On Semantic Relations:

From probabilistic systems to coalgebras and back

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Outline

- Introduction - probabilistic systems and coalgebras

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- Bisimilarity - the strong end of the spectrum

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- Bisimilarity - the strong end of the spectrum
- Application - expressiveness hierarchy

  (older result)
- Trace semantics - the weak end of the spectrum

  (new !)

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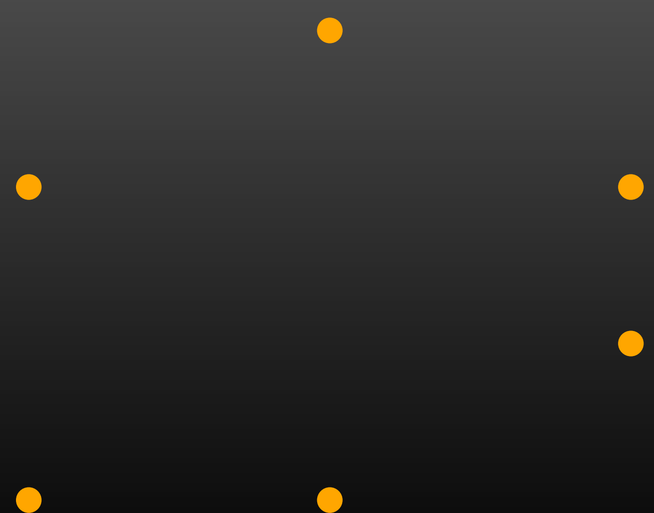

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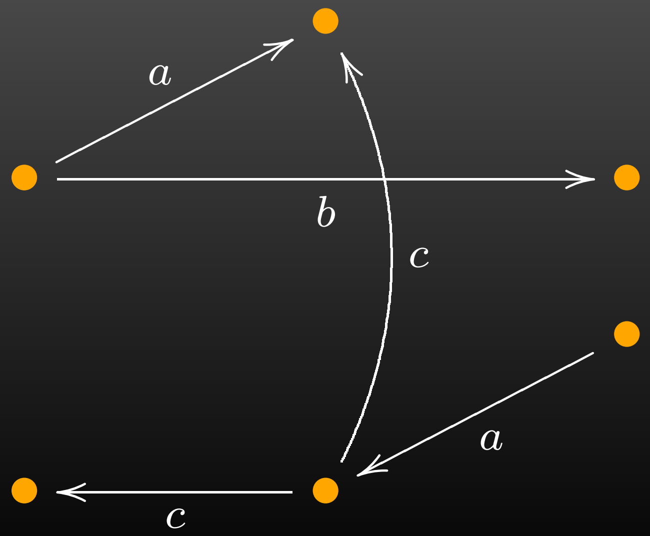

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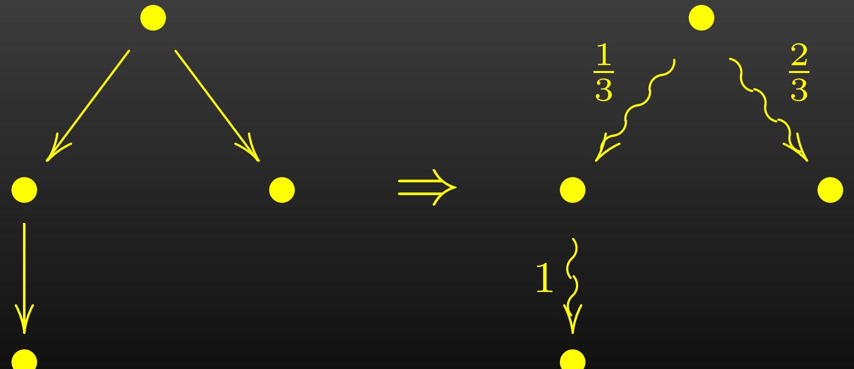

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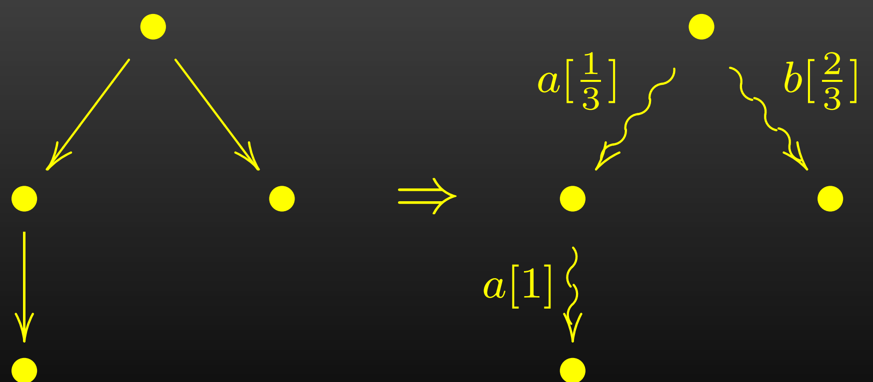

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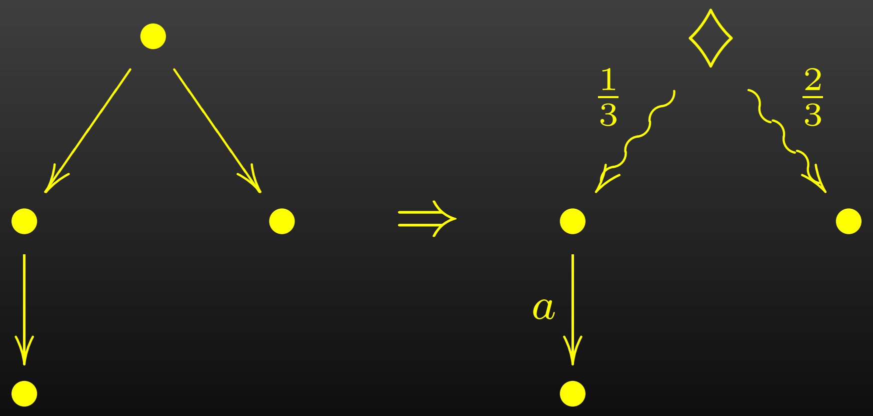

