Nawrotzki's Algorithm, for the Countable Splitting Lemma, Constructively

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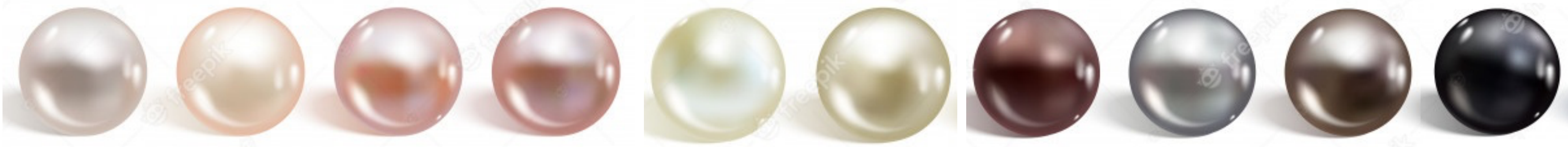

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Given: Two measures μ, ν on X, Y, respectively

Goal: Find a measure on $X \times Y$ with marginals μ, ν

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What if we require additional properties?

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topological, with Borel σ-algebra

D(X) - probability measures on X

Given: Two probability measures μ, ν on X, Y, respectively, $\Lambda \subseteq D(X \times Y)$

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$$\int_X f d\mu + \int_Y g d\nu \leqslant \sup \left\{ \int_{X \times Y} (f \oplus g) d\lambda \mid \lambda \in \Lambda \right\}$$

$$(f \oplus g)(x, y) = f(x) + g(y)$$

for bounded, measurable f,g

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and a relation $R \subseteq X \times Y$

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Stochastic dominance ?

Example: A politician can pick up strategies for the elections. One strategy ν stochastically dominates another strategy μ iff the outcome under ν is always better.

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This amounts to ★ for the set of natural numbers and any well-order on it,

and to ★ from Strassen's theorem, in this special case.

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Countable Splitting Lemma (Levy)

Kellerer's proof

1. Proves the finite case
2. Considers cutoffs μ_n, ν_n
3. Produces λ_n on $\{1, \ldots, n\} \times \{1, \ldots, n\}$ with marginals μ_n, ν_n
4. Takes the (pointwise) limit … but it does not necessarily exist.

Way out: Choose a subsequence $(n_l)_{l=0}^{\infty}$ such that the limit

$$\lambda(i,j) = \lim_{l \to \infty} \lambda_{n_l}(i,j) \text{ exists for all } i, j$$

Nawrotzki's proof

Also produces approximations, but differently - not with cutoffs

Produces λ_n with the monotonicity property:

$$i = j \Rightarrow \lambda_{n+1}(i,j) \leqslant \lambda_n(i,j)$$
$$i \neq j \Rightarrow \lambda_{n+1}(i,j) \geqslant \lambda_n(i,j)$$

non increasing on the diagonal
non decreasing off the diagonal

These approximations do not have "correct" marginals (in general).

Defines $\lambda(i,j) = \lim_{n\to\infty} \lambda_n(i,j)$

exists by monotonicity

Proves that this limit has the correct marginals.

Nonconstructiveness

Kellerer: Each approximation λ_n is computable.

Nonconstructiveness due to limit by compactness argument.

Heine-Borel: On a compact subset of real numbers, every sequence has a converging subsequence…
but how to find it ?

Nawrotski: Nonconstructiveness is in the definition of the approximations λ_n

requires computing a sum of an infinite series and evaluating suprema of infinite sets

Only λ_1 is computable, the others not.

Nonconstructiveness

Strassen's proof is super-nonconstructive — compactness comes in on every corner !

Banach-Alaoglu, Riesz-Markov representation, Krein-Milman

Our proof

follows Nawrotzki,
uses ideas of cutoffs

4/5 Nawrotzki + 1/5 Kellerer

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Still does not have ”correct” marginals.

Has computable error estimate, for fixed position i, j

ℓ^1

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Thank You !