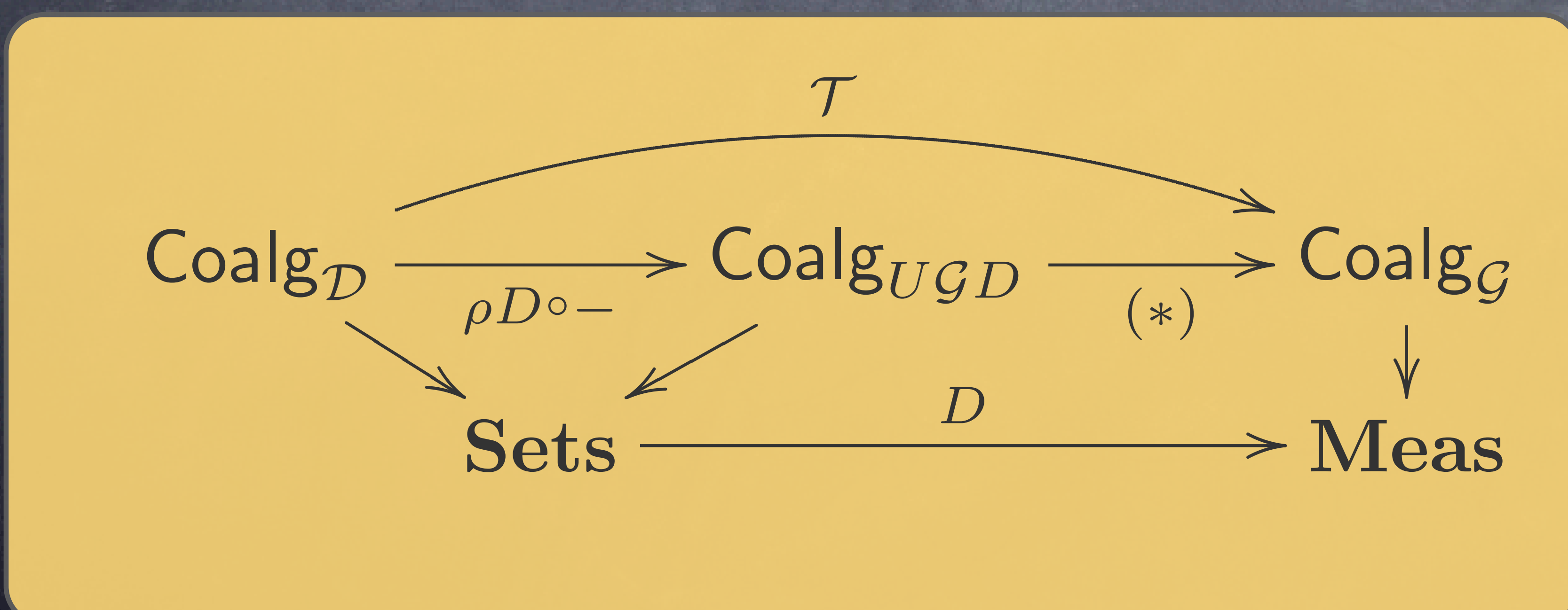
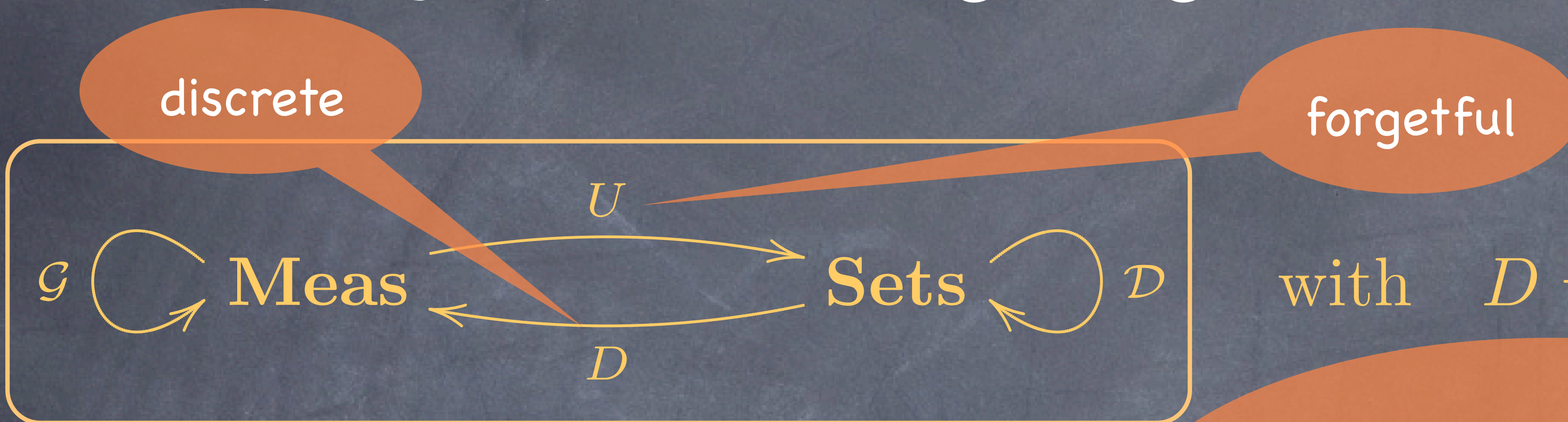