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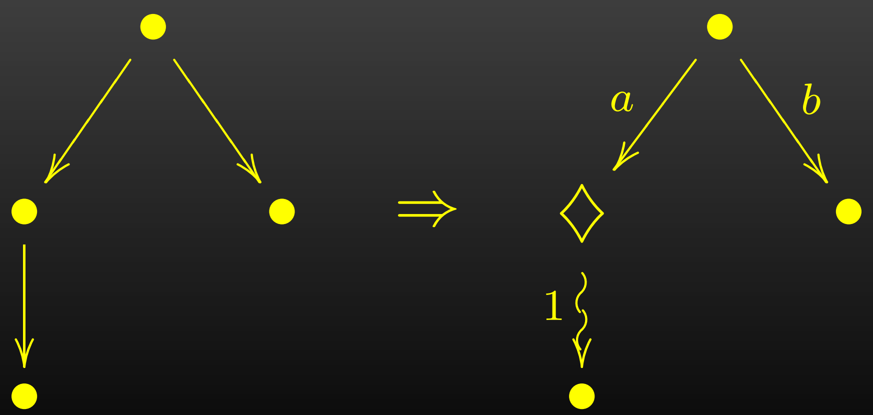

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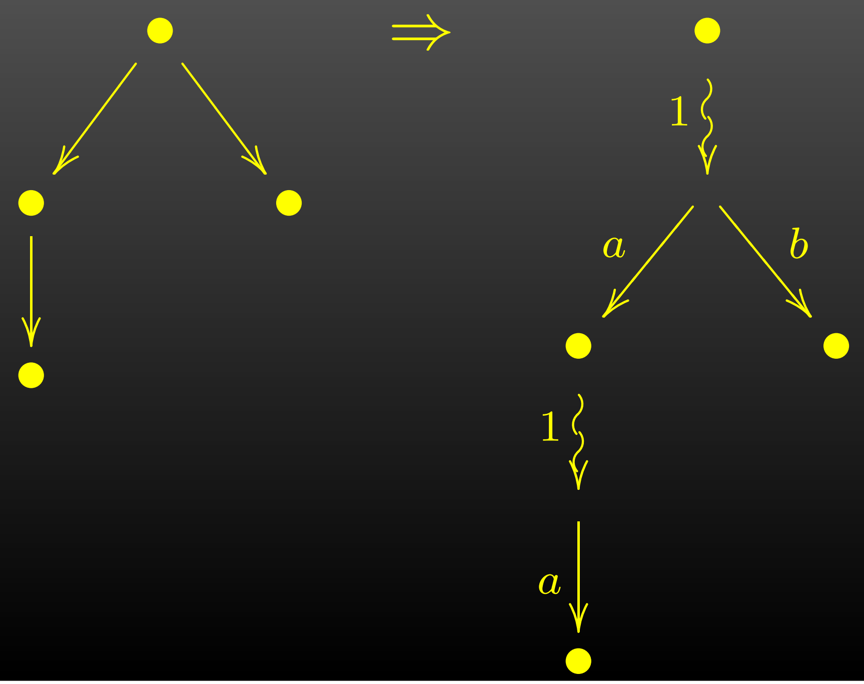

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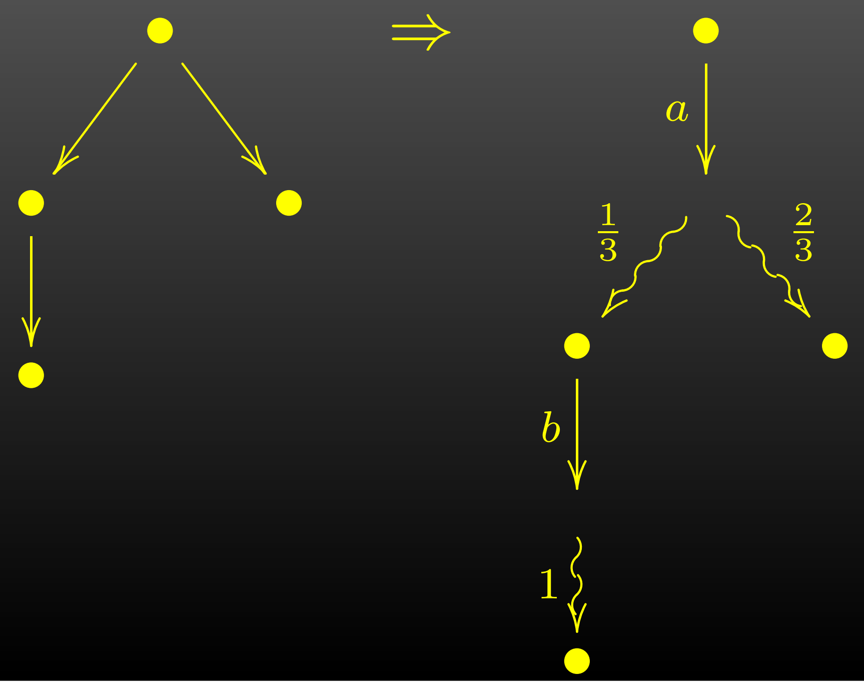

Coalgebras

are an elegant generalization of transition systems with

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- based on category theory
- provide a uniform way of treating transition systems
- provide general notions and results e.g. a generic notion of bisimulation

Examples

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!! coalgebra of the powerset functor $\mathcal{P}$

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An LTS is a pair $\langle S, \alpha : S \to \mathcal{P}S^A \rangle$

!!! coalgebra of the functor $\mathcal{P}^A$

Note: $\mathcal{P}^A \cong \mathcal{P}(A \times \_\,)$

More examples

Thanks to the probability distribution functor $\mathcal{D}$

$$\mathcal{D}S = \{\mu : S \to [0,1], \mu[S] = 1\}, \quad \mu[X] = \sum_{s \in X} \mu(x)$$

$$\mathcal{D}f : \mathcal{D}S \to \mathcal{D}T, \ \ \mathcal{D}f(\mu)(t) = \mu[f^{-1}(\{t\})]$$

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the probabilistic systems are also coalgebras ... of functors built by the following syntax

$$\mathcal{F} ::= \_ \mid A \mid \mathcal{P} \mid \mathcal{D} \mid \mathcal{G} + \mathcal{H} \mid \mathcal{G} \times \mathcal{H} \mid \mathcal{G}^A \mid \mathcal{G} \circ \mathcal{H}$$

reactive, generative

evolve from LTS - functor $\mathcal{P}(A \times \_) \cong \mathcal{P}^A$

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note: in the probabilistic case

$$(\mathcal{D}+1)^A \not\cong \mathcal{D}(A \times \_) + 1$$

Probabilistic system types

MC	$\mathcal{D}$
DLTS	$(\_ + 1)^A$
LTS	$\mathcal{P}(A \times \_) \cong \mathcal{P}^A$
React	$(\mathcal{D} + 1)^A$
Gen	$\mathcal{D}(A \times \_) + 1$
Str	$\mathcal{D} + (A \times \_) + 1$
Alt	$\mathcal{D} + \mathcal{P}(A \times \_)$
Var	$\mathcal{D}(A \times \_) + \mathcal{P}(A \times \_)$
SSeg	$\mathcal{P}(A \times \mathcal{D})$
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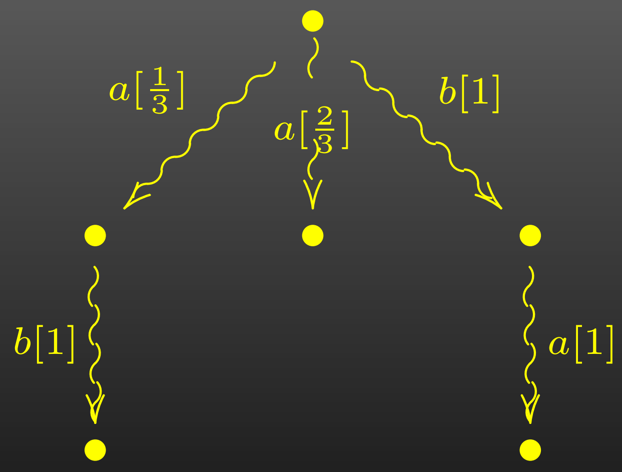