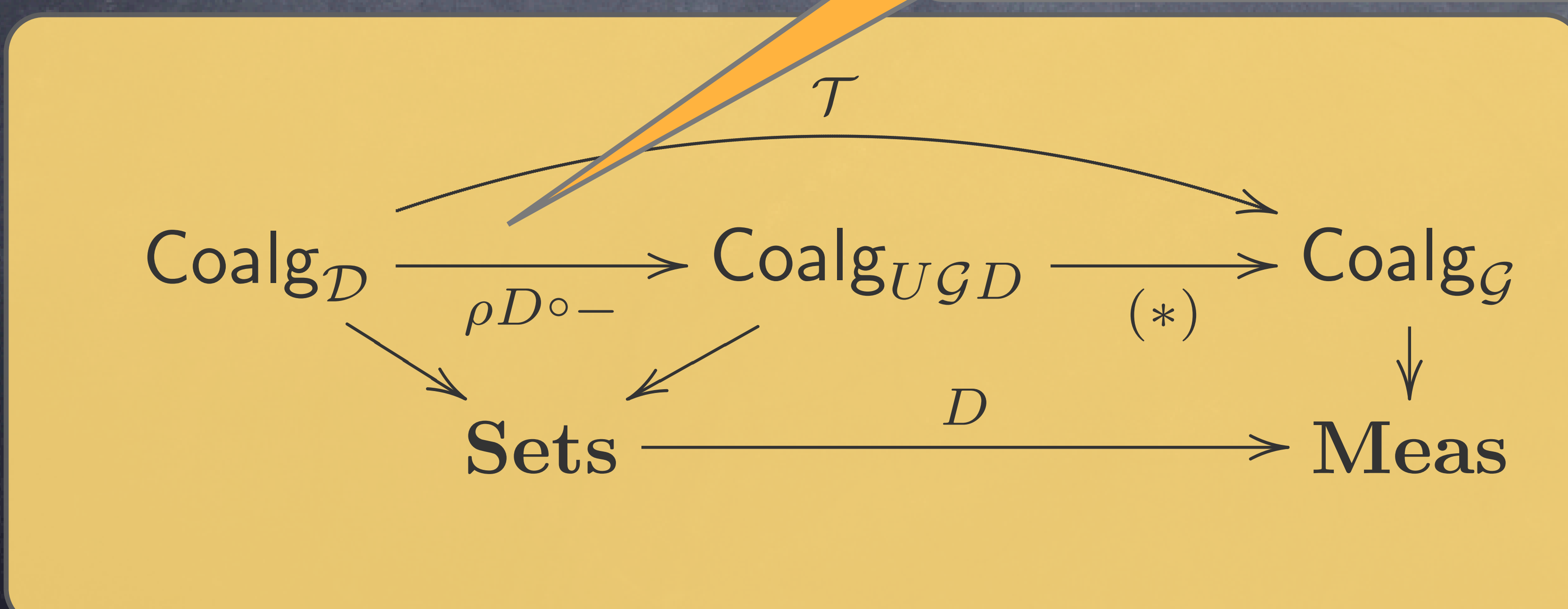
Discrete to continuous



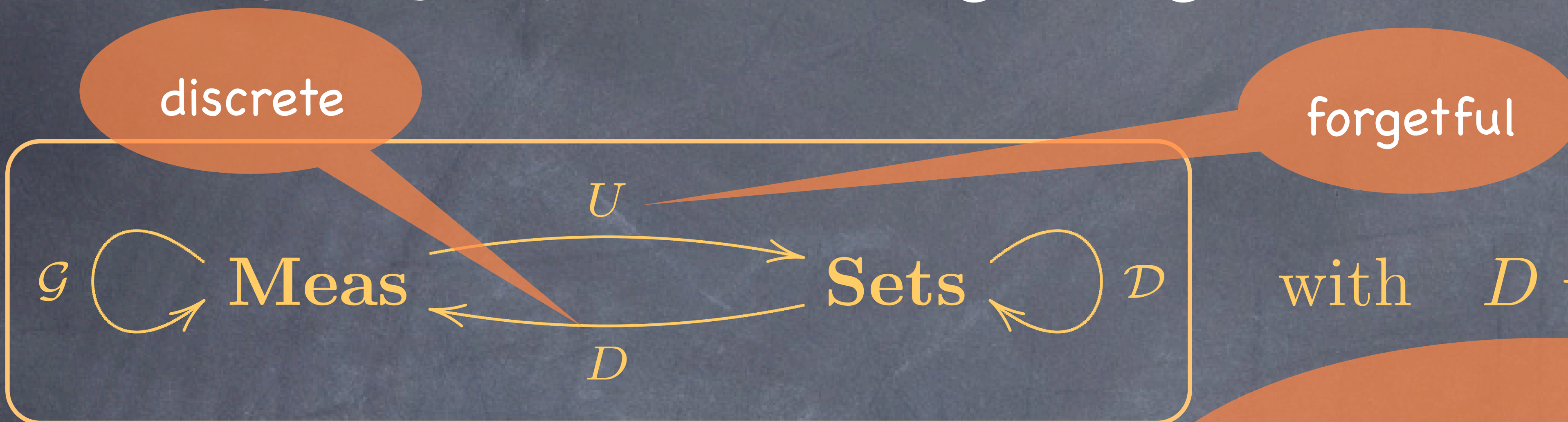
with $D \dashv U$

We can translate chains into processes:

$$\left(X \xrightarrow{c} \mathcal{D}(X) = \mathcal{D}UD(X)\right) \longmapsto \left(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\right)$$