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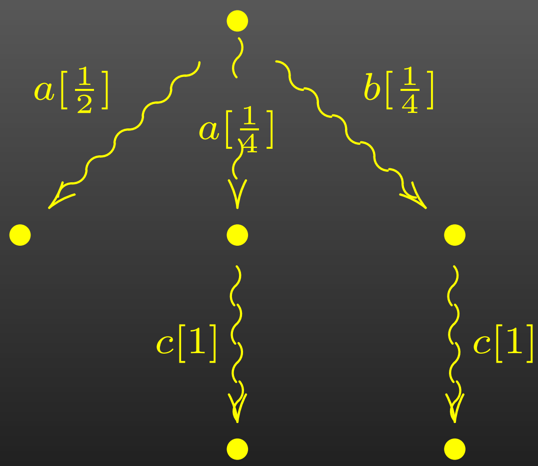

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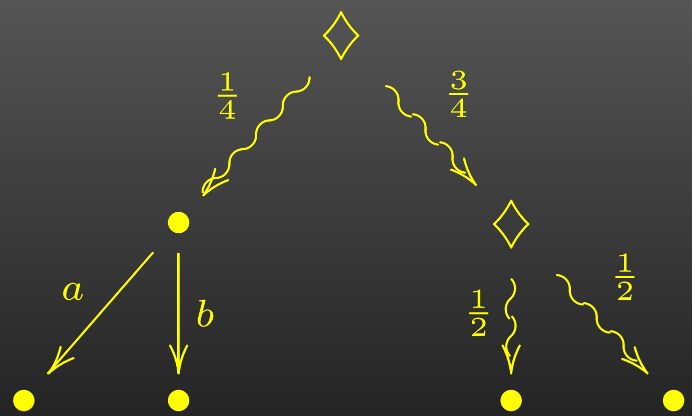

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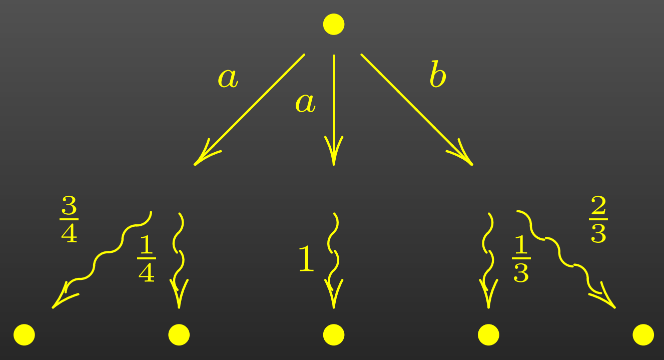

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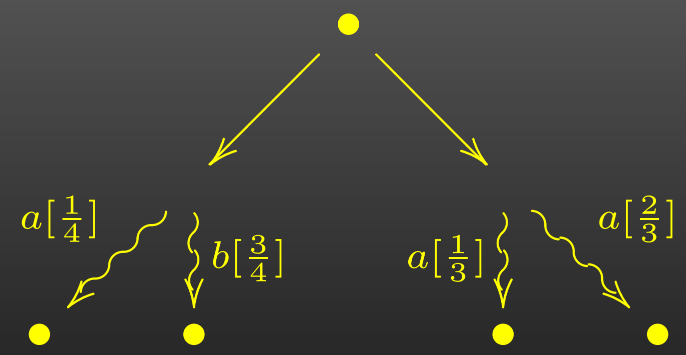
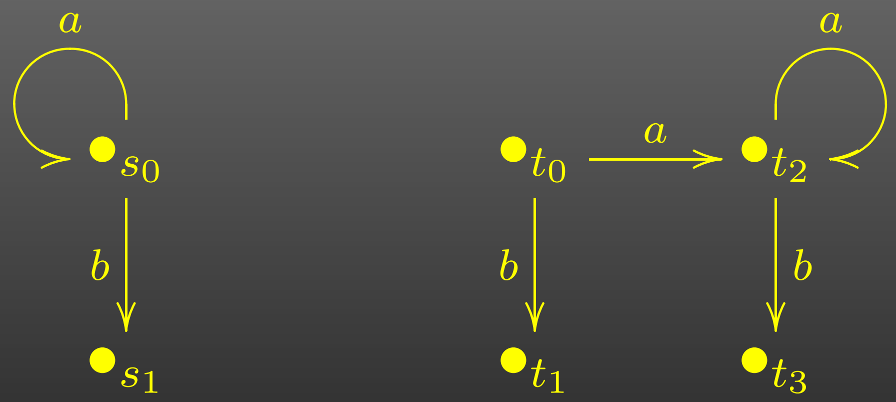


Bisimulation - LTS

Consider the LTS



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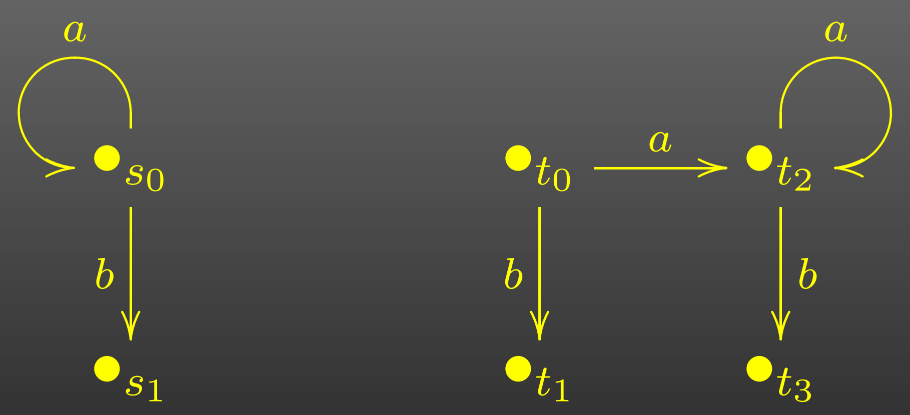