Discrete to continuous



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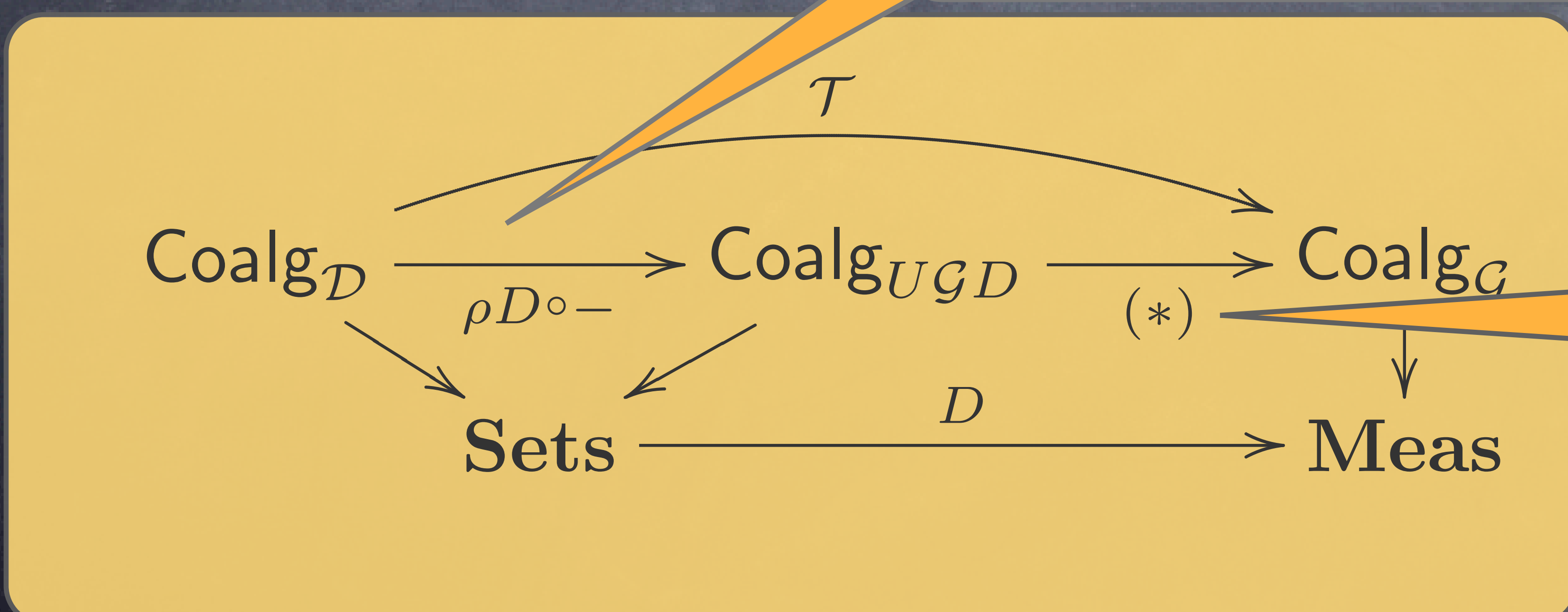
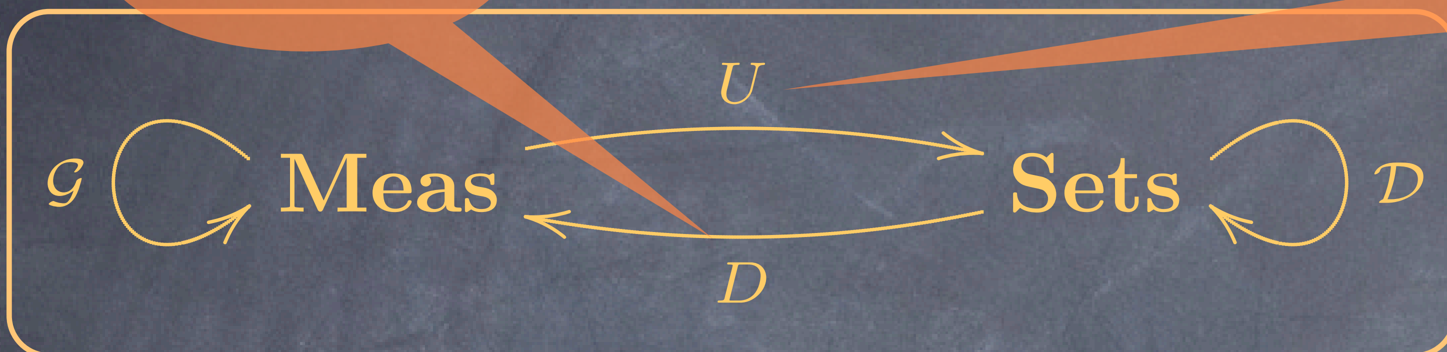


$$\frac{X \longrightarrow U\mathcal{G}D(X) \quad \text{in } \mathbf{Sets}}{D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}}$$