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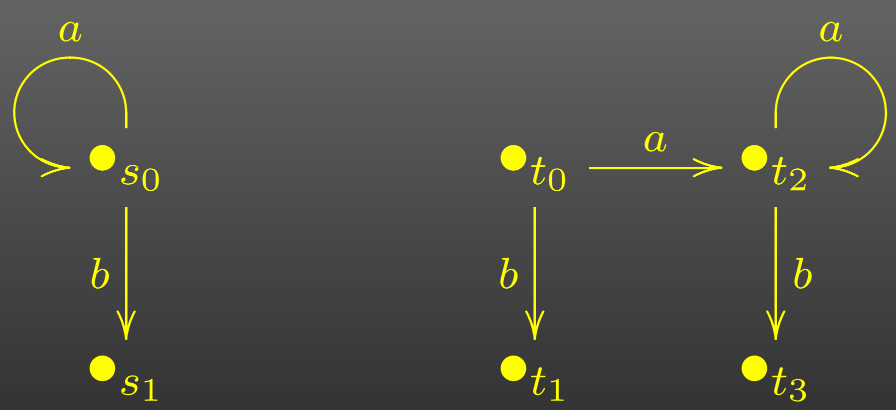

Transfer condition: $\langle s, t\rangle \in R \quad \Longrightarrow$

$$s \xrightarrow{a} s' \Rightarrow (\exists t')\ t \xrightarrow{a} t',\ \langle s', t'\rangle \in R,$$

$$t \xrightarrow{a} t' \Rightarrow (\exists s')\ s \xrightarrow{a} s',\ \langle s', t'\rangle \in R$$

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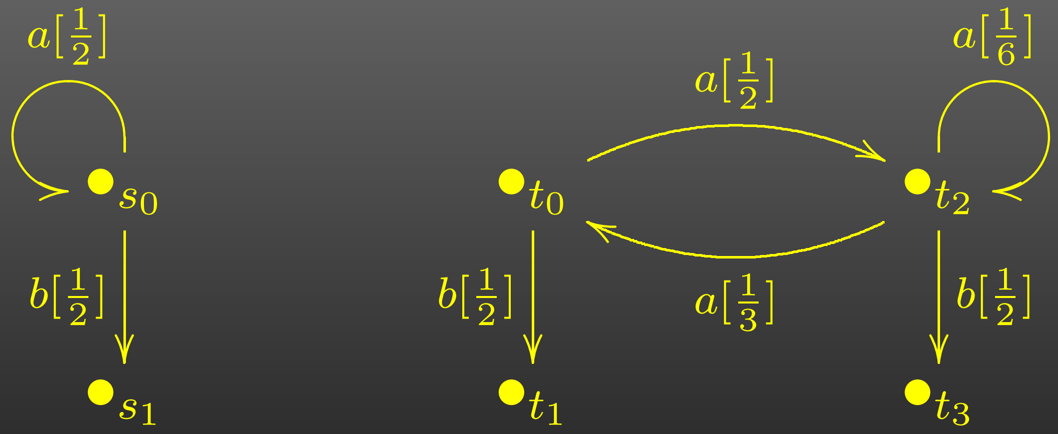

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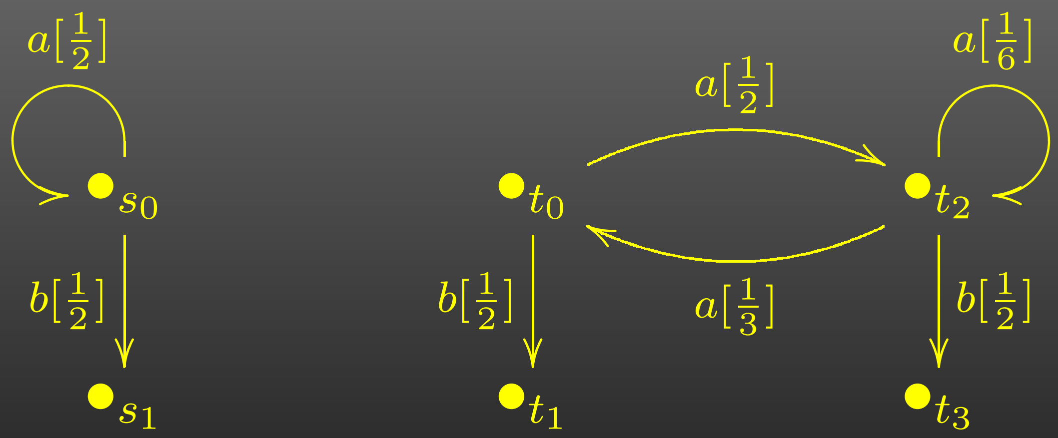

The states s_0 and t_0 are bisimilar, and so are s_0 and t_2, since there is a bisimulation R relating them...

Bisimulation - generative

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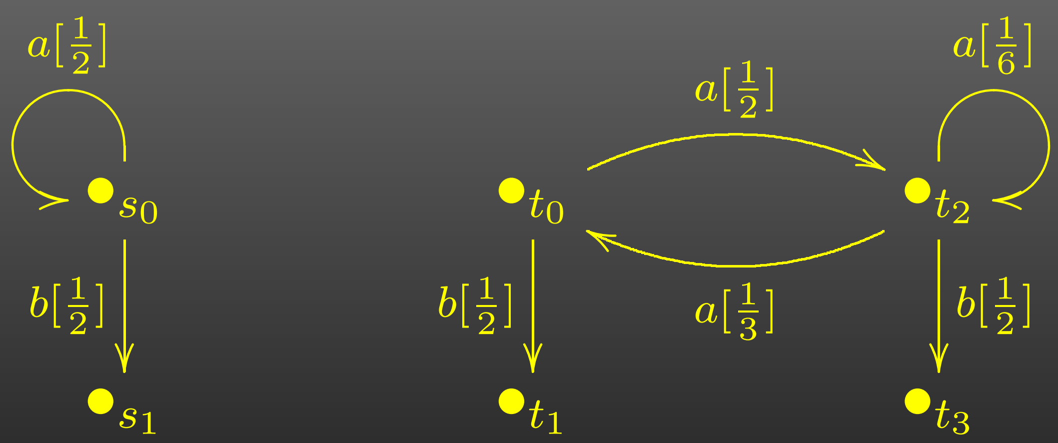

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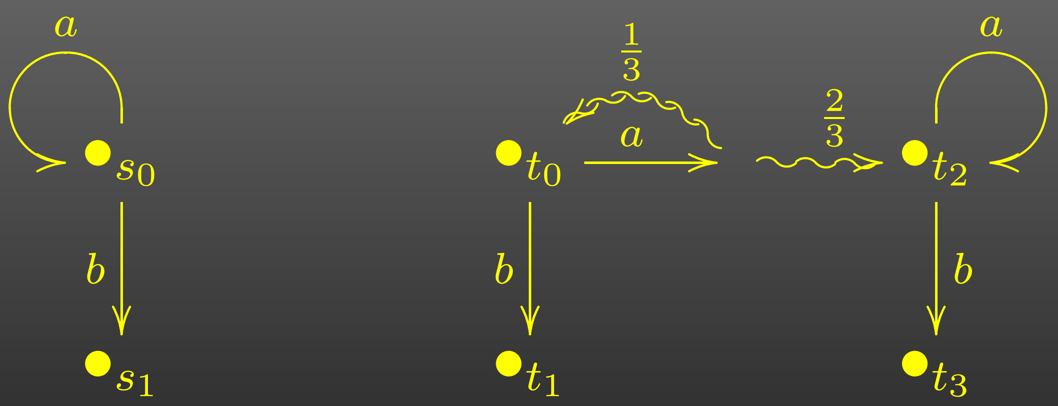
$$s \rightsquigarrow \mu \Rightarrow (\exists \mu')\ t \rightsquigarrow \mu',\ \ \mu \equiv_{R,A}\ \mu'$$

Bisimulation - simple Segala

Consider the simple Segala systems



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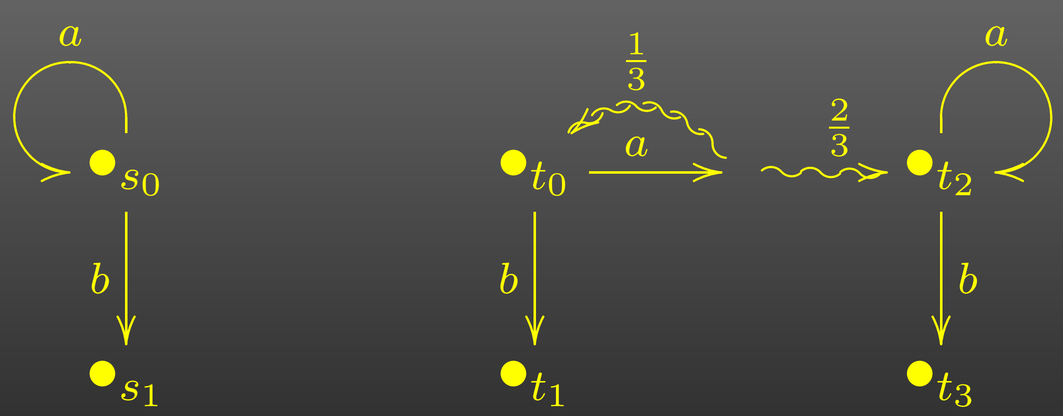

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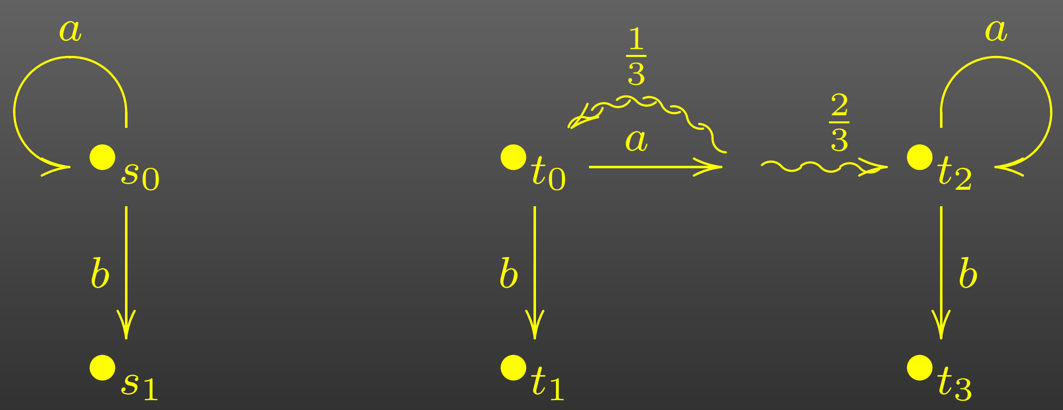

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Coalgebraic bisimulation

A bisimulation between

$$\langle S, \alpha : S \to \mathcal{F}S \rangle \text{ and } \langle T, \beta : S \to \mathcal{F}S \rangle$$

is $R \subseteq S \times T$ such that $\exists\, \gamma$:

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$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\gamma"] & T \arrow[d, "\beta"] \\
\mathcal{F}S & \mathcal{F}R \arrow[l, "\mathcal{F}\pi_1"] \arrow[r, "\mathcal{F}\pi_2"'] & \mathcal{F}T
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Transfer condition: $\langle s, t \rangle \in R \implies \langle \alpha(s), \beta(t) \rangle \in \mathrm{Rel}(\mathcal{F})(R)$

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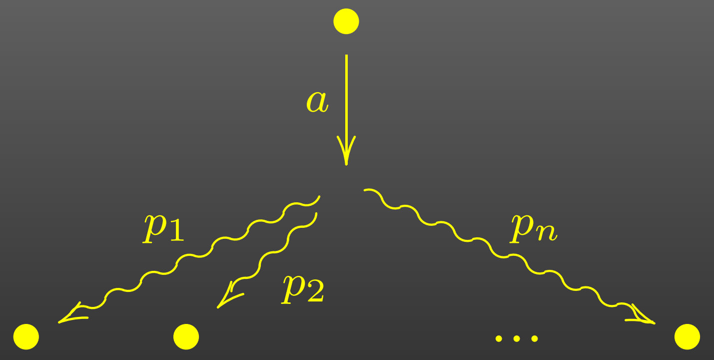
Theorem: Coalgebraic and concrete bisimilarity coincide !