Discrete to continuous

discrete

forgetful

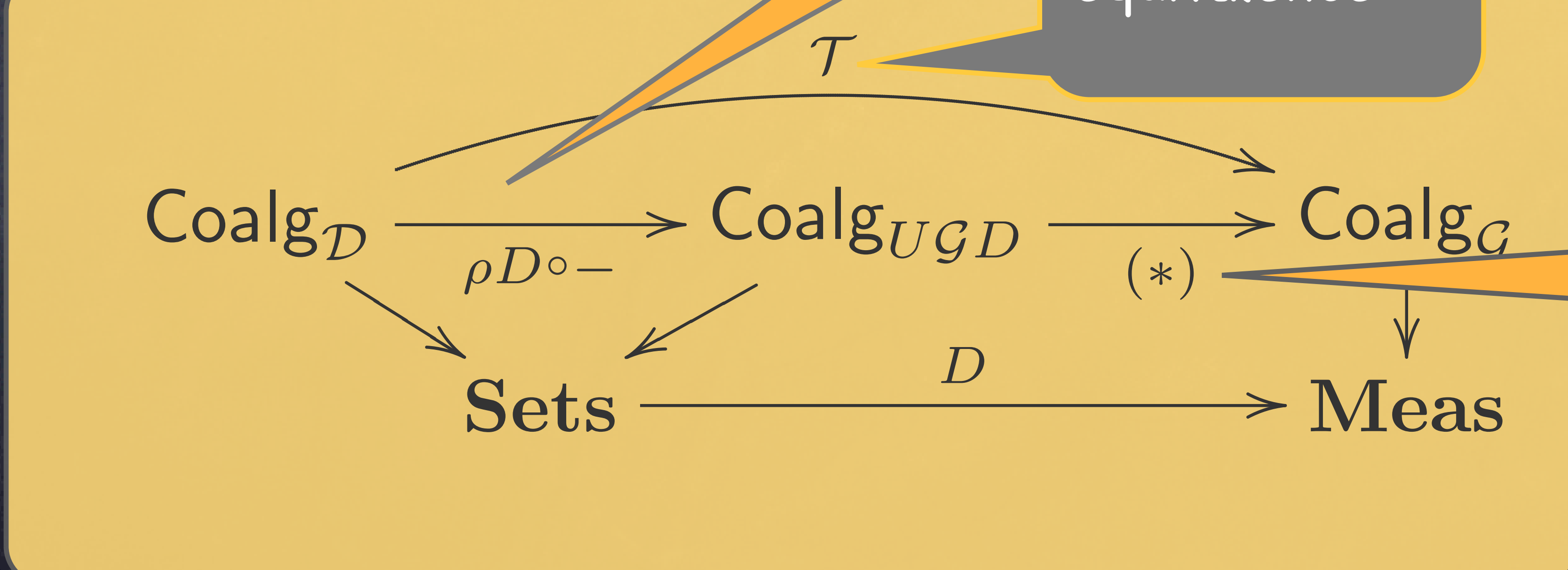


with $D \dashv U$

We can translate chains into processes:

$\big(X$ … $\big) \longmapsto \big(X \xrightarrow{c} \mathcal{D}UD(X) \xrightarrow{\rho_{DX}} U\mathcal{G}D(X)\big)$

preserves and reflects behaviour equivalence



$$\frac{X \longrightarrow U\mathcal{G}D(X) \quad \text{in } \mathbf{Sets}}{D(X) \longrightarrow \mathcal{G}D(X) \quad \text{in } \mathbf{Meas}}$$

Final message

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- Both discrete and continuous probabilistic systems are coalgebras

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often just
nice examples

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also some interesting results

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- Observation: behaviour equivalence (cospan) is more suitable than bisimilarity (span)

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- Both discrete and continuous probabilistic systems are coalgebras

- Observation: behaviour equivalence (cospan) is more suitable than bisimilarity (span)

- Measure spaces are enough, one can forget about Polish or analytic ones (unless one loves them)