Expressiveness

simple Segala system $\longrightarrow$ Segala system



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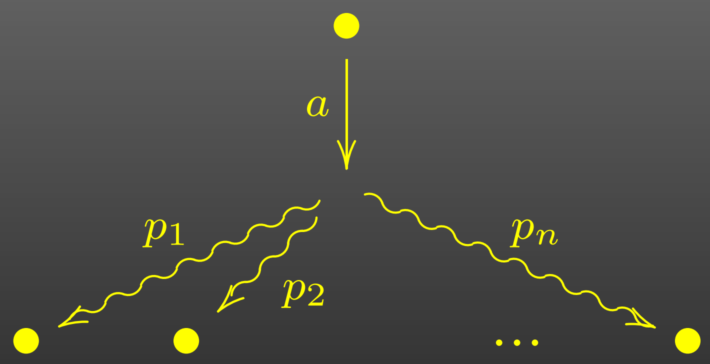

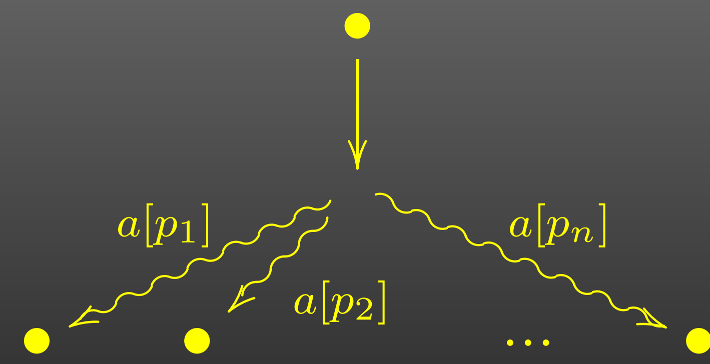

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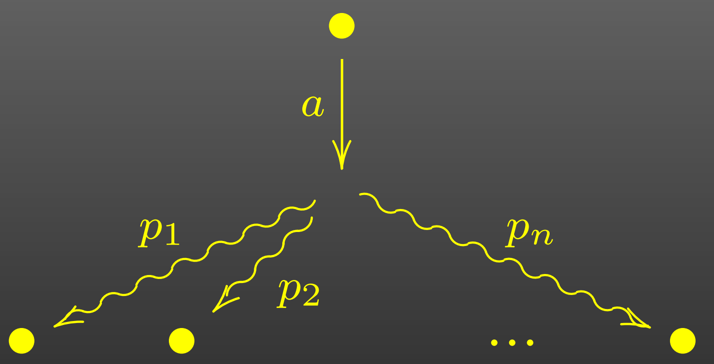

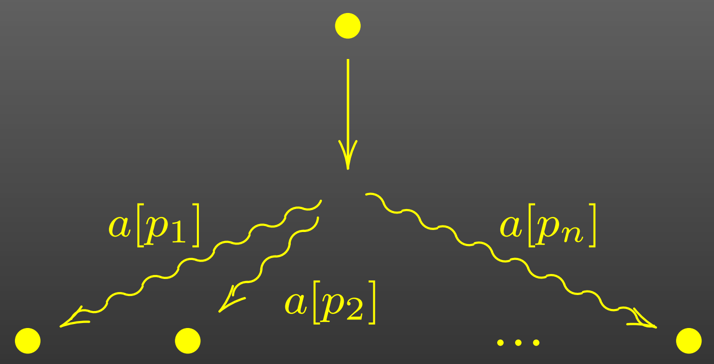

When do we consider one type of systems more expressive than another?

Comparison criterion

$$\mathsf{Coalg}_{\mathcal{F}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}$$

if there is a mapping $\langle S, \alpha : S \to \mathcal{F}S \rangle \overset{\mathcal{T}}{\mapsto} \langle S, \tilde{\alpha} : S \to \mathcal{G}S \rangle$
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$$s_{\langle S,\alpha\rangle} \sim t_{\langle T,\beta\rangle} \iff s_{\mathcal{T}\langle S,\alpha\rangle} \sim t_{\mathcal{T}\langle T,\beta\rangle}$$

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Theorem: An injective natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ is sufficient for $\mathsf{Coalg}_{\mathcal{F}} \longrightarrow \mathsf{Coalg}_{\mathcal{G}}$

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proof via cocongruences - behavioral equivalence

Example

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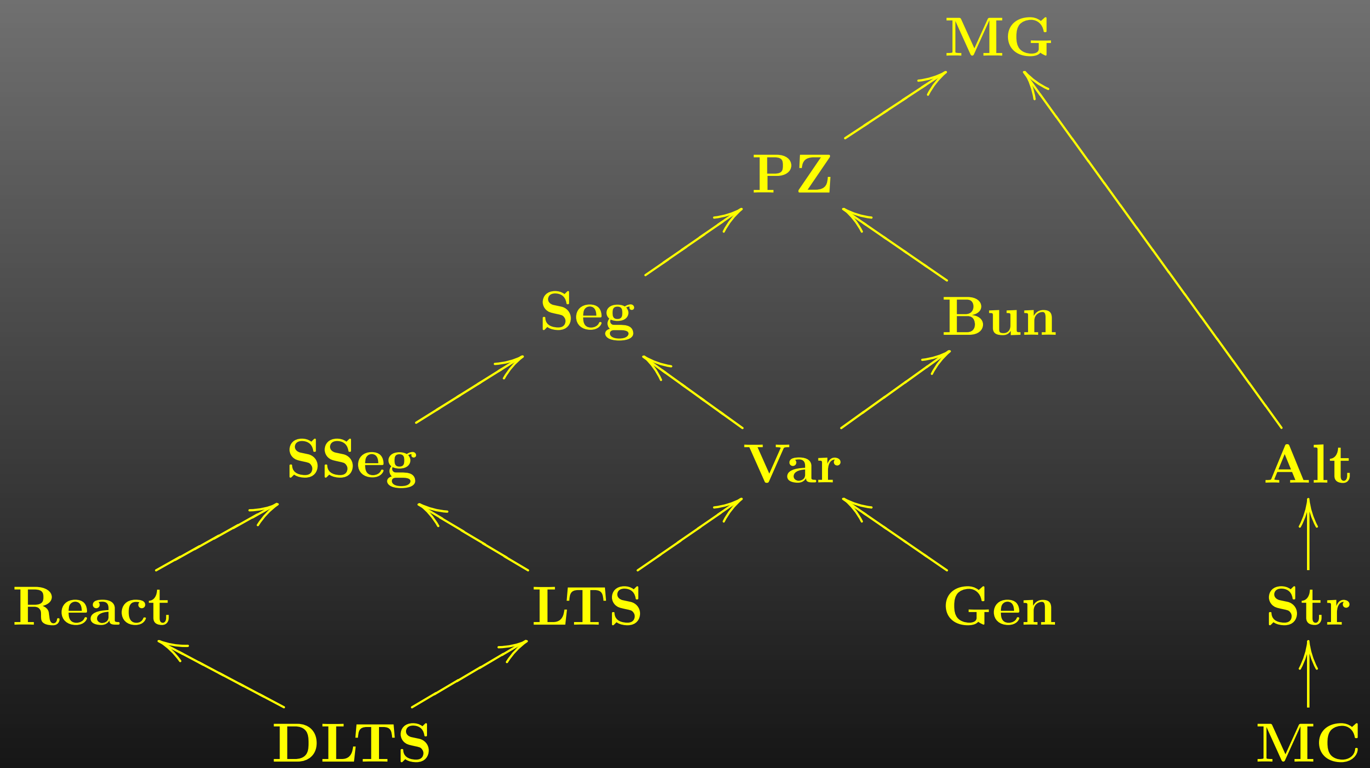
and δ_a is Dirac distribution for a

The hierarchy...



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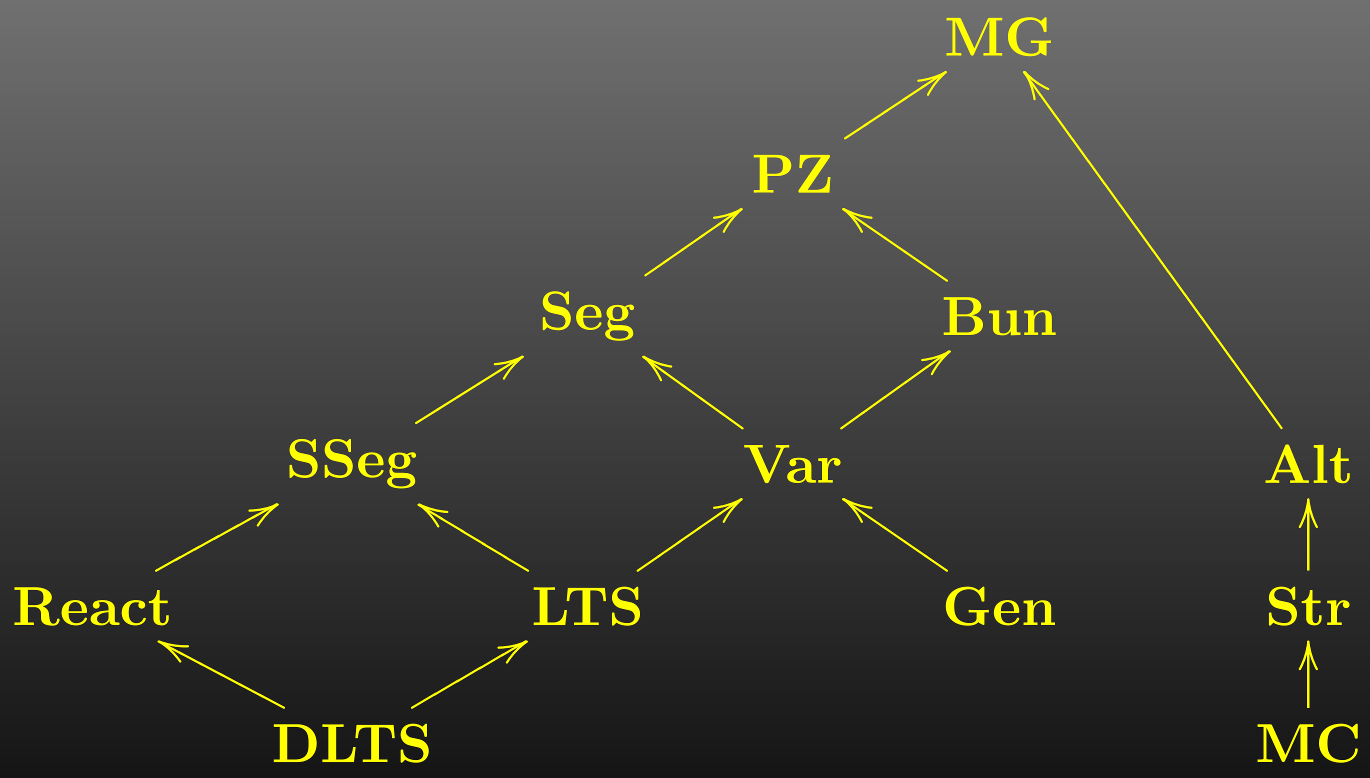


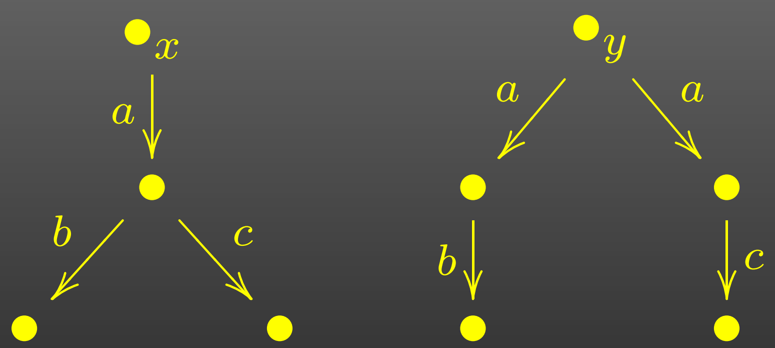
* Falk Bartels, AS, Erik de Vink

LT/BT spectrum

Bisimilarity is not the only semantics...

LT/BT spectrum

Are these non-deterministic systems equal ?



LT/BT spectrum

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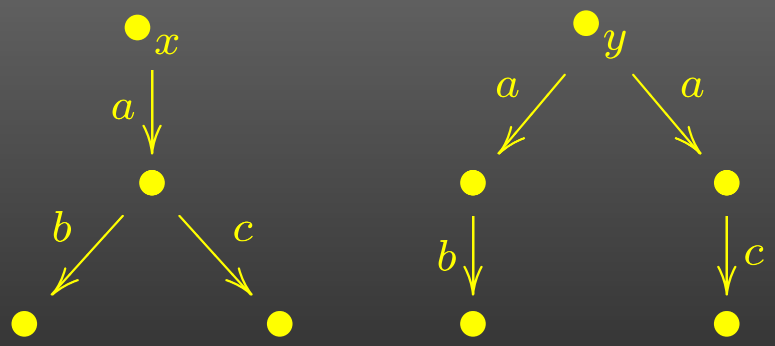

x and y are:

- different wrt. bisimilarity

LT/BT spectrum

Are these non-deterministic systems equal ?

$$
\begin{array}{ccc}
\begin{tikzcd}[column sep=small]
& x \arrow[d, "a"'] & \\
& \bullet \arrow[dl, "b"'] \arrow[dr, "c"] & \\
\bullet & & \bullet
\end{tikzcd}
& \qquad &
\begin{tikzcd}[column sep=small]
& y \arrow[dl, "a"'] \arrow[dr, "a"] & \\
\bullet \arrow[d, "b"'] & & \bullet \arrow[d, "c"] \\
\bullet & & \bullet
\end{tikzcd}
\end{array}
$$

x and y are:

- different wrt. bisimilarity, but
- equivalent wrt. trace semantics
  $\mathrm{tr}(x) = \mathrm{tr}(y) = \{ab, ac\}$

Traces - LTS

For LTS with explicit termination (NA)

trace = the set of all possible linear behaviors

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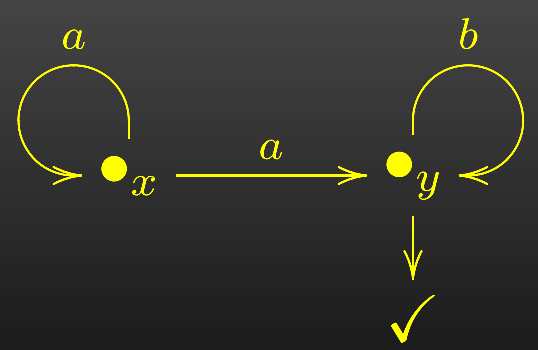

$$\mathrm{tr}(y) \;=\; b^*, \qquad \mathrm{tr}(x) \;=\; a^+ \cdot \mathrm{tr}(y) \;=\; a^+ \cdot b^*$$

Traces - generative

For generative probabilistic systems with ex. termination

trace = sub-probability distribution over possible linear behaviors

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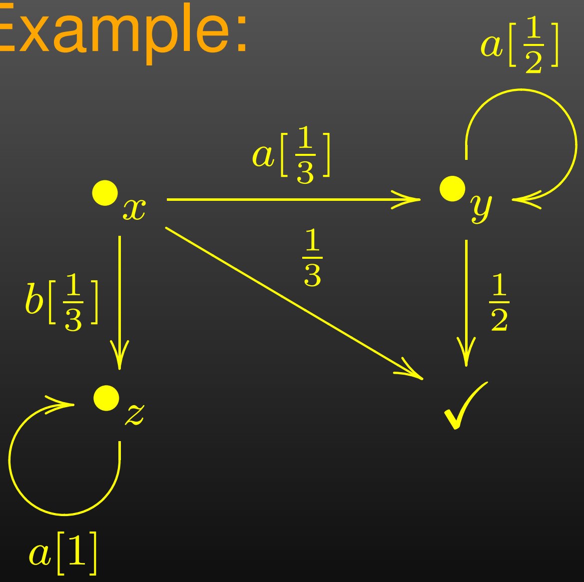

$$\mathrm{tr}(x): \quad \langle\rangle \mapsto \frac{1}{3}$$

$$a \mapsto \frac{1}{3} \cdot \frac{1}{2}$$

$$a^2 \mapsto \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$\cdots$$

Trace of a coalgebra ?

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main idea: coinduction in a Kleisli category

Coinduction

$$\begin{tikzcd} \mathcal{F}X \arrow[r, dashed, "\mathcal{F}(\mathrm{beh})"] & \mathcal{F}Z \\ X \arrow[u, "\alpha"] \arrow[r, dashed, "\mathrm{beh}"'] & Z \arrow[u, "\cong"'] \end{tikzcd}$$

system final coalgebra

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\end{tikzcd}$$

system final coalgebra

- finality = $\exists!$(morphism for any $\mathcal{F}$- coalgebra)
- beh gives the behavior of the system
- this yields final coalgebra semantics

Coinduction

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system final coalgebra

- f.c.s. in $\mathbf{Sets}$ = bisimilarity
- f.c.s. in a Kleisli category = trace semantics

When does it work?

- Monad $\mathcal{T}$ s.t. $\mathcal{K}\ell(\mathcal{T})$ is $\mathbf{DCpo}_\perp$-enriched
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- $\mathcal{F}$ preserves ω-colimits

Main Theorem

If • • •• and $a : \mathcal{F}I \xrightarrow{\cong} I$ denotes the initial $\mathbf{Sets}$-algebra, then

$$\begin{tikzcd} \mathcal{K}\ell(\mathcal{F})I \arrow[d, "\eta_I \circ a"', "\cong"] \\ I \end{tikzcd} \qquad \begin{tikzcd} \mathcal{K}\ell(\mathcal{F})I \\ I \arrow[u, "\eta_{\mathcal{F}I} \circ a^{-1}", "\cong"'] \end{tikzcd}$$

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proof: via limit-colimit coincidence Smyth&Plotkin '82

Corollary

Let • • •• and $a : \mathcal{F}I \xrightarrow{\cong} I$ denote the initial $\mathbf{Sets}$-algebra.
For $\alpha : X \to \mathcal{K}\ell(\mathcal{F})X$ in $\mathcal{K}\ell(\mathcal{T})$ i.e. $\alpha : X \to \mathcal{T}\mathcal{F}X$ in $\mathbf{Sets}$

$\exists!$ trace map $\mathrm{tr}(\alpha) : X \to \mathcal{T}I$ such that in $\mathcal{K}\ell(\mathcal{T})$:

```tikzcd
\mathcal{K}\ell(\mathcal{F})X \arrow[r, dashed, "\mathcal{K}\ell(\mathcal{F})(\mathrm{tr}(\alpha))"] & \mathcal{K}\ell(\mathcal{F})I \\
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```

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Finite traces - LTS with ✓

the finality diagram in $\mathcal{K}\ell(\mathcal{P})$

$$\begin{tikzcd}
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X \arrow[u, "\alpha"] \arrow[r, dashed, "\mathrm{tr}(\alpha)"'] & A^* \arrow[u, "\cong"']
\end{tikzcd}$$

amounts to

- $\langle\rangle \in \mathrm{tr}(\alpha)(x) \iff \checkmark \in \alpha(x)$
- $a \cdot w \in \mathrm{tr}(\alpha)(x) \iff (\exists x')\langle a, x'\rangle \in \alpha(x),\ w \in \mathrm{tr}(\alpha)(x')$

Finite traces - generative ✓

the finality diagram in $\mathcal{K}\ell(\mathcal{D})$

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amounts to $\mathrm{tr}(\alpha)(x)$:

- $\langle\rangle \mapsto \alpha(x)(\checkmark)$
- $a \cdot w \mapsto \sum_{y \in X} \alpha(x)(a, y) \cdot \mathrm{tr}(\alpha)(y)(w)$

Parallel composition

For $u, v \in \mathcal{P}(A^*)$ the (shuffle) parallel composition $u \parallel v$:

$$\langle\rangle \in u \parallel v \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad \langle\rangle \in u \quad \text{and} \quad \langle\rangle \in v$$

$$a \cdot w \in u \parallel v \quad \stackrel{\text{def}}{\Longleftrightarrow} \quad w \in \partial_a u \parallel v \quad \text{or} \quad w \in u \parallel \partial_a v$$

for $\quad \partial_a u = \{w \in \Sigma^* \mid a \cdot w \in u\}$

can be defined by coinduction

Conclusions & future work

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- Other semantics that are between bisimilarity and trace in the spectrum
- Parallel composition of ”probabilistic languages"